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LETTERS TO THE EDITOR

BONFERRONI-TYPE INEQUALITIES AND THE METHODS OF INDICATORS AND POLYNOMIALS

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Abstract

A difficulty in the application [5] of the method of polynomials as exposited by Galambos is investigated. The method, recast as the method of indicators, in a form due originally to Rényi [6], is applied to the situation of non-constant coefficients.

MONOTONICITY; NON-CONSTANT COEFFICIENTS; SOBEL–UPPULURI INEQUALITIES

1. Introduction and motivation

Let $A_1, \cdots, A_n$ be events on a probability space, $B_{r,n}$, $0 \leqq r \leqq n$, the event that exactly r of the A's occur, $S_{k,n} = \sum P(A_{i_1} A_{i_2} \cdots A_{i_k})$, summed on all $1 \leqq i_1 < i_2 \cdots < i_k \leqq n$ with $s_{0,n} = 1$. The method of polynomials is a device described in [2] for deducing linear inequalities for $P(B_{r,n})$ in terms of the $S_{k,n}$, $k = 0, 1, \cdots$ after such inequalities have been obtained in the case of Bernoulli trials (that is, independent events having the same probability $P(A_i)$ of occurrence). Galambos' statement ([2], Theorem 3) of this procedure may be given as follows. Let $\{d_k\}$ be a sequence of real numbers with r fixed. Then (1) equivalent to (2) where:

$$\binom{n}{r} p^r (1-p)^{n-r} \leqq \sum_{k=0}^{n} d_k \binom{n}{k} p^k \quad \text{for all } n \geqq 0 \text{ and all } 0 \leqq p \leqq 1; \tag{1}$$

$$P(B_{r,n}) \leqq \sum_{k=0}^{n} d_k S_{k,n} \quad \text{for all } n \geqq 0 \text{ and all events } \{A_1, \cdots, A_n\}. \tag{2}$$

It is part of this formulation that (1) hold for all n. But if the coefficients $\{d_k\}$ depend on n then the equivalence need not be valid. Thus (1), in the case of the Sobel–Uppuluri [7] bounds (taking $r = 0$ in (1)) reads

$$(1-p)^n \leqq \sum_{k=0}^{2u} (-1)^k \binom{n}{k} p^k - \frac{2u+1}{n} \binom{n}{2u+1} p^{2u+1} \tag{3}$$

(where $u \geqq 1$ is fixed and we use the convention that $\binom{n}{k} = 0$ if $n < k$), the coefficient $d_{2u+1} = -(2u+1)/n$ depending on n, so the step to (2) cannot be made with the technology

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cited. For this reason in [4], p. 263, there is a remark about the need for monotonicity of coefficients in n. (We are grateful for this and other remarks on an earlier draft to Professor J. Galambos.) However, in the case actually considered in [4] the coefficients are independent of n. We clarify this issue.

Further it is indicated in [4], through an example, that it is sufficient to obtain an inequality in the Bernoulli case only for $P(B_{0,n})$ for all $n \geqq 0$. A general version of this stated without detail in [5], Section 2 is the implication (4) $\Rightarrow$ (5) where:

(4) $(1-p)^n \leqq \sum_{k=0}^{n} d_k(n)\binom{n}{k}p^k$ for all $n \geqq 0$ and all $0 \leqq p \leqq 1$ (where $d_0(0)=1$);

(5) $P(B_{r,n}) \leqq \sum_{k=r}^{n} d_{k-r}(n-r)\binom{k}{r}S_{k,n}$ for all $n \geqq 0$ and all events $\{A_1, \cdots, A_n\}$.

The logic behind the assertion is as follows: (4) implies

$$\binom{n}{r}p^r(1-p)^{n-r} \leqq \sum_{i=0}^{n-r} d_i(n-r)\binom{n-r}{i}\binom{n}{r}p^{i+r} = \sum_{k=r}^{n} d_{k-r}(n-r)\binom{k}{r}\binom{n}{k}p^k$$

following which (1) $\Rightarrow$ (2) would result in (5). Again, the deduction from (1) to (2), giving in this case (5) from (4) is inappropriate, since the coefficient of $\binom{n}{k}p^k$, namely $d_{k-r}(n-r) \times \binom{k}{r}$, depends in general on n. In fact, it is not true that (4) implies (5). To see this let the triangular array $\{d_k(n)\}$, $0 \leqq k \leqq n$, $n \geqq 0$, be defined as follows: $d_0(0)=1$; $d_0(1)=1$, $d_1(1)=-1$; $d_0(2)=5$, $d_1(2)=-7$, $d_2(2)=10$; and for $n \geqq 3$, $d_0(n)=1$; $d_1(n)=-1$, $d_2(n)=1$, and $d_k(n)=0$ for $3 \leqq k \leqq n$. Then it is easily verified that (4) is satisfied for $n=0$ (obviously); for $n=1$ (both sides are $1-p$); for $n=2$ (the left-hand side is $(1-p)^2$ while the right-hand side is $5-14p+10p^2$, which is larger for $0 \leqq p \leqq 1$; and for $n \geqq 3$ since (4) is then just an ordinary Bonferroni inequality $P(B_{0,n}) = 1 - P(\bigcup A_i) \leqq S_{0,n} - S_{1,n} + S_{2,n}$. Yet, with $n=2$ and $r=0$, (5) asserts

$$P(B_{0,2}) \leqq d_0(2)S_{0,2} + d_1(2)S_{1,2} + d_2(2)S_{2,2} = 5 - 7S_{1,2} + 10S_{2,2}$$

while with the choices $A_1 = \Omega$, the sample space, and $A_2 = \varnothing$, the left-hand side of this inequality is 0, whereas the right-hand side is -2.

The question which needs clarification therefore is when (4) $\Rightarrow$ (5), with the triangular array $\{d_k(n)\}$, $0 \leqq k \leqq n$, $n \geqq 0$, satisfying a condition more general than: $d_k(n)$ is constant as n increases, for fixed k. We shall ultimately show that (5) holds if, in addition to (4), for fixed k

(6) $$d_k(n) \leqq d_k(n+1) \qquad n \geqq 0.$$

Further, if (4) and (6) hold with inequalities reversed, then (5) holds with its inequality reversed. Such conditions are sufficient to permit the deduction of the Sobel–Uppuluri inequalities from corresponding Bernoulli inequalities such as (3). Actually we shall find that only $p=1$ in the Bernoulli trials is required and so to arrive at such conclusions it is necessary first to consider the implication (1) $\Rightarrow$ (2) in the setting of indicator functions.

2. Results and discussion

When the coefficients d_k depend on n then we are dealing with a system of inequalities, $d_k(n)$ replacing d_k in (2), and we have shown that without additional care it is not sufficient merely to validate the case (1) of Bernoulli trials. The difficulty lies in requiring (1) to hold for all n. To deal with a triangular array it becomes appropriate to first *fix n* thereby clarifying the role played by Bernoulli trials in the passage from (1) to (2) and we shall formulate our results accordingly. This leads back to the method of indicators in a form due to Rényi [6] whose original results were stated, more generally, for inequalities involving Boolean polynomials, of which (2) is an example.

Let n be *fixed*, let r, $0 \leqq r \leqq n$, be fixed, let a be real (and positive, without loss of generality), and define $\delta_{t,r} = 1$ or 0 as $t = r$ or $t \neq r$. The method of indicators [6] states (7)$\Leftrightarrow$(8) where:

(7) $$a\delta_{t,r} \leqq \sum_{k=0}^{t} d_k \binom{t}{k} \quad \text{for all } t = 0, 1, \cdots, n;$$

(8) $$aP(B_{r,n}) \leqq \sum_{k=0}^{n} d_k S_{k,n} \quad \text{for any } n \text{ events } A_1, \cdots, A_n.$$

Note that if (8) holds for n events, by taking some of them to be Ω, the rest $\varnothing$, then (8) is seen to hold for fewer than n events. The method thus asserts the principle that an inequality is valid for n or fewer (that is, t) events if and only if it is valid when these n or fewer events are each Ω.

By applying this idea twice we deduce (9)$\Rightarrow$(10) where:

(9) $$aP(B_{0,n}) \leqq \sum_{k=0}^{n} d_k S_{k,n} \qquad \text{for any } n \text{ events } A_1, \cdots, A_n;$$

(10) $$aP(B_{r,n+r}) \leqq \sum_{k=r}^{n+r} d_{k-r} \binom{k}{r} S_{k,n+r} \qquad \text{for any } n+r \text{ events } A_1, \cdots, A_{n+r}.$$

Proof. (9) is (8) with $r = 0$, equivalent therefore by (7), to

$$a\delta_{t,0} \leq \sum_{k=0}^{t} d_k \binom{t}{k},$$

for all $t = 0, 1, \cdots, n$. But,

$$a\delta_{t,r} = a\delta_{t-r,0} \leqq \sum_{k=0}^{t-r} d_k \binom{t-r}{k} = \frac{1}{\binom{t}{r}} \sum_{k=r}^{t} d_{k-r} \binom{k}{r}\binom{t}{k}$$

$$\leqq \sum_{k=r}^{t} d_{k-r} \binom{k}{r}\binom{t}{k} \quad \text{since} \quad \binom{t}{r} \geqq 1,$$

the inequalities holding for $r \leqq t \leqq r + n$, so defining

$$f_k = \begin{cases} 0 & \text{if } k = 0, 1, \cdots, r-1 \\ d_{k-r}\binom{k}{r} & \text{if } k = r, \cdots, r+n \end{cases}$$

we get

$$a\delta_{t,r} \leqq \sum_{k=0}^{t} f_k \binom{t}{k},$$

for all $t = 0, 1, \cdots, n + r$, which forces, again from the equivalence of (7) and (8),

$$aP(B_{r,n+r}) \leqq \sum_{k=0}^{n+r} f_k S_{k,n+r} = \sum_{k=r}^{n+r} d_{k-r} \binom{k}{r} S_{k,n+r}.$$

Formulated this way this method directly applies to triangular arrays. Let $\{d_k(n);\ 0 \leqq k \leqq n,\ n \geqq 0\}$ be such an array. Then (7)$\Leftrightarrow$(8) is replaced with (7a)$\Leftrightarrow$(8a) where:

(7a) $$a\delta_{t,r} \leqq \sum_{k=0}^{t} d_k(n) \binom{t}{k} \quad \text{for all } t = 0, 1, \cdots, n \text{ and } \textit{all } n \geqq r;$$

(8a) $$aP(B_{r,n}) \leqq \sum_{k=0}^{n} d_k(n) S_{k,n} \quad \text{for arbitrary } A_1, \cdots, A_n \text{ and } \textit{all } n \geqq r.$$

Similarly (9a) $\Rightarrow$ (10a) where:

(9a) $aP(B_{0,n}) \leqq \sum_{k=0}^{n} d_k(n) S_{k,n}$ for arbitrary $A_1, \cdots, A_n$ and *all* $n \geqq 0$;

(10a) $aP(B_{r,n}) \leqq \sum_{k=r}^{n} d_{k-r}(n-r) \binom{k}{r} S_{k,n}$ for arbitrary $A_1, \ldots, A_n$ and *all* $n \geqq r$.

Both assertions follow from their previous counterparts after identifying $d_k(n)$ with d_k, fixing n, and then replacing n with $n-r$ in (10a).

Proposition. Suppose the triangular array $\{d_k(n); 0 \leqq k \leqq n, n \geqq 0\}$ satisfies the monotonicity condition:

(11) for each $k \geqq 0$ $d_k(n) \leqq d_k(n+1)$ for all $n \geqq 0$.

Then (10a) is true if and only if (7a) holds whenever $t = n$ and $r = 0$, that is

(12) $$a\delta_{t,0} \leqq \sum_{k=0}^{t} d_k(t) \binom{t}{k} \quad \text{for } all\ t = 0, 1, \cdots.$$

Proof. In the presence of (11) if (12) is true then

$$a\delta_{t,0} \leqq \sum_{k=0}^{t} d_k(n) \binom{t}{k} \quad \text{for all } n \geqq t$$

which is just a restatement of (7a) for $r = 0$, which is equivalent to (8a) with $r = 0$, namely (9a), which then implies (10a). The converse follows similarly.

We offer, as illustrations of the methodology, three distinct situations. The first involves non-constant coefficients and is a case in point where this proposition applies, the second again involves non-constant coefficients, but monotonicity is lacking, and the third is a bound with constant coefficients.

Thus consider the family (3) of Sobel–Uppuluri bounds [7] where $a = 1$ and the coefficients $d_k(n)$, which satisfy (11), are: $(-1)^k$ if $k \leqq 2u$; $-(2u+1)/n$ if $k = 2u+1$; and 0 if $2u+2 \leqq k$. Checking (12), taking $a = 1$: when $t = 0$, $d_0(t) = 1 \geqq \delta_{t,0} = 1$; when $1 \leqq t \leqq 2u$,

$$\sum_{k=0}^{t} d_k(t) \binom{t}{k} = \sum_{k=0}^{t} (-1)^k \binom{t}{k} = 0 = \delta_{t,0};$$

and when $2u + 1 \leqq t$,

$$\sum_{k=0}^{t} d_k(t) \binom{t}{k} = \sum_{k=0}^{2u} (-1)^k \binom{t}{k} - \frac{2u+1}{t} \binom{t}{2u+1}$$
$$= (-1)^{2u} \binom{t-1}{2u} - \binom{t-1}{2u} = 0 = \delta_{t,0},$$

using the identity, for $0 \leqq s < t$,

$$\sum_{k=0}^{s} (-1)^k \binom{t}{k} = (-1)^s \binom{t-1}{s}.$$

Thus providing $2u + 1 \leqq n - r$,

(13) $$P(B_{r,n}) \leqq \sum_{k=0}^{2u} (-1)^k \binom{r+k}{r} S_{r+k,n} - \frac{2u+1}{n-r} \binom{r+2u+1}{r} S_{r+2u+1,n}$$

obtained in [3].

For the second bound, by first verifying (7a) for $r = 0$, we establish that

$$P(B_{0,n}) \geqq 1 - S_{1,n} + (2n-3)\Big/\binom{n}{2} S_{2,n} - (n-2)\Big/\binom{n}{3} S_{3,n}$$

where for $n = 0$, 1, 2, 3, and $0 \leqq k \leqq n$, $d_k(n) = (-1)^k$; and for $n \geqq 4$, $d_k(n) = (-1)^k$, for $k = 0$, $1; = (2n-3)\Big/\binom{n}{2}$, $k = 2$; $= -(n-2)\Big/\binom{n}{3}$, $k = 3$; $= 0$, $k \geqq 4$. Although our results have been phrased in terms of upper bounds on $P(B_{r,n})$ parallel lower bounds are valid with a corresponding reversal in the monotonocity (11). The only difference lies in the proof of $(9) \Rightarrow (10)$ where use of $\binom{t}{r} \geqq 1$ was made to eliminate this combinatorial factor. This is no longer available since the relevant inequality is reversed. Instead we argue when $t \neq r$ the term $a\delta_{t,r}$ is 0 so $\binom{t}{r}$ can be eliminated, while when $t = r$, $\binom{t}{r} = 1$ and so can also be removed. The coefficients $d_2(n) = (2n-3)\Big/\binom{n}{2}$ decrease to 0 while $d_3(n) = -(n-2)\Big/\binom{n}{3}$ increase to 0 as n increases. Our proposition is thus not applicable and instead (7a) must be checked for each r. We can then deduce for $n \geqq r$

$$P(B_{r,n}) \geqq S_{r,n} - (r+1)S_{r+1,n} + \frac{2n-2r-3}{\binom{n-r}{2}}\binom{r+2}{2} S_{r+2,n} - \frac{n-r-2}{\binom{n-r}{3}}\binom{r+3}{3} S_{r+3,n}.$$

Finally, although the coefficients do not depend on n, we derive, using the method of indicators, the result of [1], in contrast to, and considerably more directly than, the calculus-based derivation by the method of polynomials, in [4]. The bound of interest is

$$(14) \qquad P(B_{0,n}) \leqq 1 - \frac{2}{j+1} S_{1,n} + \frac{2}{j(j+1)} S_{2,n}, \quad \text{for any integer } j \geqq 1.$$

Rényi's method of indicators now applies directly and we need only verify (7) with $r = 0$, which becomes:

$$1 \leqq 1 \quad \text{for} \quad t = 0; \qquad 0 \leqq 1 - \frac{2}{j+1} \quad \text{for} \quad t = 1;$$

$$0 \leqq 1 - \frac{2}{j+1}\binom{t}{1} + \frac{2}{j(j+1)}\binom{t}{2} \quad \text{for} \quad t \geqq 2.$$

The second inequality reduces to $1 \leqq j$ and the last inequality simplifies to $0 \leqq (j-t)^2 + (j-t)$ both of which are obviously true. Hence (14) holds for all n.

3. The method of polynomials. Concluding remarks

It is possible to phrase the preceding section in terms of Bernoulli events rather than indicators. In (8) the choices A_i, $i = 1, \cdots, t$, where A_i denotes a success on the ith of t independent Bernoulli trials with $P(A_i) = p$, $0 \leqq p \leqq 1$, while $A_{t+1} = A_{t+2} \cdots = A_n = \varnothing$, result in

$$(7p) \qquad a\binom{t}{r} p^r (1-p)^{t-r} \leqq \sum_{k=0}^{t} d_k \binom{t}{k} p^k \quad \text{for all } t = 0, 1, \cdots, n \text{ and all } 0 \leqq p \leqq 1$$

which is (7) for $p = 1$, implying (8), thereby establishing that $(7p) \Leftrightarrow (8)$, which differs slightly from $(1) \Leftrightarrow (2)$ in that n is fixed. Similar reformulations can be established with regard to (10), (8a), (10a), where now fixing n is essential, and, in particular, (12) in the proposition is replaced with

$$a(1-p)^t \leqq \sum_{k=0}^{t} d_k(t)\binom{t}{k}p^k \quad \text{for all } t=0, 1, \cdots \text{ and all } p, 0 \leqq p \leqq 1$$

from which (10a) and consequently (13) are deduced. This is precisely the set-up for the implication (4) ⇒ (5) thereby justifying its assertion in [5].

The method of polynomials requires an inequality valid for all $0 \leqq p \leqq 1$, which may be obtainable by use of calculus [4] or power series with remainder. Thereafter, however, implications may be made using the case $p = 1$ only, that is, by the method of indicators.

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