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A LIGHT-WEIGHT VERSION OF WARING'S PROBLEM

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Abstract

An asymptotic formula is established for the number of representations of a large integer as the sum of kth powers of natural numbers, in which each representation is counted with a homogeneous weight that de-emphasises the large solutions. Such an asymptotic formula necessarily fails when this weight is excessively light.

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1. Introduction

Investigations concerning the asymptotic formula in Waring's problem have played a central role in the development of the circle method since its inception by Hardy and Littlewood in the early part of the twentieth century. From this classical asymptotic relation, it is relatively straightforward to obtain a formula for the number of representations of a natural number, as the sum of a fixed number of kth powers of positive integers, in which each representation is counted with a weight that increases with the size of the integers involved in the representation. In such heavy-weight versions of Waring's problem, the larger, more typical, representations dominate, and it is these representations that the circle method most readily detects. In contrast, the light-weight versions of Waring's problem, in which representations are counted with a weight that decreases with the size of the integers occurring in the representation, pose some technical difficulties that have apparently deterred investigation. A particular case of the light-weight problem plays a fundamental role in work of Van Vu [5], and

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is resolved in essence in recent work of the author [7]. Since this light-weight version of Waring's problem lies well within the grasp of modern methods, our purpose in this paper is to establish and promote the asymptotic formulae associated with this circle of problems.

We begin with some notation. When s and k are positive integers, and ω is a real number, we define

$$R_{s,k}(n;\omega) = \sum_{\substack{x_1,\dots,x_s\in\mathbb{N}\\ x_1^k+\cdots+x_s^k=n}} (x_1\cdots x_s)^{\omega}. \tag{1.1}$$

We write $\mathfrak{s}(k)$ for the least positive integer s with the property that whenever $u \geq s$, one has for each $\varepsilon > 0$ the upper bound

$$\int_0^1 \left|\sum_{1\leq x\leq P} e(\alpha x^k)\right|^{2u} d\alpha \ll_{\varepsilon} P^{2u-k+\varepsilon}, \tag{1.2}$$

where, as usual, we write $e(z)$ for $e^{2\pi iz}$. We note for future reference that work of Hua [3] and Heath-Brown [2], respectively, establishes that $\mathfrak{s}(k) \leq 2^{k-1}$ $(k \geq 2)$ and $\mathfrak{s}(k) \leq \frac{7}{16}2^k$ $(k \geq 6)$. By employing modern versions of Vinogradov's mean value theorem (see Wooley [6]) together with work of Ford [1], moreover, one finds that for larger k one has

$$\mathfrak{s}(k) \leq \tfrac{1}{2}k^2(\log k + \log\log k + O(1)).$$

Finally, when s and k are natural numbers, we define the usual *singular series* $\mathfrak{S}_s(n) = \mathfrak{S}_{s,k}(n)$ associated with n by

$$\mathfrak{S}_{s,k}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1\\(a,q)=1}}^{q} (q^{-1}S(q,a))^s e(-na/q), \tag{1.3}$$

where

$$S(q,a) = \sum_{r=1}^{q} e(ar^k/q). \tag{1.4}$$

On considering the diagonal contribution underlying the mean value in (1.2), it is apparent that $\mathfrak{s}(k)$ satisfies the lower bound $\mathfrak{s}(k) \geq k$. The methods of Chapters 2 and 4 of Vaughan [4] then show that whenever $s \geq 2\mathfrak{s}(k)+1$, one has $0 \leq \mathfrak{S}_{s,k}(n) \ll 1$. Subject to the additional condition that, whenever k is a power of 2 with $k \geq 4$, then one has $s \geq 4k$, moreover, the aforementioned methods show also that $\mathfrak{S}_{s,k}(n) \gg 1$ uniformly in n.

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THEOREM 1.1. *Let s and k be natural numbers with $s \geq 2\mathfrak{s}(k) + 1$, and let δ be any real number with*

$$\delta < \left(\frac{s - 2\mathfrak{s}(k)}{s(s-1)}\right)\left(\frac{k}{2\mathfrak{s}(k)}\right).$$

Then there is a positive number τ, depending at most on s, k and δ, such that whenever ω is a real number with $\omega \geq -1 + k/s - \delta$, one has

$$R_{s,k}(n;\omega) = \frac{\Gamma((1+\omega)/k)^s}{\Gamma(s(1+\omega)/k)} k^{-s} \mathfrak{S}_{s,k}(n) n^{(1+\omega)s/k-1} + O(n^{(1+\omega)s/k-1-\tau}). \tag{1.5}$$

The special case of Theorem 1.1 in which $\omega = 0$ yields the classical asymptotic formula in Waring's problem, namely

$$\sum_{\substack{x_1,\dots,x_s \in \mathbb{N} \\ x_1^k + \cdots + x_s^k = n}} 1 = \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} (\mathfrak{S}_{s,k}(n) + o(1)) n^{s/k-1},$$

valid for $s > 2\mathfrak{s}(k)$. Meanwhile, the case in which $\omega = -1 + k/s$ is that central to the discussions of [5] and [7]. Here, again for $s > 2\mathfrak{s}(k)$, one obtains the pleasingly simple formula

$$\sum_{\substack{x_1,\dots,x_s \in \mathbb{N} \\ x_1^k + \cdots + x_s^k = n}} (x_1 \cdots x_s)^{-1+k/s} = k^{-s} \Gamma(1/s)^s \mathfrak{S}_{s,k}(n) + o(1).$$

It is worth noting that the formula (1.5) is established by Theorem 1.1 even for values of ω with $-1 + k/s > \omega \geq -1 + k/s - \delta$, wherein $R_{s,k}(n;\omega) \asymp n^{-\phi}$ with $\phi = 1 - (1+\omega)s/k > 0$. When k is large, and s is large enough in terms of k, the conclusion of Theorem 1.1 yields a permissible value for δ given by

$$\delta^{-1} = (1 + o(1))sk \log k. \tag{1.6}$$

Some sort of constraint on δ is certainly necessary, for the obvious representation of the integer $n = m^k + s - 1$ as the sum of s kth powers already yields the lower bound $R_{s,k}(n;\omega) \gg n^{\omega/k}$, and this exceeds the main term in (1.5) whenever $\omega < (k-s)/(s-1)$. It follows that the conclusion of Theorem 1.1 cannot be valid for all natural numbers n whenever

$$\delta > \frac{s-k}{s(s-1)},$$

though, of course, a far wider range of validity may be anticipated for almost all integers n. The latter constraint implies, for sufficiently large s, that any permissible

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value of δ must satisfy $\delta^{-1} > (1 + o(1))s$. If one is satisfied with a lower bound for $R_{s,k}(n;\omega)$ of the order of magnitude predicted by Theorem 1.1, then the use of smooth numbers leads to an acceptable value of δ satisfying the relation $\delta^{-1} = (1+o(1))s \log k$ in place of (1.6). We leave this as an exercise using the methods of [7]. Thus we see that our methods fall short of the obvious constraints on δ by a factor of $k \log k$, and $\log k$, in these respective problems. On the other hand, the widely held conjecture that (1.2) holds with $u = k$ would yield $\delta^{-1} = (2 + o(1))s$ in place of (1.6), and this would be close to best possible. Finally, on making the trivial observation that, when $s > t \geq 2\mathfrak{s}(k) + 1$, the validity of the conclusion of Theorem 1.1 implies that

$$R_{s,k}(n;\omega) \ll n^{(1+\omega)s/k-1},$$

while at the same time

$$R_{s,k}(n;\omega) \geq R_{t,k}(n-(s-t);\omega) \gg n^{(1+\omega)t/k-1},$$

it is apparent that when $\omega < -1$, then (1.5) fails for every sufficiently large integer n.

Our proof of Theorem 1.1 is based on a neoclassical application of the Hardy-Littlewood method paralleling the argument underlying our treatment (see Wooley [7]) of Vu's thin basis theorem in Waring's problem. In Section 2 we provide some auxiliary mean value estimates required in our treatment of the minor arcs. Our slightly unconventional generating functions may be analysed via partial summation, and in this way the completion of the treatment of the minor arcs in Section 2 may be reduced essentially to the familiar classical approach. The major arc treatment, which we discuss in Section 3, is more or less routine, although the analysis of the singular integral requires enough work to be deferred to Section 4. Here, for example, the convergence properties of the singular series become rather delicate in the situations wherein $\omega < -1 + k/s$.

Throughout, the letter ε will denote a sufficiently small positive number, and P will be a large real number. We use $\ll$ and $\gg$ to denote Vinogradov's notation. In an effort to simplify our account, whenever ε appears in a statement, we assert that the statement holds for every positive number ε. The 'value' of ε may consequently change from statement to statement.

2. The treatment of the minor arcs

In order to describe the application of the Hardy-Littlewood method that underlies the proof of Theorem 1.1, we begin by recording some notation. Let s and k be positive integers with $s \geq 2\mathfrak{s}(k) + 1$, let ν be a small positive number, and put

$$\delta = \left(\frac{s - 2\mathfrak{s}(k)}{s(s-1)}\right)\left(\frac{k}{2\mathfrak{s}(k)}\right) - \nu.$$

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Also, let ω be a real number with $\omega \geq -1 + k/s - \delta$. We put $w = (1+\omega)/k$, and consider the analogue of (1.1) provided by

$$\tilde{R}_{s,k}(n;w) = \sum_{\substack{x_1,\dots,x_s \in \mathbb{N} \\ x_1^k+\cdots+x_s^k=n}} (x_1 \cdots x_s)^{-1+kw}.$$

The conclusion of Theorem 1.1 then follows from the asymptotic formula

$$\tilde{R}_{s,k}(n;w) = \frac{\Gamma(w)^s}{\Gamma(sw)} k^{-s} \mathfrak{S}_{s,k}(n) n^{sw-1} + O(n^{sw-1-\tau}),$$

valid for some positive number $\tau = \tau(s,k)$, which we now seek to establish.

We consider a large natural number n, write $P = n^{1/k}$ and $P_1 = (n/s)^{1/k}$, and then define the exponential sums

$$f_w(\alpha) = \sum_{P_1 \leq x \leq P} x^{-1+kw} e(\alpha x^k) \quad \text{and} \quad g_w(\alpha) = \sum_{1 \leq x \leq P} x^{-1+kw} e(\alpha x^k).$$

Observe that whenever $x_1^k + \cdots + x_s^k = n$, with $x_i \in \mathbb{N}$ $(1 \leq i \leq s)$, then necessarily one has

$$\max_{1 \leq i \leq s} x_i \geq (n/s)^{1/k} = P_1 \quad \text{and} \quad \max_{1 \leq i \leq s} x_i \leq n^{1/k} = P. \tag{2.1}$$

By orthogonality, it follows from (2.1) that

$$\int_0^1 (g_w(\alpha) - f_w(\alpha))^s e(-n\alpha)\, d\alpha = 0,$$

and likewise we see that

$$\tilde{R}_{s,k}(n;w) = \int_0^1 g_w(\alpha)^s e(-n\alpha)\, d\alpha.$$

On substituting the former relation into the latter, it follows that

$$\begin{aligned} \tilde{R}_{s,k}(n;w) &= \int_0^1 (g_w(\alpha)^s - (g_w(\alpha) - f_w(\alpha))^s)\, e(-n\alpha)\, d\alpha \\ &= \sum_{j=1}^{s} (-1)^{j+1} \binom{s}{j} \mathscr{R}_{s,j}([0,1)), \end{aligned} \tag{2.2}$$

where we write

$$\mathscr{R}_{s,j}(\mathfrak{B}) = \int_{\mathfrak{B}} f_w(\alpha)^j g_w(\alpha)^{s-j} e(-n\alpha)\, d\alpha. \tag{2.3}$$

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For the task at hand it suffices to make use of a Hardy-Littlewood dissection that does not yield the sharpest available error terms. It is convenient to put $\tilde{w} = \min\{1/8, w/6\}$, and then to write $L = P^{\tilde{w}}$. Let $\mathfrak{M}$ denote the union of the intervals

$$\mathfrak{M}(q, a) = \{\alpha \in [0, 1) : |\alpha - a/q| \leq Ln^{-1}\},$$

with $0 \leq a \leq q \leq L$ and $(a, q) = 1$, and put $\mathfrak{m} = [0, 1) \setminus \mathfrak{M}$. We begin with an analogue of Weyl's inequality for the exponential sum $f_w(\alpha)$. In this context, it is convenient to write $\sigma(k) = 2^{1-k}\tilde{w}$.

LEMMA 2.1. *For each positive number ε, one has* $\sup_{\alpha \in \mathfrak{m}} |f_w(\alpha)| \ll P^{kw-\sigma(k)+\varepsilon}$.

PROOF. Write $F(\alpha; t) = \sum_{1 \leq x \leq t} e(\alpha x^k)$. Then an application of the classical version of Weyl's inequality (see, for example, [4, Lemma 2.4]) shows that whenever $P_1 - 1 \leq t \leq P$, one has

$$\sup_{\alpha \in \mathfrak{m}} |F(\alpha; t)| \ll t^{1+\varepsilon} L^{-2^{1-k}} \ll t^{1-\sigma(k)+\varepsilon}.$$

Applying Riemann-Stieltjes integration followed by integration by parts, we find that

$$\begin{aligned} f_w(\alpha) &= \int_{P_1-}^{P+} t^{-1+kw}\, dF(\alpha; t) \\ &= P^{-1+kw} F(\alpha; P) - P_1^{-1+kw} F(\alpha; P_1-) + \int_{P_1}^{P} (1 - kw) t^{-2+kw} F(\alpha; t)\, dt. \end{aligned}$$

It follows that whenever $\alpha \in \mathfrak{m}$, one has

$$|f_w(\alpha)| \ll P^{kw-\sigma(k)+\varepsilon} + \int_{P_1}^{P} t^{-1+kw-\sigma(k)+\varepsilon}\, dt,$$

and the conclusion of the lemma now follows immediately. □

Next we turn to mean value estimates relevant to the estimation of the minor arcs. It is here that we make use of the hypothesis (1.2), which we may assume to be valid for $u \geq s(k)$.

LEMMA 2.2. *Suppose that $t \geq 2u \geq 2s(k)$ and $\varepsilon > 0$. Then one has*

$$\int_0^1 |f_w(\alpha)|^t\, d\alpha \ll n^{tw-1+\varepsilon} \quad \text{and} \quad \int_0^1 |g_w(\alpha)|^t\, d\alpha \ll n^{\max\{tw-1,0\}+\varepsilon}.$$

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PROOF. From orthogonality, it follows that the mean value $\int_0^1 |f_w(\alpha)|^{2u}\,d\alpha$ is equal to the number of integral solutions of the equation

$$x_1^k + \cdots + x_u^k = x_{u+1}^k + \cdots + x_{2u}^k,$$

with $P_1 \le x_i \le P$ $(1 \le i \le 2u)$, and with each solution $\boldsymbol{x}$ being counted with weight

$$(x_1 x_2 \cdots x_{2u})^{-1+kw} \ll (P^{2u})^{-1+kw}.$$

Consequently, again employing orthogonality, it follows by considering the number of solutions of the underlying diophantine equations that

$$\int_0^1 |f_w(\alpha)|^{2u}\,d\alpha \ll (P^{2u})^{-1+kw} \int_0^1 |F(\alpha; P)|^{2u}\,d\alpha.$$

Thus, on making use of the trivial estimate

$$|f_w(\alpha)| \le |f_w(0)| \le \sum_{1 \le x \le P} x^{-1+kw} \ll P^{kw},$$

we obtain the upper bound

$$\int_0^1 |f_w(\alpha)|^t\,d\alpha \ll (P^{kw})^{t-2u}(P^{2u})^{-1+kw} \int_0^1 |F(\alpha; P)|^{2u}\,d\alpha.$$

In view of (1.2) and the definition of $\mathfrak{s}(k)$, therefore, we deduce that

$$\int_0^1 |f_w(\alpha)|^t\,d\alpha \ll (P^{ktw-2u})(P^{2u-k+\varepsilon}) \ll (P^k)^{tw-1+\varepsilon}.$$

The first assertion of the lemma follows on recalling that $P = n^{1/k}$.

Next write

$$h_w(\alpha; Q) = \sum_{Q/2 < x \le Q} x^{-1+kw} e(\alpha x^k).$$

Then by the same argument as in the previous paragraph, mutatis mutandis, one obtains the upper bound

$$\int_0^1 |h_w(\alpha; Q)|^t\,d\alpha \ll (Q^k)^{tw-1+\varepsilon}.$$

But it is apparent that

$$g_w(\alpha) = \sum_{\substack{j=0 \\ 2^j \le P}}^{\infty} h_w(\alpha; 2^{-j}P),$$

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and so it follows from Hölder's inequality that

$$\int_0^1 |g_w(\alpha)|^t \, d\alpha \ll (\log P)^t \max_{1 \le Q \le P} \int_0^1 |h_w(\alpha; Q)|^t \, d\alpha$$
$$\ll (\log P)^t \max_{1 \le Q \le P} (Q^k)^{tw-1+\varepsilon}.$$

The second conclusion of the lemma now follows immediately. □

At this point we record the minor arc estimate stemming from Lemmata 2.1 and 2.2.

LEMMA 2.3. *Suppose that* $s \ge 2\mathfrak{s}(k) + 1$. *Then for* $1 \le j \le s$, *one has*

$$\mathscr{R}_{s,j}(\mathfrak{m}) \ll n^{sw-1-\tau},$$

for some positive number $\tau = \tau(s, k, \nu)$.

PROOF. For the sake of convenience, write $u = \mathfrak{s}(k)$. Then on applying Hölder's inequality to (2.3), we obtain the upper bound

$$\mathscr{R}_{s,j}(\mathfrak{m}) \le \left(\int_{\mathfrak{m}} |f_w(\alpha)|^s \, d\alpha \right)^{j/s} \left(\int_0^1 |g_w(\alpha)|^s \, d\alpha \right)^{1-j/s}. \tag{2.4}$$

It follows from Lemmata 2.1 and 2.2 that

$$\int_{\mathfrak{m}} |f_w(\alpha)|^s \, d\alpha \le \left(\sup_{\alpha \in \mathfrak{m}} |f_w(\alpha)| \right)^{s-2u} \int_0^1 |f_w(\alpha)|^{2u} \, d\alpha \tag{2.5}$$
$$\ll (n^{w-\sigma(k)/k+\varepsilon})^{s-2u} n^{2uw-1+\varepsilon} \ll n^{sw-1-\sigma(k)/k+s\varepsilon}.$$

Meanwhile, the bound

$$\int_0^1 |g_w(\alpha)|^s \, d\alpha \ll n^{\max\{sw-1,0\}+\varepsilon} \tag{2.6}$$

is already immediate from Lemma 2.2. On substituting (2.5) and (2.6) into (2.4), it follows that whenever $w \ge 1/s$ one has $\mathscr{R}_{s,j}(\mathfrak{m}) \ll n^{sw-1-(j/s)\sigma(k)/k+\varepsilon}$, and this establishes the desired conclusion in the case currently under consideration.

When $1/s - \delta/k \le w < 1/s$, meanwhile, we proceed differently. Note first that our hypotheses on δ ensure that

$$1 - (s-1)w \le 1/s + (s-1)\delta/k < 1/(2u),$$

where we again write $u = \mathfrak{s}(k)$. Let $\theta = 1 - (s-1)w$. Then on applying Hölder's inequality once again to (2.3), we obtain on this occasion the upper bound

$$\mathscr{R}_{s,j}(\mathfrak{m}) \le \left(\sup_{\alpha \in \mathfrak{m}} |f_w(\alpha)| \right)^{1-2u\theta} \Upsilon_1^{\theta} \Upsilon_2^{(j-1)w} \Upsilon_3^{(s-j)w}, \tag{2.7}$$

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where

$$\Upsilon_1 = \int_0^1 |f_w(\alpha)|^{2u}\, d\alpha, \quad \Upsilon_2 = \int_0^1 |f_w(\alpha)|^{1/w}\, d\alpha, \quad \Upsilon_3 = \int_0^1 |g_w(\alpha)|^{1/w}\, d\alpha.$$

Now $1/w > s > 2\mathfrak{s}(k)$, and so it follows from Lemma 2.2 that $\Upsilon_2 \ll n^{\varepsilon}$ and $\Upsilon_3 \ll n^{\varepsilon}$. Similarly, one finds that $\Upsilon_1 \ll n^{2uw-1+\varepsilon}$. Then on recalling Lemma 2.1, and assembling these estimates within (2.7), we find that

$$\mathscr{R}_{s,j}(\mathfrak{m}) \ll n^{\varepsilon}(n^{w-\sigma(k)/k})^{1-2u\theta}(n^{2uw-1})^{\theta} \ll n^{w-\theta+\phi},$$

where $\phi = \varepsilon - (1-2u\theta)\sigma(k)/k < 0$. But $w - \theta = sw - 1$, and so we conclude in this final case that $\mathscr{R}_{s,j}(\mathfrak{m}) \ll n^{sw-1-\tau}$, for some positive number $\tau = \tau(s, k, \nu)$, thereby completing the proof of the lemma. □

3. The major arc analysis

We are able to economise in our discussion of the major arcs by appealing to the analysis of [7, Section 3]. In this context, when $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\beta \in \mathbb{R}$, we define $S(q, a)$ via (1.4), and write

$$u_w(\beta) = \int_{P_1}^{P} \gamma^{-1+kw} e(\beta\gamma^k)\, d\gamma \quad \text{and} \quad v_w(\beta) = \int_0^{P} \gamma^{-1+kw} e(\beta\gamma^k)\, d\gamma. \tag{3.1}$$

LEMMA 3.1. *Suppose that w is a positive number. Then whenever $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\beta \in \mathbb{R}$, one has*

$$f_w(\beta + a/q) - q^{-1}S(q, a)u_w(\beta) \ll qP^{-1+kw}(1 + P^k|\beta|),$$

and

$$g_w(\beta + a/q) - q^{-1}S(q, a)v_w(\beta) \ll q \max_{1 \le Q \le P} Q^{-1+kw}(1 + Q^k|\beta|).$$

PROOF. The first estimate of the lemma is essentially the conclusion of Lemma 3.5 of [7], while the second follows from the same methods. □

Next define the functions $f_w^*(\alpha)$ and $g_w^*(\alpha)$ for $\alpha \in [0, 1)$ by putting

$$f_w^*(\alpha) = q^{-1}S(q, a)u_w(\alpha - a/q) \quad \text{and} \quad g_w^*(\alpha) = q^{-1}S(q, a)v_w(\alpha - a/q),$$

when $\alpha \in \mathfrak{M}(q, a) \subseteq \mathfrak{M}$, and by taking $f_w^*(\alpha) = 0$ and $g_w^*(\alpha) = 0$ otherwise. We then write

$$\mathscr{R}_{s,j}^*(\mathfrak{B}) = \int_{\mathfrak{B}} f_w^*(\alpha)^j g_w^*(\alpha)^{s-j} e(-n\alpha)\, d\alpha. \tag{3.2}$$

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LEMMA 3.2. *For $1 \le j \le s$, one has $\mathscr{R}_{s,j}(\mathfrak{M}) - \mathscr{R}^*_{s,j}(\mathfrak{M}) \ll n^{sw-1-\tilde{w}/k}$.*

PROOF. It follows from Lemma 3.1 and the definition of $\mathfrak{M}$ that, uniformly for $\alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$, one has

$$f_w(\alpha) - f_w^*(\alpha) \ll LP^{-1+kw}(1 + LP^k n^{-1}) \ll P^{kw}L^{-4},$$

and likewise

$$\begin{aligned} g_w(\alpha) - g_w^*(\alpha) &\ll L \max_{1 \le Q \le P} Q^{-1+kw}(1 + LQ^k n^{-1}) \\ &\ll \max\{L^2,\ P^{kw}L^{-4}\} \ll P^{kw}L^{-4}. \end{aligned}$$

On making use of the trivial estimates $f_w(\alpha) \ll P^{kw}$ and $g_w(\alpha) \ll P^{kw}$, therefore, we may conclude that the estimate

$$f_w(\alpha)^j g_w(\alpha)^{s-j} - f_w^*(\alpha)^j g_w^*(\alpha)^{s-j} \ll (P^{kw}L^{-4})(P^{kw})^{s-1} \ll n^{sw}L^{-4}$$

holds uniformly for $\alpha \in \mathfrak{M}$. But the measure of the set of arcs $\mathfrak{M}$ is plainly $O(L^3 n^{-1})$, and thus we deduce from (2.3) and (3.2) that

$$\mathscr{R}_{s,j}(\mathfrak{M}) - \mathscr{R}^*_{s,j}(\mathfrak{M}) \ll \int_{\mathfrak{M}} n^{sw}L^{-4}\,d\alpha \ll n^{sw-1}L^{-1}.$$

The conclusion of the lemma is now immediate. □

We recall at this point the natural estimates for the auxiliary functions defined in (3.1).

LEMMA 3.3. *For every real number β, one has*

$$u_w(\beta) \ll P^{kw}(1 + P^k|\beta|)^{-1} \quad \text{and} \quad v_w(\beta) \ll P^{kw}(1 + P^k|\beta|)^{-\min\{1,w\}}.$$

PROOF. The claimed estimates follow by applying partial integration (compare the proof of Lemma 3.8 of [7]). □

Define next the singular integral

$$J_{s,j}(n;w) = \int_{-\infty}^{\infty} u_w(\beta)^j v_w(\beta)^{s-j} e(-\beta n)\,d\beta. \tag{3.3}$$

LEMMA 3.4. *When $1 \le j \le s$ and $w \ge 1/s - \delta/k$, the singular integral $J_{s,j}(n;w)$ is absolutely convergent, and satisfies the upper bound $J_{s,j}(n;w) \ll n^{sw-1}$. Moreover, one has*

$$J_{s,j}(n;w) - \int_{|\beta| \le Ln^{-1}} u_w(\beta)^j v_w(\beta)^{s-j} e(-\beta n)\,d\beta \ll n^{sw-1-\tilde{w}/(2k)}. \tag{3.4}$$

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PROOF. In view of the conclusion of Lemma 3.3, one has

$$\int_{-\infty}^{\infty} |u_w(\beta)^j v_w(\beta)^{s-j}| \, d\beta \ll (P^{kw})^s \int_{-\infty}^{\infty} (1 + P^k|\beta|)^{-j-(s-j)\min\{1,w\}} \, d\beta \ll n^{sw} \int_0^{\infty} (1 + n\beta)^{-1-(s-1)\min\{1,w\}} \, d\beta. \tag{3.5}$$

It therefore follows from (3.3) that the singular integral $J_{s,j}(n; w)$ is absolutely convergent, and satisfies $J_{s,j}(n; w) \ll n^{sw-1}$. In like manner, one finds that the expression on the left-hand side of (3.4) is of order

$$(P^{kw})^s \int_{Ln^{-1}}^{\infty} (1 + n\beta)^{-1-(s-1)\min\{1,w\}} \, d\beta \ll n^{sw-1} L^{-(s-1)\min\{1,w\}}.$$

The final conclusion of the lemma therefore follows on recalling our hypothesis that $w \geq 1/s - \delta/k$. □

Notice in the above argument the critical role played by the removal of the range $0 < \gamma \leq P_1$ from the variable implicit in $u_w(\beta)$. When $w \leq 1/s$, the integral on the right-hand side of (3.5) would otherwise be the divergent integral

$$\int_0^{\infty} (1 + n\beta)^{-sw} \, d\beta.$$

The next step in the analysis is the introduction of the truncated singular series

$$\mathfrak{S}_s(n; Q) = \sum_{1 \leq q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} (q^{-1} S(q, a))^s e(-na/q).$$

As is familiar in the theory of Waring's problem, the truncated singular series $\mathfrak{S}_s(n; Q)$ differs from the completed singular series, which we define via (1.3), by an amount inconsequential to our argument. Thus, since we may suppose that $s \geq 2\mathfrak{s}(k) + 1 \geq 2k + 1$, the methods of Chapters 2 and 4 of [4] demonstrate that

$$\mathfrak{S}_s(n) - \mathfrak{S}_s(n; L) \ll L^{-1/k} \ll n^{-\tilde{w}/(k^2)}, \tag{3.6}$$

and, furthermore, that $0 \leq \mathfrak{S}_s(n) \ll 1$ uniformly in n. Moreover, unless k is a power of 2 exceeding 2, the condition $s \geq 2\mathfrak{s}(k) + 1$ suffices to ensure that $\mathfrak{S}_s(n) \gg 1$ uniformly in n. When $k = 2^m$ with $m \geq 2$, meanwhile, the same conclusion holds whenever $s \geq 4k$, and also when $2\mathfrak{s}(k) + 1 \leq s \leq 4k$ provided that $n \equiv r \pmod{4k}$ for some integer r satisfying $1 \leq r \leq s$.

LEMMA 3.5. *One has* $\mathscr{R}^*_{s,j}(\mathfrak{M}) = \mathfrak{S}_s(n) J_{s,j}(n; w) + O(n^{sw-1-\tilde{w}/(k^2)})$.

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PROOF. On recalling (3.2) and the definitions of $f_w^*(\alpha)$ and $g_w^*(\alpha)$, one finds that

$$\mathscr{R}_{s,j}^*(\mathfrak{M}) = \mathfrak{S}_s(n; L) \int_{|\beta| \le Ln^{-1}} u_w(\beta)^j v_w(\beta)^{s-j} e(-\beta n)\, d\beta.$$

Consequently, on making use of (3.6) and the associated discussion, together with the conclusion of Lemma 3.4, one obtains

$$\begin{aligned}\mathscr{R}_{s,j}^*(\mathfrak{M}) &= (\mathfrak{S}_s(n) + O(n^{-\tilde{w}/(k^2)}))(J_{s,j}(n; w) + O(n^{sw-1-\tilde{w}/(2k)})) \\ &= \mathfrak{S}_s(n) J_{s,j}(n; w) + O(n^{sw-1-\tilde{w}/(k^2)}),\end{aligned}$$

and this suffices to establish the lemma. □

We summarise the discussion of this section and the last in the form of a lemma. In preparation for this lemma, we define the combined singular integral

$$J_s(n; w) = \sum_{j=1}^{s} (-1)^{j+1} \binom{s}{j} J_{s,j}(n; w).$$

LEMMA 3.6. *Whenever $w \ge 1/s - \delta/k$, one has*

$$\tilde{R}_{s,k}(n; w) = \mathfrak{S}_s(n) J_s(n; w) + O(n^{sw-1-\tau}),$$

for a positive number $\tau = \tau(s, k, \nu)$.

PROOF. By combining Lemmata 2.3, 3.2 and 3.5, one finds that there is a positive number $\tau = \tau(s, k, \nu)$ such that, for $1 \le j \le s$, one has

$$\begin{aligned}\mathscr{R}_{s,j}([0, 1)) &= \mathscr{R}_{s,j}(\mathfrak{m}) + \mathscr{R}_{s,j}(\mathfrak{M}) = \mathscr{R}_{s,j}^*(\mathfrak{M}) + O(n^{sw-1-\tau}) \\ &= \mathfrak{S}_s(n) J_{s,j}(n; w) + O(n^{sw-1-\tau}).\end{aligned}$$

The desired conclusion now follows from (2.2) by summing over j with weight $(-1)^{j+1}\binom{s}{j}$. □

4. The singular integral

The marginal convergence of the singular integral in the situation $w \le 1/s$ forces us to exercise more care than would be usual in our account. We begin by recalling that the integral $J_{s,j}(n; w)$ is absolutely convergent, so that on writing $\mathscr{B}_{s,j} = (P_1, P)^j \times (0, P)^{s-j}$, we see that

$$J_{s,j}(n; w) = \lim_{T \to \infty} \int_{\mathscr{B}_{s,j}} \int_{-T}^{T} (\gamma_1 \cdots \gamma_s)^{-1+kw} e(\beta(\gamma_1^k + \cdots + \gamma_s^k - n))\, d\beta\, d\boldsymbol{\gamma}.$$

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Consequently, by making the change of variables $\eta = n\beta$ and $v_i = \gamma_i^k/n$ $(1 \le i \le s)$, we deduce that

$$J_{s,j}(n;w) = k^{-s}n^{sw-1} \lim_{t\to\infty} \int_{\mathscr{C}_{s,j}} \int_{-t}^{t} (v_1 \cdots v_s)^{w-1} e(\eta(v_1 + \cdots + v_s - 1))\, d\eta\, d\boldsymbol{v},$$

where $\mathscr{C}_{s,j} = (s^{-1}, 1)^j \times (0,1)^{s-j}$. Making the change of variable $(v_1, \dots, v_s) \to (v_1, \dots, v_{s-1}, V)$, where $V = v_1 + \cdots + v_s$, we thus obtain

$$J_{s,j}(n;w) = k^{-s}n^{sw-1} \lim_{t\to\infty} \int_0^s \phi(V) \frac{\sin(2\pi t(V-1))}{\pi(V-1)}\, dV,$$

where

$$\phi(V) = \int_{\mathscr{D}_{s,j}(V)} (v_1 \cdots v_{s-1})^{w-1} (V - v_1 - \cdots - v_{s-1})^{w-1}\, dv_1 \cdots dv_{s-1},$$

and $\mathscr{D}_{s,j}(V)$ denotes the subset of $\mathbb{R}^{s-1}$ constrained by the inequalities

$$s^{-1} < v_i < 1 \ \ (1 \le i \le j), \qquad 0 < v_l < 1 \ \ (j+1 \le l \le s-1),$$

and

$$\begin{cases} V - 1 < v_1 + \cdots + v_{s-1} < V, & \text{when } j \ne s; \\ V - 1 < v_1 + \cdots + v_{s-1} < V - 1/s, & \text{when } j = s. \end{cases}$$

Our hypothesis that $w \ge 1/s - \delta/k$, combined with the condition $v_1 > s^{-1}$, ensures that $\phi(V)$ is a function of bounded variation, and so it follows from Fourier's integral theorem that

$$\lim_{t\to\infty} \int_0^s \phi(V) \frac{\sin(2\pi t(V-1))}{\pi(V-1)}\, dV = \phi(1).$$

Consequently, we may conclude that

$$J_{s,j}(n;w) = k^{-s}n^{sw-1} \int_{\mathscr{D}_{s,j}(1)} (v_1 \cdots v_{s-1})^{w-1} (1 - v_1 - \cdots - v_{s-1})^{w-1}\, dv_1 \cdots dv_{s-1},$$

whence

$$J_{s,j}(n;w) = k^{-s}n^{sw-1} \int_{\mathscr{E}_{s,j}} (v_1 \cdots v_s)^{w-1}\, d\mathscr{E}_{s,j},$$

where $\mathscr{E}_{s,j}$ denotes the surface defined by $v_1 + \cdots + v_s = 1$ subject to $\boldsymbol{v} \in (s^{-1}, 1)^j \times (0,1)^{s-j}$. It therefore follows that

$$J_s(n;w) = \sum_{j=1}^{s} (-1)^{j+1} \binom{s}{j} J_{s,j}(n;w) = k^{-s}n^{sw-1}\mathscr{I},$$

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where

$$\mathscr{I} = \int_{\mathscr{S}} (v_1 \cdots v_s)^{w-1}\, d\mathscr{S},$$

and $\mathscr{S}$ denotes the surface defined by $v_1 + \cdots + v_s = 1$ subject to $\boldsymbol{v} \in (0, 1)^s \setminus (0, s^{-1})^s$. But no point in $(0, s^{-1})^s$ satisfies $v_1 + \cdots + v_s = 1$, and thus we conclude that

$$\mathscr{I} = \int_{\mathscr{A}} (v_1 \cdots v_{s-1})^{w-1} (1 - v_1 - \cdots - v_{s-1})^{w-1}\, dv_1 \cdots dv_{s-1},$$

where $\mathscr{A}$ denotes the subset of $(0, 1)^{s-1}$ subject to the constraint that whenever $(v_1, \ldots, v_{s-1}) \in \mathscr{A}$, then one has $0 < v_1 + \cdots + v_{s-1} < 1$. In this way, a familiar inductive argument employing the Beta-function

$$\mathrm{B}(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}\, dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

provides the formula $\mathscr{I} = \Gamma(w)^s / \Gamma(sw)$. It is now apparent that

$$J_s(n; w) = k^{-s}\Gamma(w)^s \Gamma(sw)^{-1} n^{sw-1},$$

and this, in combination with the conclusion of Lemma 3.6, yields the desired conclusion embodied in Theorem 1.1.

References

[1] K. B. Ford, 'New estimates for mean values of Weyl sums', *Internat. Math. Res. Notices* (1995), 155–171.
[2] D. R. Heath-Brown, 'Weyl's inequality, Hua's inequality, and Waring's problem', *J. London Math. Soc. (2)* **38** (1988), 216–230.
[3] L.-K. Hua, 'On Waring's problem', *Quart. J. Math. Oxford* **9** (1938), 199–202.
[4] R. C. Vaughan, *The Hardy-Littlewood method*, 2nd edition (Cambridge University Press, Cambridge, 1997).
[5] Van H. Vu, 'On a refinement of Waring's problem', *Duke Math. J.* **105** (2000), 107–134.
[6] T. D. Wooley, 'On Vinogradov's mean value theorem', *Mathematika* **39** (1992), 379–399.
[7] ———, 'On Vu's thin basis theorem in Waring's problem', *Duke Math. J.* **120** (2003), 1–34.

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