RINGS GENERATED BY THEIR REGULAR ELEMENTS

by A. W. CHATTERS AND S. M. GINN

(Received 26 May, 1982)

1. Introduction. The units of a ring R are defined by means of a multiplicative property, but in many cases they generate R additively. For example, it is shown in [**5**, Proposition 6] that if R is a semi-simple Artinian ring then every element of R is a sum of units if and only if the ring $S=\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ is not a direct summand of R, where $\mathbb{Z}$ denotes the ring of integers. The theme of this paper is to investigate the corresponding situation concerning regular elements, i.e. elements which are not zero-divisors. We show that if R is a semi-prime right Goldie ring then every element of R is a sum of regular elements if and only if R does not have the ring S defined above as a direct summand (Corollary 2.9). We also characterise those Noetherian rings R such that every element of R is a sum of regular elements (Theorem 2.6). The characterisation is in terms of the nature of certain prime factor rings of R, and it is again the presence of the ring S, this time in a particular way as a factor ring of R, which prevents R from being generated by its regular elements. If R has no non-zero Artinian one-sided ideals or if 2 is a regular element of R, then every element of R is a sum of regular elements (Corollaries 2.5 and 2.7). As an application we show in Section 3 that, for many Noetherian rings R, the set of elements of R which are divisible by every regular element of R is a two-sided ideal of R.

2. Some rings which are generated by their regular elements. Let R be an associative ring with identity element. We use $r(x)$ and $l(x)$ to denote the right and left annihilators of an element x of R. An element c of R is said to be *right regular* if $r(c)=0$ and *regular* if $r(c)=l(c)=0$. If I is an ideal of R then $\mathscr{C}(I)$ denotes the set of all elements c of R such that $c+I$ is a regular element of R/I. Thus $\mathscr{C}(0)$ is the set of regular elements of R. Because $\mathscr{C}(0)$ is closed under multiplication, the additive subgroup A of R generated by $\mathscr{C}(0)$ is a subring of R. We say that R is *generated by its regular elements* if $A=R$, i.e. if every element of R is a sum of regular elements.

It is well-known folk-lore that every prime right Goldie ring is generated by its regular elements. We shall prove a strong form of this result which is suitable for "lifting" to a more general ring from some of its prime factor rings. Further details concerning Goldie rings can be found in [**2**, Chapter 1].

LEMMA 2.1. *Let R be a prime right Goldie ring and suppose that R has a non-zero element u such that $uRu=\{0,u\}$. Then $R\cong M_n(\mathbb{Z}/2\mathbb{Z})$ for some positive integer n.*

Proof. Because R is prime and $u\neq 0$, we have $uxu\neq 0$ for some $x\in R$. But $uxu\in uRu$ so that $uxu=u$. Set $e=ux$. Then $e^2=e$. Let $y\in R$. Then either $uxyu=0$ in which case $eye=0$, or $uxyu=u$ in which case $eye=e$. Thus eRe is a ring with only two elements, so that $eRe\cong\mathbb{Z}/2\mathbb{Z}$. Also eR is a minimal right ideal of R and $eRe\cong\operatorname{End}(eR)$. It follows from Goldie's theorem by standard arguments that R is simple Artinian and that $R \cong M_n(eRe)$ for some positive integer n.

The next result makes it possible to "lift" information from prime rings to more general rings.

LEMMA 2.2. *Let R be a prime right Goldie ring with $R \not\cong \mathbb{Z}/2\mathbb{Z}$, let $a \in R$ and let I be a non-zero ideal of R. Then there are regular elements c_1, c_2, c_3 of R such that $a = c_1 - c_2 + c_3$ and $a - c_i \in I$ for all i.*

Proof. Suppose firstly that $R \cong M_n(\mathbb{Z}/2\mathbb{Z})$ for some positive integer $n \neq 1$. Then R is a simple ring so that $I = R$. The result in this case follows from the fact that every element of R is the sum of three units [**4**, Theorem 3].

Now suppose that there is no positive integer n such that $R \cong M_n(\mathbb{Z}/2\mathbb{Z})$. For some positive integer n there are uniform right ideals $U_1, \ldots, U_n$ of R such that the sum $aR + U_1 + \ldots + U_n$ is direct and is an essential right ideal of R, and also $U_i \subseteqq I$ for all i. Because $aR + U_1 + \ldots + U_n$ contains a regular element, for each i there exists $u_i \in U_i$ such that $a + u_1 + \ldots + u_n$ is regular [**6**, Corollary 2.5]. Hence $0 = r(a + u_1 + \ldots + u_n) = r(a) \cap r(u_1) \cap \ldots \cap r(u_n)$. By Lemma 2.1, for each i there exists $y_i \in R$ such that $0 \neq u_i y_i u_i \neq u_i$.

To simplify the notation let u be one of the u_i with $0 \neq uyu \neq u$. Let $x \in R$ with $ux \neq 0$. Because uxR is an essential submodule of uR and R has zero right singular ideal, we have $l(ux) = l(u)$. But $uy \notin l(u)$ so that $uyux \neq 0$. In other words, if $x \in R$ with $uyux = 0$ then $ux = 0$. Hence $r(u) = r(uyu)$. Similarly $uy - 1 \notin l(u)$ so that $r(u) = r((uy-1)u) = r(u(yu-1))$.

Thus $r(u_i) = r(u_i y_i u_i) = r(u_i(y_i u_i - 1))$ for each i. Set

$$c_1 = a + u_1 + \ldots + u_n,$$

$$c_2 = a + u_1 y_1 u_1 + \ldots + u_n y_n u_n,$$

and

$$c_3 = a + u_1(y_1 u_1 - 1) + \ldots + u_n(y_n u_n - 1).$$

We have $a = c_1 - c_2 + c_3$. Also $a - c_i \in I$ for all i because $U_j \subseteqq I$ for all j. We know already that c_1 is regular. We have

$$r(c_2) = r(a) \cap r(u_1 y_1 u_1) \cap \ldots \cap r(u_n y_n u_n) = r(a) \cap r(u_1) \cap \ldots \cap r(u_n) = r(c_1) = 0,$$

and similarly $r(c_3) = 0$. Thus c_2 and c_3 are right regular and hence regular.

COROLLARY 2.3. *Every element of a prime right Goldie ring is the sum of three or fewer regular elements.*

We wish to thank J. C. Robson for pointing out that, by using a different method, it is possible to replace "three" by "two" in Corollary 2.3. (See the following paper.)

To avoid giving the same kind of proof twice we turn next to the case of Noetherian rings before dealing with the easier case of semi-prime right Goldie rings.

LEMMA 2.4. *Let R be a left and right Noetherian ring with prime ideals $P_1, \ldots, P_n$ such* *that* $\mathscr{C}(P_1)\cap\ldots\cap\mathscr{C}(P_n)\subseteq\mathscr{C}(0)$. *Suppose also that there is at most one i such that* $R/P_i\cong\mathbb{Z}/2\mathbb{Z}$. *Then R is generated by its regular elements.*

Proof. The numbering can be arranged so that if $i<j$ then P_i is not contained in P_j. Also if $R/P_i\cong\mathbb{Z}/2\mathbb{Z}$ for some i then P_i is a maximal ideal of R and we can arrange that $i=1$. Let $a\in R$. It follows from Corollary 2.3 applied to R/P_1 that a is a sum of elements of $\mathscr{C}(P_1)$. We shall show that if $1\le i<n$ then each element of $\mathscr{C}(P_1)\cap\ldots\cap\mathscr{C}(P_i)$ is a sum of elements of $\mathscr{C}(P_1)\cap\ldots\cap\mathscr{C}(P_i)\cap\mathscr{C}(P_{i+1})$. It will then follow by induction that a is a sum of elements of $\mathscr{C}(P_1)\cap\ldots\cap\mathscr{C}(P_n)$.

Fix i with $1\leqslant i<n$, set $S=\mathscr{C}(P_1)\cap\ldots\cap\mathscr{C}(P_i)$ and let $s\in S$. Set $K=P_1P_2\ldots P_i$ and $P=P_{i+1}$, and let $*$ denote image in $R^*=R/P$. Because P does not contain any of the ideals $P_1,\ldots,P_i$ we know that K^* is a non-zero ideal of R^*. By Lemma 2.2, there exist c_1, c_2, $c_3\in\mathscr{C}(P)$ such that $s-c_1+c_2-c_3\in P$ and $s^*-c_j^*\in K^*$ for all j. For $j=1$ or 2 choose $x_j\in K$ such that $s-c_j-x_j\in P$. Set $d_1=s-x_1$, $d_2=s-x_2$, and $d_3=s+x_1-x_2$. Then $s=d_1-d_2+d_3$. For all j we have $s-d_j\in K$. But $K\subseteq P_1\cap\ldots\cap P_i$ and $s\in S=\mathscr{C}(P_1)\cap\ldots\cap\mathscr{C}(P_i)$. Therefore $d_j\in S$ for all j. Also $c_j\in\mathscr{C}(P)$ and $d_j-c_j\in P$ so that $d_j\in\mathscr{C}(P)$ for all j. Therefore $s=d_1-d_2+d_3$ with $d_j\in S\cap\mathscr{C}(P)$ for all j, as required.

COROLLARY 2.5. *Let R be a left and right Noetherian ring with no non-zero Artinian one-sided ideals. Then R is generated by its regular elements.*

Proof. There are prime ideals $P_1,\ldots,P_n$ of R such that $\mathscr{C}(P_1)\cap\ldots\cap\mathscr{C}(P_n)\subseteq\mathscr{C}(0)$ [**7**, Proposition 1.1]. There is no i such that R/P_i is Artinian [**7**, Lemma 2.2]. Therefore Lemma 2.4 applies.

It is shown in [**6**, Corollary 2.3] that if R is any left and right Noetherian ring then there are prime ideals $P_1,\ldots,P_n$ of R such that $\mathscr{C}(P_1)\cap\ldots\cap\mathscr{C}(P_n)=\mathscr{C}(0)$.

THEOREM 2.6. *Let R be a left and right Noetherian ring with distinct prime ideals* $P_1,\ldots,P_n$ *such that* $\mathscr{C}(P_1)\cap\ldots\cap\mathscr{C}(P_n)=\mathscr{C}(0)$. *Then R is generated by its regular elements if and only if there is at most one i such that* $R/P_i\cong\mathbb{Z}/2\mathbb{Z}$.

Proof. One direction of the proof was done in Lemma 2.4.

Suppose that R is generated by its regular elements. We complete the proof by assuming that $R/P_1\cong\mathbb{Z}/2\mathbb{Z}$ and showing that $R/P_2\not\cong\mathbb{Z}/2\mathbb{Z}$. Let $a\in P_1$ with $a\notin P_2$. We have $a=c_1+\ldots+c_k$ for some $c_i\in\mathscr{C}(0)$. Because $c_i\in\mathscr{C}(P_1)$ and $R/P_1\cong\mathbb{Z}/2\mathbb{Z}$ we have $c_i-1\in P_1$ for all i. Hence $a-k1\in P_1$ so that $k1\in P_1$. Therefore k is even. But $c_i\in\mathscr{C}(P_2)$ for all i and $c_1+\ldots+c_k\notin P_2$. Hence $R/P_2\not\cong\mathbb{Z}/2\mathbb{Z}$.

COROLLARY 2.7. *Let R be a left and right Noetherian ring in which 2 is a regular element. Then R is generated by its regular elements.*

Here we are using 2 to denote $1+1$ where 1 is the identity element of R.

Proof. In the notation of Theorem 2.6 we have $2\in\mathscr{C}(P_i)$ for all i so that $R/P_i\not\cong\mathbb{Z}/2\mathbb{Z}$.

THEOREM 2.8. *Let R be a semi-prime right Goldie ring and let $P_1, \ldots, P_n$ be the minimal prime ideals of R. Then R is generated by its regular elements if and only if there is at most one i such that $R/P_i \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. We have $\mathscr{C}(P_1) \cap \ldots \cap \mathscr{C}(P_n) = \mathscr{C}(0)$. One way of establishing the result is to use the same proofs as in Lemma 2.4 and Theorem 2.6.

COROLLARY 2.9. *Let R be a semi-prime right Goldie ring. Then R is generated by its regular elements if and only if R does not have a direct summand isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.*

Proof. If R has $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ as a direct summand then it is easy to show that R is not generated by its regular elements.

Conversely suppose that R is not generated by its regular elements. Then by Theorem 2.8 there are distinct minimal prime ideals P_1 and P_2 of R such that $R/P_1 \cong \mathbb{Z}/2\mathbb{Z} \cong R/P_2$. Let I be the intersection of the other minimal primes of R (with $I = R$ if there are none). Then $P_1 \cap P_2 \cap I = 0$, and P_1 and P_2 are maximal ideals which do not contain I. Therefore $R \cong R/P_1 \oplus R/P_2 \oplus R/I$.

Results 2.8 and 2.9 can also be proved in the setting of semi-prime rings with the ascending and descending chain conditions for right annihilators, but we shall not give the more general versions because of the extra complications involved in their proofs.

3. The divisible elements of a Noetherian ring. Throughout this section let $H = \cap Rc$, where c ranges over the regular elements of R.

The left ideal H can be thought of as being the set of all elements of R which are divisible on the right by every regular element of R. It is known that H is an ideal of R if R satisfies the left Ore condition with respect to $\mathscr{C}(0)$ [**1**, proof of Theorem 1]. We shall show that H is an ideal of R whenever R is a left Noetherian ring which is generated by its regular elements.

LEMMA 3.1. *Let R be a left Noetherian ring with H as above and let c be a regular element of R. Then $H = Hc$.*

Proof. Let $h \in H$. Then $h \in Rc$ so that $h = xc$ for some $x \in R$. Let $d \in \mathscr{C}(0)$. We have $dc \in \mathscr{C}(0)$ so that $h \in Rdc$. Thus $h = ydc$ for some $y \in R$. But $h = xc$. Hence $x = yd$ so that $x \in Rd$. Therefore $x \in H$ so that $H \subseteq Hc$. Thus $H \subseteq Hc \subseteq Hc^2 \subseteq \ldots$ so that $Hc^n = Hc^{n+1}$ for some n. Therefore $H = Hc$.

COROLLARY 3.2. *Let R be a left Noetherian ring which is generated by its regular elements. Then H is an ideal of R and $H = Hc$ for every regular element c of R.*

Proof. Let $a \in R$. Then $a = c_1 + \ldots + c_n$ for some $c_i \in \mathscr{C}(0)$. We have $Ha \subseteq Hc_1 + \ldots + Hc_n$. But $Hc_i = H$ for all i, by Lemma 3.1. Therefore $Ha \subseteq H$.

THEOREM 3.3. *Let R be a left and right Noetherian ring with no non-zero Artinian one-sided ideals. Then $\cap Rc = 0$, where c ranges over the regular elements of R.*

Proof. Let H be as above. By Corollary 2.5, we know that R is generated by its regular elements. Therefore $H = 0$ by Corollary 3.2 above and Corollary 2.4 of [**3**].

We feel that there should be a more direct way of proving Theorem 3.3 but we do not know one. Also we have not found a Noetherian ring in which H is not an ideal.

REFERENCES

1. A. W. Chatters, A note on Noetherian orders in Artinian rings, *Glasgow Math. J.* **20** (1979), 125–128.

2. A. W. Chatters and C. R. Hajarnavis, *Rings with chain conditions* (Pitman, 1980).

3. A. W. Chatters, Divisible ideals of Noetherian rings, *Bull. London Math. Soc.* **13** (1981), 328–330.

4. M. Henriksen, Two classes of rings generated by their units, *J. Algebra* **31** (1974), 182–193.

5. R. Raphael, Rings which are generated by their units. *J. Algebra* **28** (1974), 199–205.

6. L. W. Small and J. T. Stafford, Regularity of zero divisors, *Proc. London Math. Soc.* (3) **44** (1982), 405–419.

7. J. T. Stafford, Noetherian full quotient rings, *Proc. London Math. Soc.* (3) **44** (1982), 385–404.

SCHOOL OF MATHEMATICS
UNIVERSITY WALK
BRISTOL
BS8 1TW

MATHEMATICS DEPARTMENT
BIRKBECK COLLEGE
MALET STREET
LONDON WC1

https://doi.org/10.1017/S0017089500005334 Published online by Cambridge University Press