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EISENSTEIN SERIES FOR REDUCTIVE GROUPS OVER GLOBAL FUNCTION FIELDS I.
The Cusp Form Case

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Introduction. Let G be the Lie group $\mathbf{SL}(2, \mathbf{R})$ and Γ a discrete subgroup of arithmetic type. The homogeneous space $\Gamma\backslash G$ can be equipped with an invariant measure so that there is a Hilbert space of square integrable functions, denoted $L^2(\Gamma\backslash G)$, on which G acts by right translations. If $\Gamma\backslash G$ is compact then this Hilbert space breaks up into a countable direct sum of irreducible representations of G, each occurring with finite multiplicity. Quite often however $\Gamma\backslash G$ is not compact, but of finite volume; in this case $L^2(\Gamma\backslash G)$ splits into a discrete spectrum L_d^2, which behaves as if $\Gamma\backslash G$ were compact, and a continuous spectrum L_c^2, which is described by the so called theory of Eisenstein series. These are generalized eigenfunctions of the Casimir operator of G, which are parametrized by a right half plane in $\mathbf{C}$, and as such are analytic functions on this half-plane; in the course of describing the continuous spectrum L_c^2 however, one analytically continues them to meromorphic functions over all of $\mathbf{C}$, and shows them to satisfy functional equations. A key role in this is played by the so called constant term of the Eisenstein series which is an intertwining operator that can also be analytically continued over $\mathbf{C}$, and also satisfies its own functional equation. Such a theory was first developed by Selberg in order to extend his trace formula to the case where $\Gamma\backslash G$ has finite volume; his ideas and methods were decisive for the later development of the subject.

More generally, let G be a connected reductive algebraic group defined over a global field F, and write $\mathbf{A}$ for the associated ring of adeles of F. In recent years there has been considerable interest in the representation of $G(\mathbf{A})$ by right translations on $L^2(G(F)\backslash G(\mathbf{A}))$ and closely related spaces, inspired by what is now called Langlands' philosophy. Roughly speaking the working hypothesis of this philosophy is that there is a correspondence between irreducible representations of $G(\mathbf{A})$ occurring in $L^2(G(F)\backslash G(\mathbf{A}))$ and irreducible representations of the Galois group of "the" algebraic closure of F. It is not our purpose here to describe the questions raised by this philosophy, or to provide evidence for it, which is considerable, but merely to point out that one of the most powerful known ways to attack these problems is by means of the trace formula, which is

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still evolving, and that, just as above, there is a continuous spectrum which must first be dealt with in some manner.

In case F is a number field, this problem was attacked and solved by Langlands some fourteen years ago, by adapting the ideas of Selberg, and using ideas of his own; for a description of the results we refer the reader to [**13**], appendix II. The aim of this paper and its sequel is to describe the continuous spectrum in case F is a global field of characteristic $p > 0$, i.e., a function field in one variable with a finite field of constants. The methods are similar in spirit to those of Selberg and Langlands; in particular, considerable use is made of the spectral theory of bounded self-adjoint operators. The argument is an inductive one and will be described in the next paper; the purpose of this paper is to start the induction, and it corresponds to § 1-6 of Langlands' notes [**13**], i.e., to Eisenstein series arising from cusp forms.

We now briefly describe the contents of this paper. Let G, $\mathbf{A}$, F be as before; although in fact we do not work with $L^2 = L^2(G(F)\backslash G(\mathbf{A}))$ we shall pretend that we do for the discussion that follows. Chapter I contains a summary of results from root systems and reduction theory together with some applications; we remark in passing that the reduction of Section 4 is due entirely to Harder [**8**], [**9**] (but cf. also [**14**]).

The object which parametrizes Eisenstein series coming from a given parabolic $P = NM$ is a complex analytic manifold $D_M(\xi)$. This is defined in Chapter 2, Section 1, along with some associated vector bundles and function spaces. In § 2 θ-series (as Godement calls them), and Eisenstein series arising from cusp forms are defined, and their convergence properties examined. Next, in § 3 the constant term of such Eisenstein series is studied, and a formula for the inner product of two θ-series is given, which involves this constant term. Such a formula is crucial for later developments, and it also provides a decomposition of L^2 into a direct sum of spaces of θ-series (2.3.4). The last part of § 3 is concerned with estimates for the constant term, and approximation arguments which are employed extensively in Chapter 3. Finally in § 4 we study to what extent the various constant terms of a reasonable function on $G(F)\backslash G(\mathbf{A})$ determine that function, and deduce miscellaneous related results. The basic observation here is also due to Harder ([**10**], Lemma 1.6.7) and should be compared with the corresponding situation in the number field case.

Initially, Eisenstein series are parametrized on an open subset of $D_M(\xi)$, and as such are analytic functions. Similarly their constant terms represent matrix valued analytic functions on the same open subset of $D_M(\xi)$. Chapter 3 is the heart of the paper: in it we analytically continue Eisenstein series $E(g, \Phi, z)$ arising from cusp forms Φ associated to maximal parabolic subgroups ($z \in \mathbf{C}^* \xrightarrow{\sim} D_M(\xi)$ in this case), and show that they satisfy the requisite functional equations. At the same time we prove similar results for the constant term $M(z)$ of $E(g, \Phi, z)$; indeed the method consists in first proving the results for $M(z)$ then passing to $E(g, \Phi, z)$ via 2.4. To begin, we construct a bounded self-adjoint operator A, and relate its spectrum to the analytic behaviour of the operator valued function $M(z)$ via the inner product formula for θ-series; this permits analytic continuation outside the unit circle except for two "whiskers" of finite length which protrude from $z = \pm 1$. In § 2 truncated Eisenstein series are discussed; they permit one to continue $M(z)$ over the unit circle by a reflection principle. It remains to deal with the whiskers, and the points $z = \pm 1, 0$.

To deal with these one makes more use of spectral theory. First, by using Stone's formula, which relates the resolvent of a self-adjoint operator to the resolution of the identity, one shows that $M(z)$ can be analytically continued along the whiskers (except for $z = \pm 1$), and has at worst a finite number of simple poles there (in fact the simple poles correspond closely to isolated points in the spectrum of A). For the points $z = \pm 1$, one makes essential use of Stone's formula, and then brute force to deduce that $M(z)$ is analytic at $z = \pm 1$; more precisely one studies the behaviour of the eigenvalues of the unitary operator $M(z)$ for $|z| = 1$ as $z \to \pm 1$. This involves a careful study of the inner product formula for truncated Eisenstein series and elementary analysis; someday when the theory of Hecke operators is better understood it will be possible to deal with $z = \pm 1$ in a less circuitous fashion. At this stage of the argument it is easy to deal with $M(z)$ as $z \to 0$; in fact as a bonus one obtains that $M(z)$ and $E(g, \Phi, z)$ are rational functions (3.5) of z.

Finally in Chapter 4 we show how the results in Chapter 3 imply similar results for Eisenstein series and their constant terms arising from cusp forms associated to non-maximal parabolics. This method is that of Langlands [**13**]; while it is not the only method available, it is certainly the quickest.

It will be clear to anyone who glances at them, that this paper owes a great deal to Langlands' manuscript [**13**], and Harder's paper [**10**], and it is a pleasure to acknowledge my debt. I would also like to thank Dr. S. J. Patterson for reading an earlier version of Chapter 3, and the referee for many helpful comments. In a less mathematical vein, I thank Professors Coates and Deligne for their help as well.

1. Preliminaries from Root Systems and Reduction Theory.

0. Some notation and conventions. Throughout this paper F will stand for a function field in one variable with finite field of constants $\mathbf{F}_q$, and $\mathbf{A}$ will denote the associated ring of adeles.

0.1. If G is defined over F with Lie algebra $\mathfrak{G}$ then ad : $G \to \mathrm{Aut}\ (\mathfrak{G})$ is the algebraic morphism defined by conjugation; in terms of points ad $(g)x = gxg^{-1}$. We define Ad : $G \to \mathrm{Aut}\, G$ in the same way. Suppose A

is an F-algebra and f is a function of some kind on $G(A)$. Then Ad gives rise to Ad*, with

$$\langle \mathrm{Ad}^*(g)f, x\rangle = \langle f^g, x\rangle := \langle f, {}^g x\rangle = \langle f, \mathrm{Ad}\,(g)x\rangle$$

where ${}^g x = \mathrm{Ad}\,(g)x$; we also write x^g for ${}^{g^{-1}}x$, so that ${}^g f = f^{g^{-1}}$.

0.2. Let P be a rational parabolic subgroup of G, with unipotent radical N. The modulus character of $P(\mathbf{A})$ is denoted by δ_P : if $p \in P(\mathbf{A})$, then $\delta_P(p)$ is $|\det\,(\mathrm{ad}\,(p))|^{1/2}$ where ad is restricted to act on the Lie algebra of N.

0.3. If V is a complex analytic manifold, we shall write ϑ_V for its structure sheaf; if $U \subset V$ is an open subset then $\Gamma(U, \vartheta_V)$ denotes the ring of sections over U.

0.4. Let H be a locally compact topological group equipped with a Haar measure. If f, g are suitable complex valued functions on H, supposed measurable, then the convolution $f * g$ is the function defined by

$$f{*}g(x) = \int_H f(xy^{-1})g(y)dy$$

("suitable" means that the integral exists e.g. f, g continuous with compact support).

The above is by no means complete, but rather a list of miscellany which appear either without comment, or with a cursory remark at some point in the paper.

1. Preliminaries from root systems.

1.1. Let $\mathscr{R} = (M, M^*, R, R^*)$ denote a root system in the sense of S.G.A.D. XXI, with Weyl group W. We shall suppose chosen a set of simple roots Δ and write R^+ for the corresponding set of positive roots. It is well known that W is generated by the fundamental reflections s_α, $\alpha \in \Delta$. We denote by $\mathfrak{l}(w)$ the minimum of

$$\{t|\ w = w_1 \ldots w_t, w_i = s_{\alpha_i}, \alpha \in \Delta\}.$$

In particular, $\mathfrak{l}(1) = 0$, $\mathfrak{l}(s_\alpha) = 1$.

Set $\Sigma_w = R^+ \cap w^{-1}(R^-)$, and define $n(w) = |\Sigma_w|$, for $w \in W$.

LEMMA. *Let $\alpha \in \Delta$, $w \in W$. Then*

(i) $n(s_\alpha w) = n(w) + 1$ *if* $w^{-1}\alpha \in R^+$.
(ii) $n(s_\alpha w) = n(w) - 1$ *if* $w^{-1}\alpha \in R^-$.
(iii) $n(ws_\alpha) = n(w) + 1$ *if* $w^{-1}\alpha \in R^+$.
(iv) $n(ws_\alpha) = n(w) - 1$ *if* $w^{-1}\alpha \in R^-$.

Proof. Observe that $\Sigma_{s_\alpha w} = \{w^{-1}\alpha\} \cup \Sigma_w$ to get (i). For (ii), replace w by $s_\alpha w$ in (i). Finally, note that $n(w) = n(w^{-1})$ to obtain (iii) and (iv).

1.2. The next result is well known.

THEOREM. *The numbers* $\mathfrak{l}(w)$, $n(w)$ *are equal.*

As a corollary, one has

1.3. COROLLARY. $w(\Delta) = \Delta$ *implies* $w = 1$.

1.4. Let $\Theta \subseteq \Delta$, and write $R_\Theta =$ (linear span Θ) $\cap R$, and similarly for $R_\Theta{}^*$. If M_Θ is the sublattice generated by Θ, there is a map induced by transposition $M^* \to (M_\Theta)^*$, and the system $(M_\Theta, M_\Theta{}^*, R_\Theta, R_\Theta{}^*)$ is a root system. The group W_Θ, generated by the s_α, $\alpha \in \Delta$ is the Weyl group for this root system.

Define D_Θ to be the set of $w \in W$ such that $w(r) \in R^+$, each $r \in \Theta$.

PROPOSITION. (i) *Each element* $w \in W$ *can be written uniquely in the form* $w_\Theta = d_\Theta w_\Theta$, $d_\Theta \in D_\Theta$, $w_\Theta \in W_\Theta$, *and* $\mathfrak{l}(w) = \mathfrak{l}(d_\Theta) + \mathfrak{l}(w_\Theta)$. *Let* Θ_1, $\Theta_2 \subseteq \Delta$, W_i *the Weyl group corresponding to* Θ_i, $i = 1, 2$. *Define*

$$D_{\Theta_1,\Theta_2} = \{w \in W \mid w^{-1}\Theta_1 > 0, w\Theta_2 > 0\}.$$

(ii) *Each element* d, $d^{-1} \in D_{\Theta_1}$ *can be written uniquely in the form*

$$d_{12}w_2,\ d_{12} \in D_{\Theta_1,\Theta_2}, w_2 \in W_{\Theta_2}, \quad \textit{and} \quad \mathfrak{l}(d) = \mathfrak{l}(d_{12}) + \mathfrak{l}(w_2).$$

Proof. (i) Induction on $\mathfrak{l}(w)$; if $\mathfrak{l}(w) = 0$, or $w \in D_{\Theta_1}$, it is trivial. If not there is $\alpha \in \Theta$ and $w\alpha < 0$. By 1.1 (iv) $\mathfrak{l}(ws_\alpha) = \mathfrak{l}(w) - 1$; induction implies $ws_\alpha = dw_1$ with $\mathfrak{l}(ws_\alpha) = \mathfrak{l}(d) + \mathfrak{l}(w_1)$. Then

$$w = dw_1s_\alpha, \quad \mathfrak{l}(w) = \mathfrak{l}(ws_\alpha) + 1 = \mathfrak{l}(d) + (\mathfrak{l}(w_1) + 1).$$

Now $\mathfrak{l}(w_1s_\alpha) \leqq \mathfrak{l}(w_1) + 1$; if there were strict inequality, one would have

$$\mathfrak{l}(w) \leqq \mathfrak{l}(d) + \mathfrak{l}(w_1s_\alpha) < \mathfrak{l}(d) + \mathfrak{l}(w_1) + 1.$$

This gives existence. For uniqueness: let $d' = dw$, $1 \neq w \in W_{\Theta_1}$. Write $w = w's_\beta$, with $\mathfrak{l}(w) = \mathfrak{l}(w') + 1$. Then $d's_\beta = dw'$ has smaller length, and by induction $s_\beta = w'$ (note $\beta \in \Theta_1$). For (ii) induct on $\mathfrak{l}(d)$; we can suppose $\mathfrak{l}(d) > 0$ and $d \notin D_{\Theta_2}$. Then there is a root $\beta \in \Theta_2$ and $d\beta < 0$. Since s_β preserves all positive roots except β we see that $ds_\beta \in D_{\Theta_1}{}^{-1}$ as well and $\mathfrak{l}(ds_\beta) = \mathfrak{l}(d) - 1$. An argument as in (i) now shows that $d = d_{12}w_2$ as desired, and uniqueness follows in the same way.

1.5. COROLLARY. *Each element* $w \in W$ *can be written uniquely in the form* $w = w_1dw_2$, $w_i \in W_i$, $d \in D_{\Theta_1,\Theta_2}$, *and* $\mathfrak{l}(w) = \mathfrak{l}(w_1) + \mathfrak{l}(d) + \mathfrak{l}(w_2)$.

Proof. Apply (i) above to w^{-1} to get $w = w_1d_1$, $d_1 \in D_{\Theta_1}{}^{-1}$. Then (ii) implies $d_1 = d_{12}w_2$. Uniqueness follows as in (i) above.

1.6. COROLLARY. *In each double coset* W_1wW_2 *there is a unique element* d *of minimal length, characterized by any of the following properties:*

(i) *Any element $w \in W_1wW_2$ can be written uniquely in the form*

$$w' = w_1dw_2, \quad \mathfrak{l}(w') = \mathfrak{l}(w_1) + \mathfrak{l}(d) + \mathfrak{l}(w_2).$$

(ii) *It is the element of least length in W_1w, wW_2.*
(iii) $d^{-1}\Theta_1 > 0$, $d\Theta_2 > 0$.

1.7. Given $\Theta_1, \Theta_2 \subseteq \Delta$, set $W(\Theta_1, \Theta_2) = \{w \in W | w\Theta_1 = \Theta_2\}$, and say that Θ_1 and Θ_2 are *associate* if this set is non empty, *associate by* $w \in W$ if $w\Theta_1 = \Theta_2$.

For $\Theta \subseteq \Delta$, define w_Θ to be the longest element in W_Θ. Such an element is unique, by the theory of Coxeter complexes. Then $\mathfrak{l}(w_\Theta) = |R_\Theta^+|$.

LEMMA. *Given $\Theta \subseteq \Psi \subseteq \Delta$, $w_0 = w_\Psi w_\Theta$. Then $\Sigma_{w_0} = R_\Psi^+ - R_\Theta^+$.*

Proof. $\mathfrak{l}(w_0) = n(w_0)$, and since $\mathfrak{l}(w_0) \leqq \mathfrak{l}(w_\Psi)$ (by definition of w in the root system coming from Ψ) we have $\Sigma_{w_0} \subseteq R_\Psi^+$. Moreover $w_0\Theta > 0$ implies $\Sigma_{w_0} \subseteq R_\Psi^+ - R_\Theta^+$. The characteristic property of w_Ψ implies

$$\mathfrak{l}(w_0) = \mathfrak{l}(w_\Psi) - \mathfrak{l}(w_\Theta).$$

Since $\mathfrak{l}(w_\Theta) = |R_\Theta^+|$, $\mathfrak{l}(w_\Psi) = |R_\Psi^+|$, we see that

$$\Sigma_{w_0} = R_\Psi^+ - R_\Theta^+.$$

1.8. If $\Theta \subseteq \Delta$, $\alpha \in \Delta\backslash\Theta$, $\Psi = \Theta \cup \{\alpha\}$, the *conjugate* of Θ in Ψ is defined to be

$$\bar{\Theta} = w_\Psi w_\Theta(\Theta) = w_\Psi(-\Theta) \subseteq \Psi.$$

Note that $\bar{\Theta}$ can be equal to Θ.

PROPOSITION. *The conjugate of Θ in Ψ is the only subset of Ψ associate to Θ by a non trivial element of W_Ψ.*

Proof. Let $w\Theta = \Theta'$, $1 \neq w \in W_\Psi$. Then $\Psi = \Theta' \cup \{\beta\}$. Now $w^{-1}\Theta' > 0$ implies $w^{-1}\beta < 0$, and $w^{-1}\beta \in R_\Psi^- - R_\Theta^-$. Lemma 1.7 implies

$$w_0(R_\Psi^- - R_\Theta^-) \subseteq R_\Psi^+.$$

Thus $w_0w^{-1}(\Theta') = \bar{\Theta} > 0$, and $w_0w^{-1}\beta > 0$. Hence $w_0w^{-1} = 1$.

2. Standard parabolic subgroups.

2.1. Let G be a connected reductive group defined over F, T' a maximal torus not necessarily defined over F. We shall suppose chosen a set of roots R', and simple roots Δ' for the pair $(G_{\bar{F}}, T_{\bar{F}})$ over an algebraic closure $\bar{F}$ of F. Let T_0 be a maximal F-split torus, $T_0 \subseteq T'$, R the set of roots of G with respect to T_0. We can choose a set of simple roots Δ for G with respect to T_0 in such a way that if $\alpha \in \Delta'$ then $\alpha|_{T_0}$ is either trivial or an element of Δ; if this is so then there is a correspondence between subsets of Δ, and Gal $(\bar{F}/F)$-stable subsets of Δ_ϕ', where Δ_ϕ' consists of those

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elements of Δ' not trivial on T_0. The set Δ corresponds to a minimal parabolic subgroup P_0 defined over F. There is a canonical Levi decomposition $P_0 = N_0M_0$, where N_0 is the unipotent radical of P_0, and M_0 is the group $Z_G(T_0)$.

Given a subset $\Theta \subset \Delta$ there is a canonical way to attach a parabolic $P_\Theta \supseteq P_0$ to Θ, which is defined over F. Indeed because of the correspondence above, we may suppose Θ is a Galois stable subset of Δ_ϕ' and work over $\bar{F}$. Then $T_\Theta \subseteq T_0$ will be the identity component of $\bigcap_{\alpha\in\Theta} \ker(\alpha)$, and $M_\Theta = Z_G(T_\Theta)$. The group N_Θ is the unipotent group generated by the root groups N_β such that β is a positive root of the form $\beta = \Sigma m_\alpha \alpha$, and where at least one of the coefficients m_α, $\alpha \in \Delta_\phi' - \Theta$ is strictly positive. The group M_Θ normalizes N_Θ, and P_Θ is defined as the semi-direct product $N_\Theta M_\Theta$; the set $\Delta_{P_\Theta} = \Delta - \Theta$ is said to be the set of simple roots of (P_Θ, T_Θ).

A parabolic subgroup constructed in this way is said to be a *standard parabolic subgroup*. The set of all standard parabolic subgroups is a set of representatives for the F-conjugacy classes of parabolic subgroups of G. In particular $P_\phi = P_0$ and $P_\Delta = G$. The standard proper maximal parabolics are defined by subsets of the form $\Delta - \{\alpha\}$; we shall sometimes simply write P^α rather than $P_{\Delta-\{\alpha\}}$ in this case.

In this paper we shall work only with standard parabolic subgroups, and the word "parabolic" will mean implicitly "standard parabolic". We shall refer to the pair (P_Θ, T_Θ) as a *parabolic pair*.

2.2. Let (P, T) be a parabolic pair, $P = NM$. We denote by $L_{M,r}$ the lattice generated by the roots of (P, T), and set

$$X_M(\mathbf{R}) = L_{M,r} \otimes_Z \mathbf{R}.$$

The elements of $X_M(\mathbf{R})$ give rise to homomorphisms

$$Z_G(\mathbf{A})M(F)\backslash M(\mathbf{A}) \to \mathbf{R}^+,$$

or alternatively, to homomorphisms

$$Z_G(\mathbf{A})Z_M(F)\backslash Z_M(\mathbf{A}) \to \mathbf{R}^+.$$

There is a homomorphism

$$H_{Z_M} : Z_M(\mathbf{A}) \to \mathrm{Mor}\,(X_M(\mathbf{R}), \mathbf{R}) = X_M^*(\mathbf{R}),$$

described as follows. If $z \in Z_M(\mathbf{A})$, then for $\chi \in X_M(\mathbf{R})$,

$$\chi(z) = q^{\langle H_{Z_M}(z), \chi\rangle}.$$

The image $H_{Z_M}(Z_M(\mathbf{A}))$ is a lattice $L_{Z_M}{}^*$ in $X_M(\mathbf{R})$. There is also a homomorphism

$$H_M : M(\mathbf{A}) \to X_M^*(\mathbf{R}) \quad \text{with } H_{M|Z_M(\mathbf{A})} = H_{Z_M},$$

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defined in the same way as before, and we set

$$M^0 = \{m \in M(\mathbf{A}) | \chi(m) = 1, \chi \in X_M(\mathbf{R})\}$$
$$Z_M{}^0 = M^0 \cap Z_M(\mathbf{A})$$
$$L_M{}^* = M^0\backslash M(\mathbf{A}).$$

We shall write $X_M(\mathbf{C}) = X_M(\mathbf{R}) \otimes \mathbf{C}$, $X_M{}^*(\mathbf{C}) = X_M{}^*(\mathbf{R}) \otimes \mathbf{C}$.

2.3. Write $X_0(\mathbf{R})$ for $X_{M_0}(\mathbf{R})$, etc. There is a natural projection

$$X_0{}^*(\mathbf{R}) \twoheadrightarrow X_M{}^*(\mathbf{R})$$

and injection

$$X_M(\mathbf{R}) \subset X_0(\mathbf{R})$$

coming from the map $M_0 \subset M$, when $P = NM$ is a parabolic containing P_0. Any root $\alpha \in X_M(\mathbf{R})$ is the restriction of a unique root $\alpha_0 \in X_0(\mathbf{R})$. Write α^* for the projection of the coroot $\alpha_0{}^*$ to $X_M^*(\mathbf{R})$. The set thus obtained is a basis for $X_M^*(\mathbf{R})$; let $L_{M,c}^*$ be the lattice in $X_M^*(\mathbf{R})$ generated by the α^* (called coroots by abuse), for $\alpha \in \Delta_P$.

The inverse image of $L_{M,c}^*$ in $Z_M(\mathbf{A})$ will be denoted by $Z_{M,c}$; it is a subgroup of finite index in $Z_M(\mathbf{A})$, and there is an exact sequence

$$Z_M{}^0 \subset Z_{M,c} \twoheadrightarrow L_{M,c}^*.$$

2.4. Let $P_2 \supseteq P_1$ be parabolics with Levi decompositions $P_i = N_iM_i$. Then $P_1 \cap M_2$ is a parabolic subgroup of M_2 with unipotent radical $N_1{}^2 = N_1 \cap M_2$. Let $\Delta_1{}^2$ be the set of roots for the pair $(P_1 \cap M_2, T_1)$; it is a subset of Δ_1. In this way, subsets of Δ_1 correspond to parabolic subgroups containing P_1, while the map $P_1 \rightarrow P_1 \cap M_2$ is a bijection between parabolics contained in P_2 and parabolic subgroups of M_2.

We shall often write $\Delta^\wedge{}_P$ for the basis dual to the basis of coroots, and refer to it as the basis of *weights*.

2.5. Let $P_i = P_{\Theta_i}$ $(i = 1, 2)$ be as above, $w \in W(\Theta_1, \Theta_2)$. Write $[\Theta_1]$ for the set of roots spanned by Θ_i, $R_i = C[\Theta_i] \cap R^+$, and similarly for $R_i{}^-$. Let

$$R' = C[\Theta_i] \cap w^{-1}R_2, \quad R'' = R_1{}^+ \cap w^{-1}(R_2{}^-).$$

Then $R_1 = R' \cup R''$, and $R_2 = wR' \cup w(-R'')$. The sets R', R'' are convex, so closed, hence generate unipotent groups N', N'' respectively, and the product map $N' \times N'' \rightarrow N_1$ is an isomorphism of varieties. Similarly for ${}^wN' \times {}^w(N''^-) \rightarrow N_2$, and we have ${}^wN' = N_2 \cap {}^wN_1$. Let δ' (resp. δ'') be the modulus character of M_1 restricted to N' (resp. N''). In additive notation, $\delta_1 = \delta' + \delta''$, $\delta_2 = w\delta' - w\delta''$, where δ_i is the modulus character for M_i.

3. Chamber decompositions.

3.1. The main result of this section is due to R. P. Langlands (see [**13**] especially Lemma 2.13), but the proof given here has been taken from unpublished notes of W. Casselman, who attributes it to J. G. Arthur.

3.2. We shall say that $P_i = N_i M_i$ $(i = 1, 2)$ are *associate* if M_1, M_2 are conjugate by an element of $G(F)$.

PROPOSITION. *Suppose P_i corresponds to Θ_i $(i = 1, 2)$. The following are equivalent:*

(i) *P_1, P_2 are associate.*

(ii) *There is $g \in G(F)$, and $gT_1g^{-1} = T_2$.*

(iii) *There is $w \in W$, and $w\Theta_1 = \Theta_2$.*

In this case one can always choose $g \in N(T_0)$ for (ii).

Proof. One need only show (ii) $\Leftrightarrow$ (iii), and clearly (iii) $\Rightarrow$ (ii). Suppose (ii) holds, then both gT_0g^{-1}, T_0 are maximal F-split tori in M_2 so conjugate by an element $m_2 \in M_2(F)$. Then $m_2 g \in N_G(T_0)$, Int $(m_2 g)T_1 = T$ so we may suppose $g \in N_G(T_0)$. Choose the unique element w of minimal length in the coset of $W_1\backslash W/W_2$ containing the image of g (1.6). Then 1.6 also implies that $w\Theta_1 > 0$, and since $gT_1g^{-1} = T_2$, this implies $wR_{\Theta_1} = R_{\Theta_2}$, so that $w\Theta_1 \subseteq R_{\Theta_2}{}^+$ whence $wR_{\Theta_2}{}^+ = R_{\Theta_2}{}^+$ and $w\Theta_1 = \Theta_2$.

3.3. For $\Theta \subseteq \Delta$ we identify

$$X_\Theta{}^*(\mathbf{R}) = \{x \in X_0{}^*(\mathbf{R}) \mid \alpha(x) = 0, \alpha \in \Theta\}.$$

For brevity, $X_\alpha{}^*(\mathbf{R}) = X_{\{\alpha\}}{}^*(\mathbf{R})$. We set

$$C_\Theta = \{x \in X_\Theta{}^*(\mathbf{R}) \mid \alpha(x) > 0, \alpha \in \Delta\backslash\Theta\}$$

$$\bar{C}_\Theta = \{x \in X_\Theta{}^*(\mathbf{R}) \mid \alpha(x) \geqq 0, \alpha \in \Delta\backslash\Theta\}.$$

The set of regular elements is the set

$$X_\Theta{}^*(\mathbf{R})^+ = X_\Theta{}^*(\mathbf{R})\backslash \textstyle\bigcup_{\alpha \in R\backslash R_\Theta} (X_\alpha{}^*(\mathbf{R}) \cap X_\Theta{}^*(\mathbf{R})).$$

It is an open subset of $X_\Theta{}^*(\mathbf{R})$.

PROPOSITION. (i) *Each connected component of $X_\Theta{}^*(\mathbf{R})^+$ has the form $w^{-1}C_{\Theta'}$ for some unique $\Theta' \subseteq \Delta$, $w \in W(\Theta, \Theta')$.*

(ii) *Given Θ, $\Theta' \subseteq \Delta$, $w \in W(\Theta, \Theta')$, there exist subsets $\Theta_1 = \Theta', \ldots$ $\Theta_n = \Theta$ and for each $i \leqq n - 1$, there exists a root $\alpha_i \in \Delta\backslash\Theta$ such that Θ_{i+1} is the conjugate of Θ_i in $\Psi_i = \Theta_i \cup \{\alpha_i\}$. If one sets $w_i = w_{\Psi_i} w_{\Theta_i}$ for $1 \leqq i \leqq n - 1$, then $w = w_{n-1} \ldots w_2 w_1$.*

3.4. Remarks. (i) Observe that the Weyl group W acts on $X_0(\mathbf{R})$ by means of the representation contragredient to the representation by which it acts on $X_0{}^*(\mathbf{R})$.

(ii) Each connected component of $X_0{}^*(\mathbf{R})^+$ is a simplicial cone ([**3**], V. 1.6).

We first prove 3.3 (i) in case C' is a component of $X_0{}^*(\mathbf{R})^+$ which has a face in common with C_Θ. Such a face must then have the form

$$X_\alpha{}^*(\mathbf{R}) \cap \bar{C}_\Theta, \alpha \in \Delta\backslash\Theta.$$

3.5. LEMMA. *Let C' be a component with $X_\alpha{}^*(\mathbf{R}) \cap \bar{C}_\Theta$ in common with $C_\Theta, \alpha \in \Delta\backslash\Theta$. Then $C' = w_0{}^{-1}C_{\bar{\Theta}}$ where $w_0 = w_\Psi w_\Theta$, $\bar{\Theta}$ = conjugate of Θ in $\Psi = \Theta \cup \{\alpha\}$.*

Proof. $w_0{}^{-1}X_{\bar{\Theta}}{}^*(\mathbf{R}) = X_\Theta{}^*(\mathbf{R})$ since $w_0{}^{-1}\bar{\Theta} = \Theta$; $w_0{}^{-1}(X_{\bar{\Theta}}{}^*(\mathbf{R})^+) = X_\Theta{}^*(\mathbf{R})^+$. Moreover $w_0{}^{-1}C_{\bar{\Theta}}$ is a connected component of $X_\Theta(\mathbf{R})^+$. We show

(i) $w_0{}^{-1}C_{\bar{\Theta}} \neq C_\Theta$; this will prove the lemma if we also show

(ii) $w_0{}^{-1}C_{\bar{\Theta}}$ has the face $X_\alpha{}^*(R) \cap \bar{C}_\Theta$.

(i) Let $\bar{\alpha}$ be the unique element of $\Theta \cup \{\alpha\}\backslash\bar{\Theta}$, then

$$w_0{}^{-1}\bar{\alpha} \in R_\Psi{}^-\backslash R_\Theta{}^-.$$

Hence, if $x \in C_{\bar{\Theta}}$ then

$$(w_0{}^{-1}\bar{\alpha})(w_0{}^{-1}x) = \bar{\alpha}(x) > 0,$$

so

$$(-w_0{}^{-1}\bar{\alpha})(w_0{}^{-1}x) < 0$$

i.e., $w_0{}^{-1}x \notin C_\Theta$.

(ii) $C_{\bar{\Theta}}$ has the face $X_{\bar{\alpha}}{}^*(\mathbf{R}) \cap \bar{C}_{\bar{\Theta}}$, and

$$w_0(X_{\bar{\alpha}}{}^*(\mathbf{R}) \cap \bar{C}_{\bar{\Theta}}) = X_\alpha{}^*(\mathbf{R}) \cap \bar{C}_\Theta.$$

3.6. We can now finish the proof of 3.3 (i), by using induction. If C' is a connected component, define $d' = d(C_\Theta, C')$ to be the minimal number of hyperplanes separating C_Θ from C'. We proceed by induction on d'.

Let H be a face of C' which is part of a chain $(X_1, \ldots, X_{d'})$ of hyperplanes of minimal length separating C_Θ from C'. Then there is a component C'' sharing this face, and $d'' = d' - 1$. We conclude that $C'' = w''^{-1}C_{\Theta''}$ for some unique $\Theta'' \subseteq \Delta$, and $w'' \in W(\Theta, \Theta'')$. Moreover $w''C'$ and $C_{\Theta''}$ share a face, so that we can apply the lemma to find $w_0{}^{-1}C_{\Theta''} = w''C'$. Then $w = w_0w''$, and $\bar{\Theta}''$ will do. As for uniqueness, if $w_1C_{\Theta_1} = w_2C_{\Theta_2}$ then this means that

$$w_2{}^{-1}w_1C_{\Theta_1} = C_{\Theta_2}$$

so that

$$w_2{}^{-1}w_1\Theta_2 = \Theta_1.$$

This implies that $w_2{}^{-1}w_1$ keeps all positive roots positive, hence $w_2{}^{-1}w_1 = 1$ and $\Theta_2 = \Theta_1$. The proof of 3.3 (ii) follows from that of (i).

3.7. Let $P_i = N_i M_i$ $(i = 1, 2)$ be parabolics,

$$\pi : P_1 \to N_1 \backslash P_1 \cong M_1$$

the natural map. Set $\bar{P}_2 = \pi(P_2 \cap P_1)$. The next lemma is straightforward to prove.

LEMMA. (i) *$\bar{P}_2$ is a parabolic subgroup of M_1 with unipotent radical* $\pi(N_2 \cap P_1)$.

(ii) *If $\bar{P}_2 = M_1$, then $M_2 \supseteq M_1$.*

4. The reduction theory of Harder.

4.1. Let c be a real number. Then we write

$$P_0(c) = \{x \in P_0(\mathbf{A}) |\ \langle \alpha, H_M(x) \rangle \geqq c, \alpha \in \Delta\}$$

where as usual

$$|\alpha(x)| = q^{\langle \alpha, H_M(x) \rangle}.$$

Let K be an open compact subgroup of the form $\Pi_p K_p$. The first main result of reduction theory is the following.

THEOREM. *There exists a constant $c_1 > -\infty$ such that*

$$G(\mathbf{A}) = G(F) P_0(c_1) \Sigma K$$

where Σ is a finite subset of $G(F)$, depending only on K.

Remark. The constant c_1 depends on the genus of F.

4.2. Suppose $y = p\xi_1 k \in P_0(c_1)\xi_1 K$, $\xi_1 \in \Sigma$; we write $|\alpha(y)| = |\alpha(p)|$ for $\alpha \in \Delta$. This is well defined.

THEOREM. (Second main theorem of reduction theory). *Let c_1 be a number such that* 4.1 *holds. There is a constant $c_2 = c_2(c_1)$ with the following property:*

If $y \in P_0(c_1)\xi_1 K \cap \gamma P_0(c_1)\xi_2 K$, $\gamma \in G(F)$, $\xi_i (i = 1, 2) \in \Sigma$ then $\gamma \in P_{\Delta - \Delta'}(F)$ if $|\alpha(y)| > q^{c_2}$ for each $\alpha \in \Delta' \subseteq \Delta$.

4.3. The next question to consider is that of compactness.

THEOREM. *$M \subseteq G(\mathbf{A})$ is relatively compact modulo $Z(\mathbf{A})G(F)$ if and only if*

$$M \subseteq G(F) P_0(c_1, c') \Sigma K,$$

for some c', where

$$P_0(c_1, c') = \{x \in P_0(\mathbf{A}) |\ q^{c'} \geqq |\alpha(x)| \geqq q^{c_1}, \alpha \in \Delta\}.$$

4.4. Let $X_G(\mathbf{R})$ be the group of quasi characters $G(\mathbf{A}) \to \mathbf{R}^+$. This is a finite dimensional vector space over $\mathbf{R}$; write $X_G^*(\mathbf{R})$ for its dual. In

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the same way as before, there is a homomorphism

$$G(\mathbf{A}) \to X_G^*(\mathbf{R})$$

and we write G^0 for its kernel.

Recall that G is anisotropic if and only if G contains no proper parabolic subgroups.

THEOREM. *The following conditions are equivalent.*

(i) *G is anisotropic.*

(ii) *$G(F)\backslash G^0/K$ is a finite set.*

(iii) *$G(F)\backslash G^0$ is compact.*

4.5. Let N be a soluble linear algebraic group defined over F. The following result is proved in [**5**] Théorème 2, cf. [**9**] Satz 2.2.1.

PROPOSITION. *$N(F)\backslash N^0$ is compact.*

4.6. It will be convenient to reformulate some of these results. From 2.2 we have a homomorphism

$$M_0(\mathbf{A}) \xrightarrow{H_0} X_{M_0}^*(\mathbf{R}),$$

and lattices

$$L_{M_0}^* \supseteq L_{Z_0}^* \supseteq L_{0,c}^*.$$

It follows that the group $Z_0(\mathbf{A})M_0{}^0$ has finite index in $M_0(\mathbf{A})$; let $\mathfrak{l}_i$ be a set of coset representatives so that

$$M_0(\mathbf{A}) = \dot{\bigcup}_i M_0{}^0 Z_0(\mathbf{A})\mathfrak{l}_i.$$

Then $H_0(m)$ can be written in the form

$$H_0(m) = H_0(z) + H_0(\mathfrak{l}_i)$$

where $\mathfrak{l}_i$, $H_0(z)$ are unique. Therefore

$$M_0(\mathbf{A}) = M_0{}^0 Z_0(\mathbf{A})\,\Lambda, \quad \Lambda \text{ finite}$$
$$P_0(\mathbf{A}) = N_0(\mathbf{A})M_0{}^0 Z_0(\mathbf{A})\,\Lambda.$$

Theorem 4.4 implies that $M_0{}^0 = M(F)Z(\mathbf{A})\nu$, where ν is a compact set, and Theorem 4.5 implies that $N_0(\mathbf{A}) = N_0(F)\omega$, where ω is a compact set. Combining these results we see that

$$P_0(\mathbf{A}) = N_0(\mathbf{A})M_0(F)\nu Z_0(\mathbf{A})\,\Lambda = P_0(F)\omega Z_0(\mathbf{A})\nu\,\Lambda.$$

Moreover, if we replace c_1 by $c_1 - \inf_{i,\alpha} |\alpha(\mathfrak{l}_i)|$, then $P_0(c_i)$ can be replaced by $P_0(F)\omega Z_0(c_1)\nu\,\Lambda$, and

$$G(\mathbf{A}) = G(F)\omega Z_0(c_1)C$$

where $C = \nu\xi\Sigma K$ is a compact subset of $G(\mathbf{A})$.

Similarly, by changing $c_2(c_1)$ if necessary we can reformulate Theorem 4.2 in this fashion.

Finally, suppose K is a maximal compact subgroup, chosen so that $G(\mathbf{A}) = P_0(\mathbf{A})K$ (such K always exist from the work of Bruhat-Tits); then changing c_1 again if necessary, we can eliminate the finite set Σ from the fundamental domain obtained above.

We may write $Z_0(c_1)$ in the form $Z_0^0 \cdot L^*(c_1)$ where $L^*(c_1)$ maps isomorphically onto a subset of $L^*_{Z_M}$. Then we can put the fundamental domain above in the form $\omega L^*(c_1)C$. We shall write $\mathfrak{S}(c_1, \omega, C)$ for the subset $\omega L^*(c_1)C$ or sometimes $\mathfrak{S}(c_1, \omega)$ when no confusion can occur (this will imply that a subset C has been chosen in the manner above). On occasion we shall use these same symbols to denote the subset $\omega Z_0(c_1)C$.

As a corollary of Theorem 4.1 one has

4.7. COROLLARY. *$G(F)Z(\mathbf{A})\backslash G(\mathbf{A})$ has finite volume.*

Proof. Set $\mathfrak{S} = \omega Z_0(c_1)C$, and choose a sequence $\mathfrak{S}_n \subseteq \mathfrak{S}$ $(n \geqq 1)$ such that (i) $\mathfrak{S}_n$ is compact mod $Z(\mathbf{A})$ (ii) $\cup \mathfrak{S}_n = \mathfrak{S}$. Let f_n be the characteristic function of $\mathfrak{S}_n$. Then

$$\int_{Z(\mathbf{A})\backslash G(\mathbf{A})} f_n(g)dg = \int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} \sum_{G(F)} f_n(\gamma g)dg.$$

The left hand side is just the volume of $(Z(\mathbf{A})\backslash\mathfrak{S}_n)$; the right hand side is at least the volume of the image of $\mathfrak{S}_n$ in $G(F)Z(\mathbf{A})\backslash G(\mathbf{A})$. Now use monotone convergence.

5. Applications.

5.1. We begin with some remarks concerning compact subgroups of $G(\mathbf{A})$. From now on, in this paper, K will always denote a fixed maximal compact subgroup of the form $\Pi_{\mathfrak{p}} K_{\mathfrak{p}}$, and K' will stand for an open compact subgroup $K' \subseteq K$, also of the form $\Pi_{\mathfrak{p}} K_{\mathfrak{p}}'$. Moreover, we suppose K is chosen as in 4.6, so that $G = P_0(\mathbf{A}) \cdot K$.

Suppose $\rho : K \rightarrow \mathbf{GL}(m, \mathbf{C})$ is a continuous (irreducible) representation. Since K is totally disconnected, ρ factors through $H\backslash K$ where H is an open compact subgroup; it is clear that H contains a subgroup K' as above. Thus if V is a finite dimensional complex vector space on which K acts then V is fixed pointwise by some normal subgroup K', and the representation factors through the finite group $K'\backslash K$.

5.2. Let $\varphi : G(F)\backslash G(\mathbf{A}) \rightarrow \mathbf{C}$ be a measurable function. Then φ is *K-finite* if the space $V(\varphi)$ generated by right K-translates of φ is a finite dimensional $\mathbf{C}$-vector space. In this case $V(\varphi)$ is fixed pointwise by an open compact subgroup K', and in particular consists of locally constant (hence continuous) functions.

We shall say that φ is *cuspidal* if for every proper parabolic $P = NM$, one has

$$\varphi^P(g) := \int_{N(F)\backslash N(\mathbf{A})} \varphi(ng)dn \equiv 0.$$

5.3. The next proposition has been known for some time, cf. [**7**], [**10**].

PROPOSITION. *Let φ be a K-finite cuspidal function on $G(F)\backslash G(\mathbf{A})$. Then φ has compact support modulo $Z(\mathbf{A})G(F)$.*

In order to prove this result we need the following result, which we shall then use to prove a lemma.

5.4. Let $P = NM$ be a parabolic corresponding to $\Theta \subseteq \Delta$. For each root $\beta = \sum a_j\alpha_j, \alpha_j \in \Delta$, set

$$a(\beta) = \sum_{\alpha_j\in\Delta\setminus\Theta} a_j,$$

and let R_i be the set of roots β such that $a(\beta) > i$, for each integer $i \geqq 0$. Then each R_i is closed, and $R_i \cap (-R_i) = \emptyset$ hence gives rise to a subgroup $N_i \subseteq N$. This gives a chain

$$\mathrm{N} = \mathrm{N}_0 \supseteq N_1 \supseteq \ldots \supseteq N_n = \{1\}.$$

The proof of the following proposition can be found in SGAD XXVI Proposition 2.1, cf. also [**2**] Théorème 3.17.

PROPOSITION. (i) *Each N_i is smooth, connected, and characteristic and closed in P. For each F algebra A, one has the commutator relation*

$$(N_i(A), N_j(A)) \subseteq N_{i+j+1}(A).$$

(ii) *Each $N_{i+1}\backslash N_i \cong W_i$, a vector group on which P acts linearly.*
(iii) dim (N) = dim (Lie $(N_{\bar{F}})) \geqq n$.

5.5. LEMMA. *There is a constant c' such that if $t \in T_0(c_1)$ and if for some $\alpha \in \Delta$ one has $|\alpha(t)| \geqq c'$,*

$$N^\alpha(F)(t(K' \cap N^\alpha(\mathbf{A}))t^{-1}) = N^\alpha(\mathbf{A})$$

where $P^\alpha = N^\alpha M^\alpha$. Here c_1 is as in 4.1.

Proof. We prove the lemma by climbing up the chain exhibited in 5.4 for the group N^α.

First consider the group $W_i(\mathbf{A})$. According a well known variant of the Riemann-Roch theorem, if Ω is an open subgroup of the vector group $V(\mathbf{A})$, there is a constant $c_\Omega = c$ such that for each $a \in \mathbf{A}^*$ with $|a| > q^c$, one has $a\Omega + V(F) = V(\mathbf{A})$. The construction of W_i implies that as schemes

$$W_i \cong \bigoplus\nolimits_{a(\beta)=i+1} N_\beta,$$

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where in this case $\beta = a(\beta)\alpha + \beta'$. Furthermore it follows from the construction that under the action of T_0, the vector group W_i breaks up into a sum of weight spaces W_β,

$$W_i = \bigoplus_{a(\beta)=i+1} W_\beta$$

such that on W_β, T_0 acts via β. For each β, write $\beta = a(\beta)\alpha + \beta'$ where β' contains no multiple of α. Letting β vary, one can choose a number c_3 such that $|\beta'(t)| > q^{c_3}$, $t \in T_0(c_1)$. Then

$$|\beta(t)| = |\alpha(t)|^{a(\beta)}|\beta'(t)| > |\alpha(t)|^{a(\beta)}q^{c_3}.$$

Taking projections we see that $\bar{K}' = \Pi(K' \cap N_i(\mathbf{A}))$ is an open subgroup of $W_i(\mathbf{A})$ and that $K' \cap W_\beta(\mathbf{A})$ is open compact in $W_\beta(\mathbf{A})$. Set $\bar{K}_\beta' = \bar{K}' \cap W_\beta(\mathbf{A})$. Then

$$t\bar{K}_\beta' t^{-1} = \alpha(t)^{a(\beta)}\beta'(t)\bar{K}_\beta'.$$

Choose

$$\bar{c}' = \sup\left(\frac{c_{\Omega_\beta} - c_3}{a(\beta)}\right).$$

Then for $|\alpha(t)| > \bar{c}'$,

$$t\bar{K}'t^{-1} + W_i(F) = W_i(\mathbf{A}).$$

Here we have taken $\Omega_\beta = \bar{K}_\beta'$.

Now let $n \in N_i(\mathbf{A})$; set

$$N' = \prod_{a(\beta)=i+1} N_\beta$$

where N_β as before is the unipotent group generated by those (absolute) positive roots restricting to the relative root β. There are isomorphisms of schemes

$$N_{i+1} \times N' \to N_i \quad \text{(given by multiplication)}$$
$$N' \to W_i.$$

In particular $n \in N_i(\mathbf{A})$ can be written $n = n_{i+1}n'$, $n_{i+1} \in N_{i+1}(\mathbf{A})$, $n' \in N'(\mathbf{A})$. The preceding remarks imply that

$$n' = n_F'k',\ n_F' \in N'(F), \quad k' \in t(K' \cap N'(\mathbf{A}))t^{-1};$$

hence $n = n_F' n_{i+1}{}^{n_F'}k'$. Now $n_{i+1}{}^{n_F'} \in N_{i+1}$ by the commutator formula in 5.4 (i), so by induction,

$$n_{i+1}{}^{n_F'} = \bar{n}_{i+1}k_i \quad \text{for } |\alpha(t)| \geqq q^{c_{i+1}}.$$

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Thus

$$n = n_i n' = n_F{}' \bar{n}_{i+1F} k_i k',$$

$$n_F{}' \bar{n}_{i+1F} \in N_i(F), \quad \text{and}$$

$$k_i k' \in tK't^{-1} \cap N_i(\mathbf{A}).$$

So if we take $c_i = \max(c_{i+1}, \bar{c}')$ we get the lemma for $N_i(\mathbf{A})$.

5.6. Proof of Proposition 5.3. From 4.1 it is enough to show that given a compact set Ω, one can find a constant c' such that if $t \in T_0(c_1)$, $\varphi(tg) \neq 0$ implies $t \in T_0(c_1, c')$ for $g \in \Omega$. For each $g \in \Omega$, the function $x \mapsto \varphi(xg)$ satisfies the same conditions as φ. Further, since Ω/K' is finite one may find K'' so that all the functions $x \mapsto \varphi(xg)$ are K'' invariant. So it is enough to show that $\varphi(t) \neq 0$ implies $t \in T_0(c_1, c')$ whenever φ is K' invariant, cuspidal.

For each $n \in N^\alpha(\mathbf{A})$, we may write $n = \gamma t n' t^{-1}$, $\gamma \in N^\alpha(F)$, $n' \in N^\alpha(F) \cap K'$, by the lemma. Then

$$\varphi(nt) = \varphi(\gamma t n') = \varphi(t), \quad \text{and}$$

$$\phi(t) = \int_{N^\alpha(F)\backslash N^\alpha(\mathbf{A})} \phi(nt)dn \equiv 0 \quad \text{by assumption.}$$

5.7. For ξ a character of $Z(F)\backslash Z(\mathbf{A})$, define $\mathscr{L}(\{G\}, \xi)$ to be the space of measurable functions

$$\varphi : G(F)\backslash G(\mathbf{A}) \rightarrow \mathbf{C}$$

satisfying

(i) $\varphi(zg) = \xi(z)\varphi(g), \quad z \in Z(\mathbf{A}).$

(ii) $\phi^P(g) = \int_{N(F)\backslash N(\mathbf{A})} \phi(ng)dn \equiv 0,$

all proper parabolics $P = NM$.

(iii) $\int_{G(F)Z(\mathbf{A})\backslash G(\mathbf{A})} |\phi(g)|^2 dg < \infty.$

PROPOSITION. *Let K' be a compact open subgroup of $G(\mathbf{A})$. The set of $\varphi \in \mathscr{L}(\{G\}, \xi)$ invariant on the right by K' is a vector space of finite dimension.*

Proof. (Jacquet-Godement). The proof of the lemma shows there is a compact subset $C \subset G(\mathbf{A})$ such that all functions right invariant by K' have support in $Z(\mathbf{A})G(F)C$; moreover C is evidently a finite union of right cosets of K'. Therefore these functions are determined by their values on the finite set C/K', and the result follows.

5.8. From 5.7 it follows that for each f in the Hecke algebra (cf. [7]), the convolution operator f acting on $\mathscr{L}(\{G\}, \xi)$ has an image of finite rank. This implies the following.

THEOREM. *Let ρ_0 be the action of $G(\mathbf{A})$ on $\mathscr{L}(\{G\}, \xi)$ by right translations. Then ρ_0 is a discrete sum of irreducible unitary representations, each of which occurs with finite multiplicity.*

5.9. We conclude this section with some amplifications of 5.3.

Firstly, observe that one can prove a version of Lemma 5.5 for any proper parabolic subgroup, using Proposition 5.4.

Secondly, suppose that φ is not cuspidal, but merely K-finite. Then using the version of Lemma 5.5 just mentioned one sees that there is a constant c' so that if $|\alpha(t)| > c'$, $\alpha \in \Theta$ then $\varphi(t) = \varphi^P(t)$ for $t \in Z_0(c')$ where P is the parabolic of type $\Delta\backslash\Theta$. This leads us to our next observation.

Choose $\mathfrak{S} = \mathfrak{S}(c_1, \omega, C)$ as in 4.6, and let $g = ntx$ where $n \in \omega$, $t \in L_0(c_1)$, $x \in C$. Then $g = t(t^{-1}nt)x$, and $\bigcup_{t\in L(c_1)} t^{-1}\omega t$ is relatively compact. Thus $g = ty$, $y \in C'$ where C' is compact. Using the argument of 5.6 together with the remark above we find that if $nmk = g \in \mathfrak{S}$ satisfies $|\alpha(m)| > c'$ for c' large enough, $\alpha \in \Theta$, then $\varphi(g) = \varphi^P(g)$, P the standard parabolic of type $\Delta\backslash\Theta$.

Finally, note that both φ and φ^P give rise to functions on $P(F)\backslash G(\mathbf{A})$. Thus by projecting $\mathfrak{S}$ into $P(F)\backslash G(\mathbf{A})$ one can interpret the above in terms of the functions on $P(F)\backslash G(\mathbf{A})$.

6. Weights and heights.

6.1. Let F_s be the separable algebraic closure of F; if H is an algebraic object defined over F, we write H_s for the corresponding object obtained by extending scalars to F_s. The next two results in their present form are proved in [**16**], Théorème 2.5, 3.3. In this section we write Λ_{s+} for the set of those dominant weights of G_s with respect to some root system, which are characters for the maximal torus.

6.2. THEOREM. *For each $\lambda \in \Lambda_{s+}$ there is an irreducible F_s-representation having λ as dominant weight, which occurs with multiplicity one (i.e. it is simple). There is a one to one correspondence between elements of Λ_{s+} and F_s-isomorphism classes of absolutely irreducible F_s-representations of G_s. The dimension of such a representation is no greater than the dimension of the corresponding representation over the complex numbers.*

6.3. Let V be such a representation with dominant weight λ. It is well known and easy to see that there is a unique line D_λ in V upon which B, the Borel group corresponding to the chosen set of simple roots Δ, acts via λ. In fact if P_λ is the standard parabolic obtained from those simple coroots orthogonal to λ, then P_λ is the stabilizer of D_λ. In particular, let $m\lambda_\alpha$ be a multiple of one of the fundamental dominant weights, then the corresponding parabolic is simply P^α.

6.4. Recall that if G is defined over F, then $\sigma \in \mathrm{Gal}\,(F_s/F) = \Gamma$ acts upon the root system, and on the weight group; the action we take is the **-action*. Namely, for $\sigma \in \Gamma$, the natural action gives another set of simple roots $\sigma(\Delta)$, so there is a unique element w_σ of the Weyl group which takes $\sigma(\Delta)$ back to Δ, and we set $\sigma^*\alpha = w_\sigma(\sigma(\alpha))$.

Finally, if D is a finite dimensional division algebra over F, then a *D-module* V is an F-module equipped with a D-action. We define $\mathbf{GL}_D(V)$ to be the algebraic group such that for each F-algebra A,

$$\mathbf{GL}_D(V)(A) = \mathrm{Aut}_{D \underset{F}{\otimes} A}(V \otimes A),$$

where $\mathrm{Aut}_{D\underset{F}{\otimes} A}$ means "automorphisms preserving the $D \underset{F}{\otimes} A$ action." In particular $\mathbf{GL}_{n,D}$ refers to $\mathbf{GL}_D(V)$, $V = (D)^n$.

THEOREM (Tits). *Let $\lambda \in \Lambda_{s+}{}^\Gamma$. There exists a division algebra, central over F, and a D-representation $\rho : G \to \mathbf{GL}_m, D$, which is absolutely irreducible with dominant weight λ; D is unique up to F-isomorphism. For given D, the representation is unique up to D isomorphism. If $\lambda \in \Lambda_{s0}$ (group generated by roots, and by weights which are trivial on the semi simple part of T_s, T a maximal torus defined over k), or if G is quasi-split, then $D = F$.*

6.5. The group $\Lambda_s{}^\Gamma$ is a free submodule of Λ_s with basis given by the elements $\{\lambda_{\alpha_1} + \ldots + \lambda_{\alpha_r} | \alpha_i \in \Gamma(\alpha_1)\}$, where $\Gamma(\alpha_1)$ is the orbit under Γ of α_1. Let λ correspond to $\Gamma(\alpha)$. If $\alpha|_{T_0}$ gives a relative root $\bar{\alpha}$, then

$$(\lambda|_{T_0}, \bar{\alpha}^*) = r(\lambda_\alpha, \alpha^*) = r.$$

On the other hand, the relative F-weights are generated by the $\lambda_{\bar{\alpha}_i}$. From this it follows immediately that to each $\lambda_{\bar{\alpha}}$, α a simple F-root, there exists an integer $m_{\bar{\alpha}} \geqq 1$ so that $m_{\bar{\alpha}}\lambda_{\bar{\alpha}}$ corresponds to an irreducible D representation with dominant weight $m_{\bar{\alpha}}\lambda_{\bar{\alpha}}$. Furthermore there is a unique line $D_{m_{\bar{\alpha}}\lambda_{\bar{\alpha}}}$ stabilized by the maximal F-parabolic $P^{\bar{\alpha}}$. This follows from the fact that the representation corresponding to $\lambda_{\alpha_1} + \ldots + \lambda_{\alpha_r}$ corresponds to the conjugacy class of parabolics of type $P_\Delta - \{\alpha_1, \ldots, \alpha_r\}$, which by descent corresponds to the maximal parabolic $P^{\bar{\alpha}}$.

6.6. As we shall be considering $G(F)\backslash G(\mathbf{A})$ it will be convenient for us to consider representations V of G contragredient to the above. Suppose that we have a representation ρ_λ' as in 6.5. Then we get a right G action ρ_λ' (in pointwise language) in the usual way by setting

$$(v^*\rho_\lambda(g), v) = (v^*, \rho_\lambda'^{-1}(g)v).$$

As a variant of 6.5 we obtain a representation $\rho_{m_\alpha\lambda_\alpha}$ corresponding to a negative integer $m_\alpha \leqq -1$.

6.7. Let V be a finite dimensional F-vector space. We say that $v \in V(\mathbf{A})$ is *primitive* if there is a $g \in \mathbf{GL}(V(\mathbf{A}))$ such that $0 \neq vg \in V(F)$. We define a *height function* $\| \quad \|$ on the set of primitive elements in the following way. Choose on each $V(F_\mathfrak{p})$ ($F_\mathfrak{p}$ being the local field corresponding to the place $\mathfrak{p}$), a norm compatible with the field norm $| \quad |_\mathfrak{p}$ on $F_\mathfrak{p}$, such that, for almost all $\mathfrak{p}$, and almost all $v_\mathfrak{p} \in V(F_\mathfrak{p})$ one has

$$\|v_\mathfrak{p}\|_\mathfrak{p} = \sup_i |\xi_i|_\mathfrak{p}$$

where

$$v_\mathfrak{p} = \sum_{i=1}^{n} \xi_i v_i,$$

v_i a base of $V(F)$ fixed once and for all. Then for v primitive, one defines

$$\|v\| = \prod_\mathfrak{p} \|v_\mathfrak{p}\|_\mathfrak{p}.$$

The properties of such a function are enumerated in [**5**] § 1.1; we recall only those of immediate interest:

(i) Fix $g \in \mathbf{GL}(V(\mathbf{A}))$. Then $\{v \in V(F) | \|vg\| < c\}$ is finite modulo F^*, and thus $\|vg\|$ achieves its minimum on $V(F)$.

(ii) $\|\lambda v\| = |\lambda| \|v\|$, all $\lambda \in \mathbf{A}^*$, $v \in V(\mathbf{A})$ primitive.

(iii) If $C \subset \mathbf{GL}(V(\mathbf{A}))$ is compact, there are constants $c_1, c_2 > 0$ such that $c_1\|v\| \leqq \|vc\| \leqq c_2\|v\|$, all $c \in C$, $v \in V(\mathbf{A})$ primitive.

(iv) If $\| \quad \|_1$, $\| \quad \|_2$ are two heights, then $\|v\|_1/\|v\|_2$ remains in a fixed compact subset of $\mathbf{R}^+\backslash\{0\}$, when v varies.

7. A partition.

7.1. Let P and P' be parabolic subgroups corresponding to Θ, Θ' respectively. In the following lemma we shall suppose that $|\Theta| = |\Theta'|$ and that $0 \leqq b < 1$ is fixed. For each $\alpha_i \in \Delta$ we write (V_i, ρ_i) for the rational representation in 6.6 with lowest weight $m_i\bar{\omega}_i$. For brevity we write $\chi_i = m_i\bar{\omega}_i$.

Finally, if $g \in P_0(c_1)K$ as in § 4, write $g = p_g k_g$ with $p_g \in P_0(c_1)$, $k_g \in K$.

LEMMA. *Suppose that g and $g' = \gamma g \in P_0[c_1]K$, with $\gamma \in G(F)$. There are constants t, t' such that if $\alpha, \beta \in \Delta$ and*

$$|\alpha(p_g)| > t, \alpha \notin \Theta$$
$$|\alpha(p_g)|^b > |\beta(p_g)|, \alpha \notin \Theta, \beta \in \Theta$$

and

$$|\alpha(p_{g'}')| > t', \alpha \notin \Theta'$$
$$|\alpha(p_{g'}')| > |\beta(p_{g'}')|, \alpha \notin \Theta', \beta \in \Theta'$$

then

$$P = P'.$$

Proof. From Theorem 4.2 one knows that t_0 may be chosen so that $\gamma \in P(F)$, and similarly for t_0'; we suppose this is the case. In particular, if $v_i \in V_i$ is a rational vector transforming by χ under the action of the maximal parabolic $P^i = P^{\alpha_i}$, then

$$|\chi(g)|\,\|v_i k_g\| = \|v_i \rho_i(g)\| = \|v_i \rho_i(g')\| = |\chi(g')|\,\|v_i k_{g'}\|$$

where we are writing $|\chi(g)| = |\chi(p_g)|$. Thus there are numbers $d_1, d_2 > 0$ and

$$d_1 \leqq |\chi(g)|/|\chi(g')| \leqq d_2.$$

Since $\alpha_j = \sum c_{ji}\bar{\omega}_i$, we readily deduce the existence of constants $d_1', d_2' > 0$ such that

$$d_1'|\alpha_j(g')| \leqq |\alpha_j(g)| \leqq d_2'|\alpha_j(g')| \quad \text{for each } \alpha_j \in \Delta.$$

Order the simple roots such that

$$|\alpha_1(g')| \geqq |\alpha_2(g')| \geqq \ldots \geqq |\alpha_n(g')|.$$

Our assumptions on g' imply then that $\Theta' = \{\alpha_{q+1}, \ldots, \alpha_n\}$ some $q \geqq 1$. If $i \leqq q < j$, then

$$|\alpha_i(g)| \geqq d_1'|\alpha_i(g')| = d_1'|\alpha_i{}^b(g')|\,|\alpha_i{}^{1-b}(g')| > |\alpha_j(g')| d_1' t_0'^{1-b}.$$

Choose t_0' so big that $d_1' t_0'^{1-b} > 1$. Then

$$|\alpha_i(g)| > \alpha_j(g') d_1' t_0'^{1-b} \geqq 1/d_2'\alpha_j(g) \cdot d_1' t_0'^{1-b} > |\alpha_j(g)|$$

for $i \leqq q < j$ and t_0' large enough.

If now we order the simple roots $\beta_i \in \Delta$ so that

$$|\beta_i(g)| \geqq |\beta_2(g)| \geqq \ldots \geqq |\beta_n(g)|$$

then the preceding argument implies that

$$\{\alpha_1, \ldots, \alpha_q\} = \{\beta_1, \ldots, \beta_q\}.$$

Thus $\Theta = \Theta'$ and $P = P'$.

7.2. Given $\mathfrak{S}(c_1, \omega, c) = \mathfrak{S}$ as in 4.6, numbers $0 \leqq b < 1$ and $t > 0$, and $\Theta \subseteq \Delta$, we write

$$\mathfrak{S}(b, \Theta, t) = \left\{ g \in \mathfrak{S} \,\middle|\, \begin{array}{l} |\alpha(g)| \geqq t, \alpha \in \Delta\backslash\Theta \quad \text{and} \\ |\alpha(g)|^b \geqq |\beta(g)|, \alpha \in \Delta\backslash\Theta, \beta \in \Theta \end{array} \right\}.$$

Write $G(\mathbf{A}) = N(A)M(\mathbf{A})K$ where $P = NM$ is a maximal parabolic. We suppose in the sequel that $P = P^\alpha, \alpha \in \Delta$. Given $T > 0$ let

$$G(T) = \{g |\ q^{\langle H_M(g), \alpha\rangle} \geqq T\}.$$

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Fix b, and put

$$\mathfrak{S}(b, a) = \{g \in \mathfrak{S} |\ |\alpha(g)|^b \geqq |\beta(g)|, \beta \neq \alpha\}.$$

Finally $\mathfrak{S}_\alpha(T) = G(T) \cap \mathfrak{S}(b, \alpha)$.

LEMMA. *There is a constant $T' = T'(T, b)$ so that*

$$\mathfrak{S}_\alpha(T) \subseteq \mathfrak{S}(b, \Delta\backslash\{\alpha\}, T'\}.$$

Proof. Let (V, ρ) be the rational representation corresponding to P, with lowest weight $\chi = m_\alpha\bar{\omega}_\alpha$. We let v be the v_i in the preceding lemma. If $g \in \mathfrak{S}$, $g = p_0k_0 = pk \in P(\mathbf{A})K$ with $p_0 \in \omega M_0(c_1)$, $k_0 \in K$, then

$$|\chi(p_0)|\,\|vk_0\| = |\chi(p)|\,\|vk\|.$$

Set $p = z\mathfrak{l}m$, where $m \in M^0(\mathbf{A})$, $\mathfrak{l}$ represents a coset of $L_{Z_M}{}^*\backslash L_M{}^*$, and $z \in Z_M(\mathbf{A})$. The equality becomes

$$|\chi(p_0)|\,\|vk_0\| = |\chi(z)|\,\|vk\|\,|\chi(\mathfrak{l})|.$$

Since the number of possible $\mathfrak{l}$ is finite, we obtain numbers $d_1, d_2 > 0$ such that

$$d_1 \leqq |\chi(p_0)|/|\chi(z)| \leqq d_2.$$

On the other hand $\bar{\omega}_\alpha = \sum b_{\alpha\beta}\beta$, $b_{\alpha\beta} \geqq 0$ and on Z_M, $m_\alpha\bar{\omega}_\alpha = m_\alpha b_{\alpha\alpha}\alpha$, $b_{\alpha\alpha} > 0$. Thus

$$|\chi(p_0)| = \prod_{\beta\neq\alpha} |\beta(p_0)|^{m_\alpha b_{\alpha\beta}}\cdot |\alpha(p_0)|^{m_\alpha b_{\alpha\alpha}}$$

$$|\chi(z)| = |\alpha(z)|^{m_\alpha b_{\alpha\alpha}}.$$

Since p_0 is supposed to be in $\mathfrak{S}_\alpha(T)$, we have

$$|\alpha^b(p_0)| \geqq |\beta(p_0)|, \beta \neq \alpha$$

so that

$$\prod_{\beta\neq\alpha} |\beta(p_0)|^{m_\alpha b_{\alpha\beta}} > \prod |\alpha^b(p_0)|^{m_\alpha b_{\alpha\beta}}.$$

Thus

$$|\alpha(p_0)|^c \leqq |\chi(p_0)| \leqq d_2\chi(z) \leqq d_4(T)$$

since $z \in \mathfrak{S}_\alpha(T)$. Here c is a negative $(\neq 0)$ number. The result follows.

7.3. *Remark.* Note that T' increases if T increases, and $T > 1$.

7.4. Let

$$\begin{tikzcd} G(\mathbf{A}) \arrow[r, "\Pi_p'"] \arrow[d, "\Pi"'] & P(F)\backslash G(\mathbf{A}) \arrow[dl, "\Pi_P"] \\ G(F)\backslash G(\mathbf{A}) & \end{tikzcd}$$

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be the obvious projections and write

$$S_\alpha(T) = \Pi(\mathfrak{S}_\alpha(T)), S_\alpha'(T) = \Pi_P'(S_\alpha(T)).$$

LEMMA. *We can choose T so that if $\alpha \neq \beta$ then $S_\alpha(T) \cap S_\beta(T)$ is empty and such that Π_P is a bijection of $S_\alpha'(T)$ onto $S_\alpha(T)$.*

This lemma follows immediately from 7.1 and 7.2.

2. A Preliminary Decomposition.

1. The complex analytic manifold $D_M(\xi)$.

1.1. Let $P = NM$ be a parabolic, fixed in the sequel. Let ξ be a character of $Z(F)\backslash Z(\mathbf{A})$. We write $D_M(\xi)$ for the set of those quasi-characters of $Z_M(F)\backslash Z_M(\mathbf{A})$ which prolong ξ.

The group $X_M(\mathbf{C})$ acts on $D_M(\xi)$ via

$$\chi \mapsto \chi \cdot q^{\langle H_{Z_M}(\cdot),\omega\rangle}, \ \chi \in D_M(\xi), \ \omega \in X_M(\mathbf{C}).$$

The stabilizer of χ is just iL_{Z_M}, where

$$L_{Z_M} = \{\omega \in X_M(\mathbf{C}) | \langle H_M(z), \omega\rangle \in 2\pi\mathbf{Z}/\log q, z \in Z_M(\mathbf{A})\}$$

and where i is the square root of -1 which lies in the upper half plane.

Consequently, there is a structure of complex analytic manifold on $D_M(\xi)$ characterized by the fact that each orbit is an open and closed subvariety.

If $\zeta \in D_M(\xi)$, one defines $\operatorname{Re}\zeta$ by $\operatorname{Re}\zeta = |\zeta|$; it is an element of $X_M(\mathbf{R})$. By definition, the set of *characters* of $D_M(\xi)$ is the set of ζ for which $|\zeta| = 1$; we denote this set by $D_M{}^0(\xi)$.

1.2. We note in passing some observations on the orbit structure. To begin, observe that if two elements χ, ζ of $D_M(\xi)$ restrict to the same element on $Z_M{}^0$ then they are in the same orbit. In effect $\chi\zeta^{-1}$ is trivial on $Z_M{}^0$, so can be viewed as a quasicharacter on $L_{Z_M}{}^*$.

Similarly, if ω is a character of $Z_M{}^0$, then the set of quasicharacters prolonging ω is just the coset $\bar{\omega}X_M(\mathbf{C})$, $\bar{\omega}$ is any character extending ω.

On the other hand, the set of quasicharacters of $Z_M{}^0$ which extend ξ is a countable set of characters. Indeed ξ is unitary, and we have

$$0 \to Z(F)\backslash Z(\mathbf{A}) \to Z_M(F)\backslash Z_M{}^0 \to Z(\mathbf{A})Z_M(F)\backslash Z_M{}^0 \to 0.$$

By Pontrjagin duality, and the fact that $Z(\mathbf{A})Z_M(F)\backslash Z_M{}^0$ is compact, the set of quasicharacters prolonging ξ is just

$$\omega(Z(\mathbf{A})Z_M(F)\backslash Z_M{}^0)^\wedge$$

where ω is a character chosen to prolong ξ.

Henceforth, we shall suppose chosen a set $\{\omega\}$ of characters representing the orbits $\Omega(\omega)$ of $D_M(\xi)$ under the action of $X_M(\mathbf{C})$.

1.3. In 1.2.2 the lattice L_M^* was defined; it contains as a sublattice the lattice $L_{Z_M}^*$. Let $\mathfrak{l}_1, \ldots, \mathfrak{l}_a$ be a set of coset representatives for $L_{Z_M}^* \backslash L_M^*$. If

$$g = nmk \in N(\mathbf{A})M(\mathbf{A})K = G(\mathbf{A}),$$

write $H_M(g) = H_M(m)$; this is a locally constant map, invariant by $N(\mathbf{A})P(F)$. Write $H_M(g) = \bar{H}_M(g) + \mathfrak{l}_g$, where $\bar{H}_M(g) \in L_{Z_M}^*$, and $\mathfrak{l}_g \in \{\mathfrak{l}_1, \ldots, \mathfrak{l}_a\}$; such a decomposition is evidently unique. The map $g \mapsto \bar{H}_M(g)$ is evidently locally constant, invariant by $N(\mathbf{A})P(F)$, because the map $g \mapsto H_M(g)$ is.

1.4. With these considerations out of the way, we now begin the study of the space $\mathscr{L}(\xi)$, which is the main object of interest in this paper. It is defined to be the space of measurable functions $\varphi : G(F)\backslash G(\mathbf{A}) \rightarrow \mathbf{C}$ such that

(i) $\varphi(zg) = \xi(z)\varphi(g)$, $z \in Z(\mathbf{A})$, $g \in G(\mathbf{A})$

(ii) $\displaystyle\int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} |\phi(g)|^2 dg < \infty.$

The object of the theory of Eisenstein series, to be described in this, and a subsequent, paper is to describe the action of $G(\mathbf{A})$ by right translations on $\mathscr{L}(\xi)$ in terms of representations induced from standard parabolic subgroups. To do this one must introduce a number of auxiliary spaces defined in terms of these subgroups.

Suppose then that $P = NM$ is a proper parabolic; the subgroup $M \neq G$ is a reductive group as well, and we define $\mathscr{L}_M(\xi)$ to consist of those measurable functions φ on $M(F)\backslash M(\mathbf{A})$ which satisfy

(i) $\varphi(zm) = \xi(z)\varphi(m)$, $z \in Z(\mathbf{A})$

(ii) $\displaystyle\int_{Z(\mathbf{A})M(F)\backslash M(\mathbf{A})} |\phi(m)|^2 dm < \infty.$

The closed, invariant (by $M(\mathbf{A})$) subspace $\mathscr{L}(\{M\}, \xi)$ is defined in terms of proper parabolic subgroups of M, as in 1.5. We observe that one can also define the space $\mathscr{L}(\{M^0\}, \xi)$, where M^0 is defined in 1.2.2, because $Q(\mathbf{A}) \cap M^0$ always contains the unipotent radical of $Q(\mathbf{A})$, if $Q \subseteq M$ is a proper parabolic.

Since $M(\mathbf{A})$ acts on $\mathscr{L}(\{M\}, \xi)$ by right translations we may consider the representation,

$$\mathrm{Ind}_{P(\mathbf{A})}^{G(\mathbf{A})} (\mathscr{L}\{M\}, \xi))$$

where $P(\mathbf{A})$ acts on $\mathscr{L}(\{M\}, \xi)$ via the natural projection $P \rightarrow M$. This representation of $G(\mathbf{A})$ acts on the space $\mathscr{C}(P, \xi)$ of functions,

$$\varphi : N(\mathbf{A})P(F)\backslash G(\mathbf{A}) \rightarrow \mathbf{C}$$

such that

(i) $m \mapsto \varphi(mg) \in \mathscr{L}(\{M\}, \xi)$, each $g \in G(\mathbf{A})$

(ii) $\int_{Z(\mathbf{A})N(\mathbf{A})P(F)\backslash G(\mathbf{A})} |\phi(g)|^2 dg < \infty$.

In 1.3.2 we have defined the notion of an associate class of parabolics. Write $\{P\}$ for the class of parabolics associate to P.

We set

$$\mathscr{C}(\{P\}, \xi) = \bigoplus_{P \in \{P\}} \mathscr{C}(P, \xi).$$

1.5. We shall work extensively with a subspace $\mathscr{C}_0(P, \xi)$ of $\mathscr{C}(P, \xi)$. It is defined to consist of those functions φ such that

(i) φ is (right) K-finite.

(ii) For each $g \in G(\mathbf{A})$, the support of $m \mapsto \varphi(mg)$ is compact modulo M^0.

(iii) The space of right translates of φ by $G(\mathbf{A})$, viewed as a space of functions on M^0, is a finite sum of irreducible subspaces of $\mathscr{L}(\{M^0\}, \xi)$.

The result proved in 1.5.3, together with (ii) and (iii) implies that φ has compact support modulo $N(\mathbf{A})P(F)Z(\mathbf{A})$, whence compact support mod $P(F)Z(\mathbf{A})$.

There is a simple variant of $\mathscr{C}_0(P, \xi)$ which we shall often use. Namely, if $K' \subseteq K$ is as described in 1.5.1, then we replace (i) by (i)$'$ $\varphi(gk') = \varphi(g)$, $k' \in K'$.

The resulting space is denoted $\mathscr{C}_0(P, K', \xi)$; we have

$$\mathscr{C}_0(P, \xi) = \cup_{K'} \mathscr{C}_0(P, K', \xi).$$

In particular any element of $\mathscr{C}_0(P, \xi)$ is locally constant.

1.6. The next thing we must define in this setting is a Fourier transform. Let $\varphi \in \mathscr{C}_0(P, \xi)$, $\nu \in D_M(\xi)$; then ν can be written (ω, ζ) where

$$\zeta \in \mathrm{i}L_{Z_M}{}^*\backslash X_M(\mathbf{C}) \quad \text{and} \quad \omega \in \{\omega\}.$$

We set

$$\Phi(g; \nu) = \Phi(g; (\omega, \zeta)) = \int_{Z_M(F)Z(\mathbf{A})\backslash Z_M(\mathbf{A})} \phi(zg)(\omega \cdot \zeta)^{-1}(z)dz.$$

The assumption (ii) in 1.5 implies that this integral is finite. For each ν, $\Phi(g; \omega \cdot \zeta) = \Phi(g; \nu)$ is a function satisfying the conditions (i) and (iii) in 1.5 and is such that

(ii)$'$ $\Phi(zg, \omega \cdot \zeta) = \omega(z)\zeta(z)\Phi(g, \omega\zeta)$, $z \in Z_M(\mathbf{A})$;

(iv) For each g, the function

$$m \mapsto \Phi(mg; \zeta)$$

satisfies

$$\int_{Z_M(\mathbf{A})M(F)\backslash M(\mathbf{A})} |\Phi(mg;\zeta|^2 e(m, -2\zeta_0)dm < \infty, \quad \operatorname{Re}\zeta = \zeta_0.$$

This implies

$$\text{(v)} \int_{Z_M(\mathbf{A})N(\mathbf{A})P(F)\backslash G(\mathbf{A})} |\Phi(g;\zeta|^2 e(g, -2\zeta_0 + 2\delta_P)dg < \infty$$

where for $g = nmk$, we write as usual,

$$e(g, \zeta) = e(m, \zeta) = \exp\{\log q(\bar{H}_M(m), \zeta)\}.$$

The property (iv) follows from the Plancherel formula and Fubini's theorem cf. the argument in 3.9 below.

We shall denote this space of functions by $\mathscr{C}_0(P, \omega \cdot \zeta)$. If we fix g for the moment, and let z vary, it follows from standard commutative harmonic analysis, that

$$\varphi(g) = \int_{\operatorname{Re}\zeta=\zeta_0} \Phi(g;\zeta)d\zeta.$$

1.7. Let $\mathscr{C}_0(P, K', \omega\zeta\delta_P)$ be the subspace of $\mathscr{C}_0(P, \omega\zeta\delta_P)$ analogous to the subspace $\mathscr{C}_0(P, K', \xi)$. Then $\mathscr{C}_0(P, K', \omega\zeta\delta_P)$ is finite dimensional by 1.5.7; we remark in passing that $\mathscr{C}_0(P, K', \xi)$ is not finite dimensional.

In particular, if $\varphi \in \mathscr{C}_0(P, K', \xi)$, we obtain functions

$$\Phi_\omega(g, \zeta\delta_P) \in \mathscr{C}_{0,\omega}(P, K', \zeta\delta_P).$$

There is a map,

$$\mathscr{C}_0(P, \omega \cdot \delta_P) \rightarrow \mathscr{C}_0(P, \omega \cdot \zeta\delta_P)$$
$$\Phi \rightarrow e(g, \zeta)\Phi.$$

From its construction, this map is an isomorphism. Thus we obtain a finite dimensional complex analytic vector bundle over $D_M(\xi)$, via

$$X_M(\mathbf{C}) \times_{\mathfrak{i}L_{Z_M}} \mathscr{C}_0(P, K', \omega \cdot \delta_P) \rightarrow \bigcup_{\zeta\in\mathfrak{i}L_{Z_M}X_M(\mathbf{C})} \mathscr{C}_0(P, K', \omega \cdot \zeta\delta_P)$$

where the space on the right inherits its analytic structure from the space on the left. At (ω, ζ) the fibre of this vector bundle is $\mathscr{C}_0(P, K', \omega \cdot \zeta\delta_P)$.

In fixing K', one ensures that only finitely many ω can intervene. Indeed K' has finite index in K, and if we restrict ω to $Z_M{}^0$, we see that it can only be one of a finite number of possibilities, since $Z_M(F)Z(\mathbf{A})\backslash Z_M{}^0$ is compact. Consequently the vector bundle just constructed has fibre $\{0\}$ outside a finite number of components. We denote it by $\mathscr{C}_0(P, K')$.

Let us fix a single component for the moment; then a holomorphic global section over it can be viewed as a holomorphic function

$$\Phi_\omega : iL_{Z_M}\backslash X_M(\mathbf{C}) \to \mathscr{C}_0(P, K', \omega \cdot \delta_P).$$

Now the Fourier transform constructed above assigns to each function $\varphi \in \mathscr{C}_0(P, K', \xi)$ a section, via

$$\varphi \mapsto e(g, -\zeta)\Phi(g, \omega \cdot \zeta\delta_P).$$

It follows by standard arguments that this section is holomorphic. In fact if one chooses a basis for $X_M(\mathbf{C})$ then the space of sections obtained can be identified with the space of trigonometric polynomials with values in $\mathscr{C}_0(P, K', \omega \cdot \delta_P)$.

A holomorphic global section over the whole bundle can thus be viewed as a function

$$(1.7.1) \quad \Phi(\zeta) : iL_{Z_M}\backslash X_M(\mathbf{C}) \to \bigoplus_\omega \mathscr{C}_0(P, K', \omega \cdot \delta_P)$$

where the sum on the right is finite, by an earlier remark. We write $\Gamma(\mathscr{C}_0(P, K'))$ for this space of sections.

1.8. *Conventions.* We pause to make some conventions, and remarks.

(i) We have been viewing an element $\Phi \in \Gamma(\mathscr{C}_0(P, K'))$ both as a function

$$\Phi(\zeta) : iL_{Z_M}\backslash X_M(\mathbf{C}) \to \bigoplus_\omega \mathscr{C}_0(P, K', \omega \cdot \delta_P)$$

and as a family of functions Φ_ζ, each of which belongs to

$$\bigoplus_{\{\omega\}} \mathscr{C}_0(P, K', \omega \cdot \zeta\delta_P).$$

From now on, we shall write $\Phi(\zeta)$ to mean we are considering things in the first way, and we shall write Φ_ζ when we are looking at matters from the second viewpoint.

(ii) Each $w \in W(M, M')$ induces isomorphisms

$$X_M(\mathbf{C}) \to X_{M'}(\mathbf{C})$$
$$D_M(\xi) \to D_{M'}(\xi).$$

We shall always write this action on the left; it is given by

$$\langle {}^w\zeta, x\rangle = \langle \zeta, w^{-1} \cdot x\rangle,$$

where the action on the right is the one induced by Ad : $w \cdot x = wxw^{-1}$. We remark that we shall indiscriminately write quasicharacters in additive or multiplicative form. Thus for example $({}^w\zeta)^{-1}$ will correspond to $-w\zeta$ under this convention.

(iii) A word is in order concerning the space $\mathscr{C}_0(P, \omega \cdot \zeta\delta_P)$. The group $G(\mathbf{A})$ acts by right translation on this space. If we equip $\mathscr{C}_0(P, \omega \cdot \zeta\delta_P)$ with the norm

$$\int_{N(\mathbf{A})Z_M(\mathbf{A})P(F)\backslash G(\mathbf{A})} |\zeta|^{-2}(g)|\Phi_\zeta(g)|^2 dg$$

then the action of $G(\mathbf{A})$ is unitary with respect to this norm if $\operatorname{Re}(\omega\cdot\zeta\delta_P) = \delta_P$. The point is that

$$\zeta(g) = q^{\langle H_M(g),\zeta\rangle} \neq q^{\langle \bar{H}_M(g),\zeta\rangle}$$

so the norm above is equivalent to, but not the same as, the norm given in 1.6 (v).

2. The functions $\varphi^\wedge$.

2.1. Let $P = NM$ be a parabolic with roots $\alpha_1, \ldots, \alpha_r$; there are corresponding fundamental weights $\bar{\omega}_1, \ldots, \bar{\omega}_r$. For each such weight $\bar{\omega}_i$, one can find a rational representation

$$\rho_i : G \to \mathbf{GL}(V)$$

such that $G(\mathbf{A})$ acts on $V(\mathbf{A})$ to the right, a negative integer m_i, and a rational vector v such that $P^{\alpha_i}(\mathbf{A})$ acts on v via $m_i\bar{\omega}_i$ (cf. 1.6). In particular $P_0 \subset P^{\alpha_i}$ acts on v this way.

If $x \in G(\mathbf{A}) = P(\mathbf{A})K$, we put

$$|\bar{\omega}_i(x)| = |\bar{\omega}_i(p)| \quad \text{for } x = pk.$$

This is well defined since there is an integer m_i such that $m_i\bar{\omega}_i$ is a rational character, and $|\bar{\omega}_i^{m_i}(x)|$ takes values in $\mathbf{R}^+$.

LEMMA. *Let C be a subset of $G(\mathbf{A})$, compact modulo $Z(\mathbf{A})$. Then,*

$$\sup_{\substack{\gamma\in G(F)\\ c\in C}} |\bar{\omega}_i(\gamma c)| < \infty.$$

Proof. Take v, m_i as above, $\chi = m_i\bar{\omega}_i$. It is enough to show

$$\inf |\chi(\gamma c)| = \inf |\bar{\omega}_i^{m_i}(\gamma c)| < \infty.$$

Since $G(\mathbf{A}) = P(\mathbf{A})K$, there are constants $d_1, d_2 > 0$ such that

$$d_1|\chi(g)| \leqq \|vg\| \leqq d_2|\chi(g)|,\ g \in G(\mathbf{A}).$$

From 1.6.7 (i), one has a constant $d_3 > 0$ such that

$$\|v\gamma\| \geqq d_3,\ \gamma \in G(F).$$

These two remarks imply that there is a $d_4 > 0$, and $|\chi(\gamma)| \geqq d_4$ for $\gamma \in G(F)$. If now $c \in C$, then

$$\|v\gamma c\| = |\chi(\gamma)|\,\|vkc\|,$$

where $\gamma = pk$, so that

$$\|v\gamma c\| \geqq d_4 \cdot d > 0,$$

since kc varies over a compact set.

2.2. Let $\lambda : Z_M(F)\backslash Z_M(\mathbf{A}) \to \mathbf{C}^*$ be a quasicharacter supposed unitary on $Z(\mathbf{A})$. For the next lemma we shall consider functions

$$\Phi : P(F)N(\mathbf{A})\backslash G(\mathbf{A}) \to \mathbf{C}$$

such that

(i) $\Phi(zg) = \lambda(z)\Phi(g)$, $z \in Z_M(\mathbf{A})$, $g \in G(\mathbf{A})$

(ii) Φ is (right) K-finite

(iii) Φ is bounded on subsets compact modulo M^0.

Recall that the *Weyl chamber* C_P associated to P consists of those elements $\lambda \in X_M(\mathbf{R})$ such that

$$(\lambda, \alpha_i^*) \geqq 0, \alpha_i \in \Delta_P.$$

We write $\lambda > \mu$ if $\lambda - \mu \in C_P$.

LEMMA. (Godement) *The series* $\sum_{\gamma \in P(F)\backslash G(F)} \Phi(\gamma g)$ *converges uniformly on subsets of* $G(\mathbf{A})$ *compact modulo* $Z(\mathbf{A})$, *provided*

$$\operatorname{Re} \lambda - 2\delta_P \in C_P.$$

Proof. We recall the function $g \to \bar{H}_M(g)$ of 1.3; it gives rise to a function $g \to \lambda(g) = e(g, \lambda)$ (1.6). Since

$$G(\mathbf{A}) = N(\mathbf{A})Z_M(\mathbf{A})BM^0K$$

where B is a finite set, and since

$$|\Phi(nzbm^0k)| \leqq |\lambda(z)|\,|\Phi(bm^0k)|$$

we may, by (iii) above, suppose that λ is real valued, and that $\Phi(g) > 0$ with $\Phi(p^0g) = \Phi(g)$ if $p^0 \in N(\mathbf{A})M^0$. Consequently, it follows that for Ω compact modulo $Z(\mathbf{A})$, one can find constants $d_1, d_2 > 0$ such that

$$d_1\Phi(g) \leqq \Phi(g\omega) \leqq d_2\Phi(g), \omega \in \Omega, g \in G(\mathbf{A}).$$

This implies that it is enough to prove convergence at the point 1. Let K' be any open compact subgroup of $G(\mathbf{A})$ such that Φ is right K'-invariant; then convergence at 1 is equivalent to convergence of

$$\sum_{P(F)\backslash G(F)} \int_{Z(\mathbf{A})\backslash K'Z(\mathbf{A})} \Phi(\gamma k')dk'.$$

Choose K' so small that $G(F) \cap K' = \{1\}$, then one only has to show that

$$\int_{P(F)Z(\mathbf{A})\backslash G(F)K'Z(\mathbf{A})} \Phi(g)dg$$

converges. Now, Lemma 2.1 implies that

$$G(F)K'Z(\mathbf{A}) \subseteq N(\mathbf{A})M(t)K, \quad \text{some } t > 0$$

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where

$$M(t) = \{m \in M(\mathbf{A}) \mid |\bar{\omega}_i(m)| \leqq q^N = t, \text{ each } \bar{\omega}_i\}.$$

The integral

$$\int_{P(F)Z(\mathbf{A})\backslash N(\mathbf{A})M(t)K} \Phi(nmk)\delta_P^{-2}(m)dndmdk$$

may be written as

$$\int_{P(F)Z(\mathbf{A})\backslash N(\mathbf{A})M(\mathbf{A})K} \Phi(nmk)\delta_P^{-2}(m)\chi(nmk)dndmdk$$

where χ is the characteristic function for $N(\mathbf{A})M(t)K$. In turn, this can be written

$$\int_{L_{M,c}^* \cap L_M^*/L_M^*} \int_{L_{M,c}^* \cap L_M^*} \int_{M(F)Z(\mathbf{A})\backslash M^0} \int_{N(F)\backslash N(\mathbf{A})} \int_K .$$

By assumption (ii), everything reduces to computing (after adjusting t, perhaps)

$$\int_{L_{M,c}^* \cap L_M^*} \chi(l)\lambda\delta_P^{-2}(l)dl.$$

The lattice $L_{M,c}^* \cap L_M^*$ can be coordinatized by positive multiples of the $\alpha_i^*, \alpha_i \in \Delta_P$. If we note that

$$\lambda\delta_P^{-2}(z) = q^{\langle H_M(z),\lambda-2\delta_P\rangle} \quad \text{if } z \in Z_{M,c} = \prod_i q^{\langle H_M(z),\bar{\omega}_i\rangle\langle\alpha_i^*,\lambda-2\delta_P\rangle}$$

then the resulting integral is just

$$\text{2.2.1)} \quad \prod_i \frac{q^{\langle\alpha_i^*,\lambda-2\delta_P\rangle N r_i}}{1 - q^{r_i\langle\alpha_i^*,\lambda-2\delta_P\rangle}}, \quad r_i > 0.$$

2.3. Now let $\varphi \in \mathscr{C}_0(P, K', \xi)$, and define $\varphi^\wedge$ by

$$\varphi^\wedge(g) = \sum_{\gamma\in P(F)\backslash G(F)} \varphi(\gamma g).$$

LEMMA. *The series $\varphi^\wedge(g)$ converges uniformly on subsets of $G(\mathbf{A})$ compact modulo $Z(\mathbf{A})G(F)$.*

Proof. Let Ω be compact modulo $Z(\mathbf{A})G(F)$; one may as well suppose φ non negative, and Ω compact. For each $g \in \Omega$, gK' is an open neighbourhood of g; we conclude that there is a finite number of $g \in \Omega$, $g_1, \ldots, g_s$ say, and $\Omega \subseteq \bigcup_{1\leqq i\leqq s} g_iK'$. If $g \in \Omega$ then $g \equiv g_i (\text{mod } K')$. Thus

$$\varphi^\wedge(g) = \sum_\gamma \varphi(\gamma g_i),$$

and a term $\varphi(\gamma g_i)$ is non zero if $\gamma g_i \in \operatorname{supp} \varphi$. Now supp φ is compact modulo $P(F)Z(\mathbf{A})$ (cf. the remark in 1.5) so we conclude that for each i, the number of such γ is finite mod $P(F)Z(\mathbf{A})$. Since $1 \leqq i \leqq s$, we are done.

2.4. LEMMA. *The support of* $\varphi^\wedge$ *is compact modulo* $Z(\mathbf{A})G(F)$.

Proof. As remarked earlier, supp φ is compact modulo $P(F)Z(\mathbf{A})$. Let Ω be compact such that supp $\varphi \subseteq P(F)Z(\mathbf{A})\Omega$. Then, by reduction theory,

$$\Omega \subseteq P_0(c_1', c_2')\Sigma K', \quad \Sigma \text{ finite}.$$

One may choose c_2'' so that Theorem 1.4.2 holds for c_1', c_2''. If α is a simple root, let (V, ρ), χ be as in 1.7.1, and let $\delta = \inf_{g \in \Omega} \|vg\|$.

By an argument used in 1.5.5 one can arrange for c_2'' to be such that $\langle H_0(g), \alpha\rangle > c_2''$ implies $\|vg\| < \delta$, $g \in P_0(c_1')\Sigma K'$ (here one uses $\chi = m_\alpha b_{\alpha\alpha}\alpha + \sum_{\beta \neq \alpha} m_\alpha b_{\alpha\beta}\beta$, where $m_\alpha < 0$, $b_{\alpha\beta} \geqq 0$, and $b_{\alpha\alpha} > 0$). Now suppose $g \in P_0(c_1')\Sigma K'$, and $\langle H_0(g), \alpha\rangle > c_2''$, $\alpha \in \Delta$, so that $\|vg\| < \delta$. If $\varphi^\wedge(g) \neq 0$ then there is $\gamma \in G(F)$, and

$$\gamma g \in \Omega \subseteq P_0(c_1')\Sigma K'.$$

Theorem 1.4.2 implies $\gamma \in P^\alpha(F)$, but then $\|v\gamma g\| = \|vg\| > \delta$ and this is a contradiction.

2.5. COROLLARY. $\varphi^\wedge \in \mathscr{L}(\xi)$.

2.6. Suppose now that $\Phi_\omega \in \mathscr{C}_0(P, K', \omega \cdot \delta_P)$. Recall that for $\zeta \in X_M(\mathbf{C})$ we have constructed a function

$$T_\zeta \Phi_\omega(g) = q^{\langle \bar{H}_M(g), \zeta\rangle} \Phi_\omega(g) = \Phi_{\omega \cdot \zeta}(g).$$

Since $q^{\langle \bar{H}_M(g), \zeta\rangle} \equiv 1$ if and only if $\zeta \in iL_{Z_M}$, one can work with $iL_{Z_M} \backslash X_M(\mathbf{C})$ equally well, and

$$\Phi_{\omega.\zeta} \in \mathscr{C}_0(P, K', \omega \cdot \zeta\delta_P).$$

Now define

$$E(g, \Phi_{\omega\cdot\zeta}) = \sum_{\gamma \in P(F) \backslash G(F)} \Phi_{\omega\cdot\zeta}(\gamma g).$$

Lemma 2.2 implies that this series converges uniformly on subsets of $G(\mathbf{A})$ compact modulo $Z(\mathbf{A})G(F)$ provided Re $\zeta - \delta_P \in C_P$. It is a holomorphic function with values in $\mathscr{L}_{\text{loc}}(\xi)$, where, for $\mathscr{L}_{\text{loc}}(\xi)$ one takes the space of measurable functions

$$\varphi : G(F)\backslash G(\mathbf{A}) \to \mathbf{C}$$

such that

(i) $\varphi(zg) = \xi(z)\varphi(g)$, $z \in Z(\mathbf{A})$, $g \in G(\mathbf{A})$;

(ii) For each compact subset $C \subseteq G(F)Z(\mathbf{A})\backslash G(\mathbf{A})$, one has

$$\int_C |\varphi(g)|^2 dg < \infty.$$

The function $E(g, \Phi_{\omega.\zeta})$ is evidently K finite.

2.7. Given $\varphi \in \mathscr{C}_0(P, K', \xi)$, we have constructed a Fourier transform $\Phi(g; \omega \cdot \zeta\delta_P)$ in 1.6. In the notation above, this function can be written $\Phi_{\omega.\zeta}(g)$. Set

$$\Phi_\zeta = \bigoplus_\omega \Phi_{\omega.\zeta}.$$

This is a finite sum, since a non zero summand only occurs if $\omega|_{K' \cap Z_{M^0}} = 1$ by Schur orthogonality. We set

$$E(g, \Phi_\zeta) = \bigoplus_\omega E(g, \Phi_{\omega.\zeta}).$$

LEMMA.

$$\varphi^\wedge(g) = \int_{\mathrm{Re}\ \zeta=\zeta_0} E(g, \Phi_\zeta)d\zeta.$$

Proof. The result follows from the formula

$$\varphi(g) = \int_{\mathrm{Re}\ \zeta=\zeta_0} \Phi_\zeta(g)d\zeta$$

(which follows from the usual formula, and the remark above about Schur orthogonality), and from interchanging sum and integral, which can be done because φ is very well behaved. The details are left to the reader.

2.8. The next proposition says that the $\varphi^\wedge$ form a dense subspace of $\mathscr{L}(\xi)$.

PROPOSITION. *Let $\phi \in \mathscr{L}(\xi)$ and suppose that $(\phi, \psi) \equiv 0$ for each ψ as ψ runs through $\bigcup_P \mathscr{C}_0(P, \xi)$. Then $\phi \equiv 0$.*

Proof. One may suppose ϕ is right K'-invariant for some K'; indeed $G(\mathbf{A})$ has a countable base of neighbourhoods of the identity which consist of open compact subgroups, so that there is a sequence $\phi_n \to \phi$ (in $\mathscr{L}(\xi)$) where ϕ_n is K_n'-invariant, and also satisfies the same properties as ϕ. Then

$$0 = \int_{G(F)Z(\mathbf{A})\backslash G(\mathbf{A})} \phi(g)\overline{\psi(g)}dg = \int_{P(F)N(\mathbf{A})Z(A)\backslash G(\mathbf{A})} \phi^P(g)\overline{\psi(g)}.$$

We are finished as soon as we have recalled the following.

2.9. Let f be a measurable locally bounded function on $P(F)N(\mathbf{A})\backslash G(\mathbf{A})$ transforming by ξ. Then for any $h \in \mathscr{C}_0(P, \xi)$ the integral

$$\int_{N(\mathbf{A})P(F)Z(\mathbf{A})\backslash G(\mathbf{A})} f(g)\bar{h}(g)dg$$

exists. We shall say that the cuspidal component of f is zero if the above integral vanishes for all such h.

PROPOSITION (Langlands). *Suppose ϕ is a continuous function on $G(F)\backslash G(\mathbf{A})$ such that the cuspidal component of ϕ^P is zero for every P, including $P = G$. Then $\phi \equiv 0$.*

The proof of this is by now well known, see for example [**1**] (Proposition 3.2 (actually, to use Springer's proof, let

$$\psi \in \mathscr{L}(\{M\}, M \cap K', \xi)$$

and define a function ψ' on $N(\mathbf{A})P(F)\backslash G(\mathbf{A})$ by demanding ψ' to be K'-invariant on the right, $N'(\mathbf{A})$ invariant on the left, equal to $\psi\delta_P{}^{+2}$ on $M(\mathbf{A})$ and zero outside $N(\mathbf{A})M(\mathbf{A})K'$. Then

$$0 = \int_K \int_{M(F)Z(\mathbf{A})\backslash M(\mathbf{A})} \phi^P(mk)\psi(mk) = \int_{M(F)Z(\mathbf{A})\backslash M(\mathbf{A})} \phi^P(m)\psi(m)$$

and then one can argue as Springer does).

3. The constant term.

3.1. Recall that for $\Phi \in \Gamma(\mathscr{C}_0(P, K'))$ we have constructed the locally integrable function $E(g, \Phi_\zeta)$, $\zeta \in iL_{Z_M}\backslash X_M(\mathbf{C})$, $\operatorname{Re}\zeta - \delta_P \in C_P$.

Let $P' = N'M'$ be another parabolic. We set

$$E^{P'}(g, \Phi_\zeta) = \int_{N'(F)\backslash N'(\mathbf{A})} E(n'g, \Phi_\zeta)dn'.$$

Since $N'(F)\backslash N'(\mathbf{A})$ is compact and $E(g, \Phi_\zeta)$ continuous, this integral does exist. This expression is usually referred to as "the constant term".

LEMMA. *Suppose P, P' have the same rank. Then*

$$E^{P'}(g, \Phi_\zeta) = \begin{cases} 0 & \text{if } P' \notin \{P\} \\ \displaystyle\sum_{W(M,M')} \int_{wN(\mathbf{A})w^{-1} \cap N'(\mathbf{A})\backslash N'(\mathbf{A})} \Phi(w^{-1}n'g)dn', & \text{if not.} \end{cases}$$

Proof. Use

$$\sum_{P(F)\backslash G(F)} = \sum_{P(F)\backslash G(F)/N'(F)}$$

times

$$\sum_{N'(F) \cap \gamma^{-1}P(F)\gamma\backslash N'(F)}$$

to see that

$$E^{P'}(g, \Phi_\zeta) = \sum_{P(F)\backslash G(F)/N'(F)} \int_{\gamma^{-1}P(F)\gamma \cap N'(F)\backslash N'(\mathbf{A})} \Phi_\zeta(\gamma n'g)dn' \tag{3.1.1}$$

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(Note

$$\int_{\gamma^{-1}P(F)\gamma \cap N'(F)\backslash N'(\mathbf{A})} |\Phi_\zeta|$$
$$= \int_{N'(F)\backslash N'(\mathbf{A})} \sum_{\gamma^{-1}P(F)\gamma \cap N'(F)\backslash N'(F)} |\Phi_\zeta| < \infty .)$$

(3.1.2) $$= \sum \int_{\gamma^{-1}P(\mathbf{A})\gamma \cap N'(\mathbf{A})\backslash N'(\mathbf{A})} \times \int_{\gamma^{-1}P(F)\gamma \cap N'(F)\backslash \gamma^{-1}P(\mathbf{A})\gamma \cap N'(\mathbf{A})} \Phi_\zeta(\gamma u n' g) du dn'.$$

The last integral may be written as

$$\int_{P(F) \cap \gamma N'(F)\gamma^{-1}\backslash P(\mathbf{A}) \cap \gamma N'(\mathbf{A})\gamma^{-1}} \Phi_\zeta(u\gamma n' g) du.$$

Here one is integrating over the unipotent radical of the parabolic subgroup $M \cap \gamma P'\gamma^{-1}$ of M. On the other hand $m \mapsto \Phi(mg)$ is cuspidal as a function of M^0. We conclude that

$$\int_{P(F) \cap \gamma N'(F)\gamma^{-1}\backslash P(\mathbf{A}) \cap \gamma N'(\mathbf{A})\gamma^{-1}} \Phi_\zeta(u\gamma n' g) du \equiv 0$$

unless the unipotent radical is trivial, i.e., $P \cap \gamma N'\gamma^{-1} \subseteq N$. In this case 1.3.7 implies that P and P' are associate by γ^{-1}, i.e., $\gamma^{-1} \in W(M, M')$, and we may replace the sum over the double cosets by a sum over $W(M, M')$.

Suppose then that P, P' are associate. Then 3.1.2 becomes

$$\Sigma \int_{\gamma^{-1}P(\mathbf{A})\gamma \cap N'(\mathbf{A})\backslash N'(\mathbf{A})} \Phi_\zeta(\gamma n' g) dn'$$

and 3.1.1 becomes

$$\sum_{W(M,M')} \int_{\gamma P(\mathbf{A})\gamma^{-1} \cap N'(\mathbf{A})\backslash N'(\mathbf{A})} \Phi_\zeta(\gamma^{-1} n' g) dn'$$
$$= \sum_{W(M,M')} \int_{\gamma N(\mathbf{A})\gamma^{-1} \cap N'(\mathbf{A})\backslash N'(\mathbf{A})} \Phi_\zeta(\gamma^{-1} n' g) dn'$$

as claimed.

3.2. Set

$$N(w, \zeta)\Phi_\zeta(g) = \int_{wN(\mathbf{A})w^{-1} \cap N'(\mathbf{A})\backslash N'(\mathbf{A})} \Phi_\zeta(w^{-1} n' g) dn'$$

for $w \in W(M, M')$.

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LEMMA. *The map* $\Phi_\zeta \mapsto N(w, \zeta)\Phi_\zeta$ *is a linear transformation*

$$N(w, \zeta) : \mathscr{C}_0(P, K', \omega \cdot \zeta\delta_P) \to \mathscr{C}_0(P', K', {}^w\omega \cdot {}^w\zeta\delta_{P'}).$$

Proof. One can suppose that P, P' are associate. Let

$$P = P_{\Theta_1} = P_1, P' = P_{\Theta_2} = P_2.$$

If $N(w, \zeta) \not\equiv 0$, then

$$w^{-1}N_2 w \cap P_1 \subset N_1,$$

and

$$N(w, \zeta)\Phi_\zeta(g) = \int_{wN_1(\mathbf{A})w^{-1} \cap N_2(\mathbf{A})\backslash N_2(\mathbf{A})} \Phi_\zeta(w^{-1}n_2 g)dn_2.$$

Thus

$$N(w, \zeta)\Phi_\zeta(z_2 g) = \int \Phi_\zeta(w^{-1}n_2 z_2 g)dn_2, \quad z_2 \in Z_{M_2}$$

$$= \int \Phi_\zeta(w^{-1}z_2 w \cdot w^{-1}(z_2^{-1}n_2 z_2)g)dn_2$$

and one only has to see what happens when the change of variable

$$n_2 \to z_2^{-1}n_2 z_2$$

is effected. The discussion in 1.2.5 implies that the integral may be written

$${}^w\omega(z_2)q^{\langle H_{M_2}(z_2), -w\zeta+\delta_{P_2}\rangle}\int_{wN_1(\mathbf{A})w^{-1} \cap N_2(\mathbf{A})\backslash N_2(\mathbf{A})} \Phi_\zeta(w^{-1}n_2 g)dn_2.$$

This implies that $N(w, \zeta)\Phi_\zeta$ transforms in the desired way, and the remaining conditions in the definition of $\mathscr{C}_0(P', K', {}^w\omega \cdot {}^w\zeta\delta_{P'})$ are easy to check.

3.3. Let $\varphi \in \mathscr{C}_0(P, K', \xi)$, $\psi \in \mathscr{C}_0(P', K', \xi)$. Taking inner products in $\mathscr{L}(\xi)$, we have

$$(\phi^\wedge, \psi^\wedge) = \int_{G(F)Z(\mathbf{A})\backslash G(\mathbf{A})} \phi^\wedge(g)\overline{\psi^\wedge(g)}dg$$

$$= \int_{P'(F)Z(\mathbf{A})\backslash G(\mathbf{A})} \phi^\wedge(g)\overline{\psi(g)}dg$$

$$= \int_K \int_{M'(F)Z(\mathbf{A})\backslash M(\mathbf{A})} \phi^{P'}(mk)\psi(mk)\delta_{P'}^{-2}(m')dmdk.$$

Since

$$\phi^\wedge(g) = \int_{\mathrm{Re}\,\zeta=\zeta_0} E(g, \Phi_\zeta)d\zeta = \bigoplus_{\{\omega\}} \int_{\mathrm{Re}\zeta=\zeta_0} E(g, \Phi_{\omega\cdot\zeta})d\zeta$$

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for $\zeta_0 > \delta_P$, this expression can be replaced by

$$\int_{\mathrm{Re}\,\zeta=\zeta_0}\int_K\int_{M'(F)Z(\mathbf{A})\backslash M'(\mathbf{A})} E^{P'}(mk,\,\Phi_\zeta)\overline{\psi(mk)}\delta_{P'}{}^{-2}(m)dmdkd\zeta.$$

This expression is zero unless P and P' are associate; in this case from 3.1, 3.2, one obtains

$$\int_{\mathrm{Re}\,\zeta=\zeta_0}\int_K\int_{M'(F)Z(\mathbf{A})\backslash M'(\mathbf{A})} \sum N(w,\,\zeta)\Phi_\zeta(mk)\psi(mk)\delta_{P'}{}^{-2}(m)$$

$$= \sum_{W(M,M')}\bigoplus_{\{\omega\}}\int_{\mathrm{Re}\,\zeta=\zeta_0}\int_K\int N(w,\,\zeta)\Phi_\zeta(\dot{m}k)$$

$$\times\int \overline{\psi(z\dot{m}k)\overline{{}^w\omega\cdot{}^w\zeta\delta_{P'}}^{-1}(z)\delta^{-2}(\dot{m})}$$

(first integral over $M'(F)Z_{M'}(\mathbf{A})\backslash M'(\mathbf{A})$, second over $Z(\mathbf{A})Z_{M'}(F)\backslash Z_{M'}(\mathbf{A})$)

$$= \sum\int_{\mathrm{Re}\,\zeta=\zeta_0}\int_K\int N(w,\,\zeta)\Phi_\zeta(\dot{m}k)\overline{\Psi_{\overline{(w\zeta)}^{-1}}(\dot{m}k)}\delta_{P'}{}^{-2}(\dot{m})d\dot{m}dkd\zeta$$

since we may suppose that $W(M,\,M')$ preserves the set $\{\omega\}$.

Set $\langle N(w,\,\zeta)\Phi_\zeta,\,\Psi_{\overline{(w\zeta)}^{-1}}\rangle$ equal to

$$(3.3.1)\quad \int_K\int_{M'(F)Z_{M'}(\mathbf{A})\backslash M'(\mathbf{A})} N(w,\,\zeta)\Phi_\zeta(mk)\overline{\Psi_{(w\zeta)^{-1}}(\dot{m}k)}\delta_{P'}{}^{-2}(\dot{m})d\dot{m}dk.$$

Then one can condense the formula for $(\varphi^\wedge,\,\psi^\wedge)$ into

$$(3.3.2)\quad \sum_{W(M,M')}\int_{\mathrm{Re}\,\zeta=\zeta_0}\langle N(w,\,\zeta)\Phi_\zeta,\,\Psi_{(w\zeta)^{-1}}\rangle d\zeta.$$

3.4. Write $\mathscr{L}(\{P\},\,\xi)$ for the closure in $\mathscr{L}(\xi)$ of $\{\varphi^\wedge|\varphi\in\mathscr{C}_0(\{P\},\,\xi)\}$ (cf. 1.4.). Combining 3.1–3.3 and 2.8, we have:

PROPOSITION. *The spaces $\mathscr{L}(\{P\},\,\xi)$ are orthogonal to each other for different $\{P\}$, and*

$$\mathscr{L}(\xi) = \bigoplus_{\{P\}}\mathscr{L}(\{P\},\,\xi).$$

If $\varphi\in\mathscr{C}_0(P,\,K',\,\xi)$, $\psi\in\mathscr{C}_0(P',\,K',\,\xi)$ with P associate to P' the scalar product in $\mathscr{L}(\xi)$ of $\varphi^\wedge$, with $\psi^\wedge$ is given by 3.3.2.

3.5. If we regard the expression (3.3.1) for a moment we see that it comes from a sesquilinear pairing

$$\langle\ ,\ \rangle : \mathscr{C}_0(P',\,K',\,\zeta\delta_{P'})\times\mathscr{C}_0(P',\,K',\,\overline{\zeta^{-1}}\delta_{P'})\to\mathbf{C}$$

$$(3.3.3)\quad \langle\Phi_\zeta,\Psi_{\overline{\zeta^{-1}}}\rangle = \int_K\int_{M'(F)ZM(\mathbf{A})\backslash M'(\mathbf{A})}\Phi_\zeta(\dot{m}k)\Psi_{\zeta^{-1}}(\dot{m}k)\delta_{P'}{}^{-2}(\dot{m})d\dot{m}dk.$$

This can of course be interpreted in another way, which is also useful. Namely (3.3.3) can be written as

$$\int_{K'}\int_{M'(F)Z_M(\mathbf{A})\backslash M'(\mathbf{A})}\Phi'(\dot{m}k)\overline{\Psi(\dot{m}k)}\delta_{P'}{}^{-2}(m)dmdk$$

where

$$\Phi'(g) = q^{-\langle \bar{H}_M(g),\zeta\rangle}\Phi_\zeta(g) \in \bigoplus_{\{\omega\}} \mathscr{C}_0(P', K', \omega\cdot\delta_P)$$

and similarly for $\Psi(g)$.

3.6. Next, we shall interpret $N(w, \zeta)$ in terms of vector bundles, and the trivialization with fibre $\mathscr{C}_0(P', K', \omega\cdot\delta_P)$.

First, given an element $w \in W(M, M')$, there is an induced map

$$w : D_M(\xi) \to D_{M'}(\xi)$$
$$(\omega\cdot\zeta) \mapsto ({}^w\omega\cdot{}^w\zeta).$$

If we take the vector bundle $\mathscr{C}_0(P', K')$ over $D_{M'}(\xi)$ we obtain a bundle $w^*\mathscr{C}_0(P', K')$ over $D_M(\xi)$ with fibre at $\omega\cdot\zeta$ the space $\mathscr{C}_0(P', K', ({}^w\omega\cdot{}^w\zeta\delta_{P'}))$. Thus one may view the $N(w, \zeta)$ as giving a holomorphic section $N(w)$ for the bundle

$$\mathrm{Hom}\ (\mathscr{C}_0(P, K'), w^*\mathscr{C}_0(P', K'))$$

at least when one restricts to the open subset of $D_M(\xi)$ defined by $\operatorname{Re}\zeta > \delta_P$.

We shall eventually prove that $N(w)$ extends to a meromorphic global section over $D_M(\xi)$.

Secondly, let us see what this means if we trivialize at the point $\omega \in \{\omega\}$.

There are maps

$$\begin{array}{ccc}
\mathscr{C}_0(P, K', \omega\cdot\zeta\delta_P) & \xrightarrow{N(w,\,\omega\cdot\zeta)} & \mathscr{C}_0(P', K', {}^w\omega\zeta\delta_{P'}) \\
\uparrow T_\zeta & & \downarrow T_{{}^w\zeta^{-1}} \\
\mathscr{C}_0(P, K', \omega\cdot\delta_P) & & \mathscr{C}_0(P', K', {}^w\omega\cdot\delta_P)
\end{array}$$

which induces a linear map

$$M(w, \zeta) : \oplus_{\{\omega\}}\mathscr{C}_0(P, K', \omega\delta_P) \to \oplus_{\{\omega\}}\mathscr{C}_0(P', K', {}^w\omega\delta_{P'}).$$

Since there is a non degenerate pairing

$$\mathscr{C}_0(P', K', \omega\delta_P) \times \mathscr{C}_0(P', K', {}^w\omega\delta_{P'}) \to \mathbf{C}$$

one may certainly speak of the adjoint $M^*(w, \zeta)$ of $M(w, \zeta)$.

3.7. Define $X_M(\mathbf{R})^+$ by

$$X_M(\mathbf{R})^+ = C_P + \delta_P$$

where C_P is the Weyl chamber associated to the standard parabolic P, and δ_P is the modulus character, as usual.

LEMMA. *The adjoint of* $M(w, \zeta)$ *is* $M(w^{-1}, ({}^w\bar{\zeta})^{-1})$ *both holomorphic on*

$$\{\zeta | \operatorname{Re} \zeta \in \textit{convex hull}\, \{X_M(\mathbf{R})^+ \cup (w)^{-1}X_{M'}(\mathbf{R})^+\}\}.$$

Proof. Given $\varphi \in \mathscr{C}_0(P, K', \xi)$ define a function φ_w by

$$\varphi_w(g) = \sum_{wP(F)w^{-1} \cap P'(F) \backslash P'(F)} \varphi(w^{-1}\gamma g).$$

Since $\varphi^\wedge(g)$ exists, and since

$$G(F) = \bigcup_{W(M,M')} P(F)wP'(F)$$

one concludes that $\varphi_w(g)$ exists, invariant by $P'(F)$. A straightforward computation using the by now familiar double coset decomposition implies that

$$\int_{P'(F)Z(\mathbf{A})\backslash G(\mathbf{A})} \psi(g)\overline{\varphi_w(g)}dg = \int_{P(F)Z(\mathbf{A})\backslash G(\mathbf{A})} \psi_{w^{-1}}(g)\overline{\varphi(g)}dg.$$

Each of these integrals is readily computed by means of familiar manipulations. The left one is just

$$\int_{\operatorname{Re} \zeta=\zeta_0} \langle\Psi(-w\bar{\zeta}), M(w, \zeta)\Phi(\zeta)\rangle d\zeta, \quad \zeta_0 \in C_P + \delta_P$$

which implies that the right side is given by

$$\int_{\operatorname{Re} \chi=\chi_0} \langle\Psi(\chi), M(w^{-1}, \chi)^*\Phi(-w^{-1}\bar{\chi})\rangle, \quad \chi_0 \in C_{P'} + \delta_{P'}.$$

Set $\chi' = -w^{-1}\bar{\chi}$, then the second integral becomes

$$\int_{\operatorname{Re} \chi'=\chi'_0} \langle\Psi(-w\bar{\chi}'), M(w^{-1}, -w\bar{\chi}')^*\Phi(\chi')\rangle d\chi',$$

$$\bar{\chi}_0' \in -w^{-1}(C_{P'} + \delta_{P'}).$$

We shall refer to the two integrals as (3.7.1), (3.7.2) respectively.

Now let

$$G_1(\zeta) = \langle\Psi(-w\bar{\zeta}), M(w, \zeta)\Phi(\zeta)\rangle$$
$$G_2(\chi) = \langle\Psi(-w\bar{\chi}), M(w^{-1}, -w\bar{\chi})^*\Phi(\chi)\rangle.$$

Let v be any holomorphic function

$$v : iL_{Z_M}\backslash X_M(\mathbf{C}) \to \bigoplus_{\{\omega\}} \mathscr{C}_0(P, K', \omega)$$

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which is a Laurent series (e.g. a finite sum $\sum_i f_i q^{\zeta(m_i)}$, $m_i \in L_{Z_M}{}^*$). Then v gives rise to a function on $iL_{Z_M}\backslash X_M(\mathbf{C})$, and it is evident by standard Fourier analysis that the Fourier transform of $\Phi_\zeta \cdot v(\zeta)$ is an element of $\mathscr{C}_0(P, K', \xi)$. Applying this remark to (3.7.1) and (3.7.2) one obtains

$$\int v(-w\bar{\zeta}) \cdot G_1(\zeta) = \int v(-w\bar{\chi}) \cdot G_2(\chi).$$

From this, and the Plancherel formula for the group $L_{Z_M}{}^*$, one deduces the existence of a function h on $L_{Z_M}{}^*$ such that

$$h(z) = \int G_1(\zeta_1 + i\mu) q^{\langle z, \zeta_1 + i\mu\rangle} d\mu = \int G_2(\zeta_2 + i\mu) q^{\langle z, \zeta_2 + \mu\rangle} d\mu$$

with $\zeta_1 \in C_P + \delta_P$, $\zeta_2 \in -w^{-1}(C_{P'} + \delta_{P'})$, and h is independent of $\zeta_i (i = 1, 2)$. The conclusion follows easily from this, since the Fourier transform of h is holomorphic on the desired convex hull.

3.8. Our next task is to estimate $\|M(w, \zeta)\|$. In view of the definition of the pairing (3.3.2) it is enough to estimate $|M(w, \zeta)\Phi(mk)|$ for $m \in M^0$, $k \in K$. One can certainly choose a compact set C in M^0K such that each $\Phi_\zeta \in \mathscr{C}_0(P, K', \zeta\delta_P)$ has support in C mod $Z_M(\mathbf{A})M(F)$, and then C/K' is a finite set. Thus it is enough to estimate $|M(w, \zeta)\Phi(g_i)|$, for a finite set $\{g_i\}$. Let ω be a compact set such that $N'(\mathbf{A}) = N'(F)\omega$, and let Ω be compact so that $\omega g_i \subseteq \Omega$ each g_i. Then

$$|M(w, \zeta)\Phi(g_i)| \leqq |{}^w\zeta\delta_{P'}(g_i)|^{-1} \int_{N'(F)\backslash N'(\mathbf{A})} \sum_{P(F)\backslash G(F)} |\Phi_\zeta(\gamma n' g_i)|$$
$$\leqq c \sup_{n' \in \omega} \left\{ \sum_{P(F)\backslash G(F)} |\Phi_\zeta(\gamma n' g_i)| \right\} \leqq c \sup_{g \in \Omega} \sum |\Phi_\zeta(\gamma g)|.$$

Since $|\Phi|$ is bounded on $G(\mathbf{A}) = N(F)\omega' M^0 Z_M(\mathbf{A})\Sigma K'$ (where Σ is a finite set and ω' is compact), the last inequality becomes

$$\leqq \|\Phi\|_\infty \sum q^{\langle \bar{H}_M(\gamma g), \zeta_0\delta_P\rangle}, \ \operatorname{Re} \zeta = \zeta_0, \quad \text{and}$$
$$\|\Phi\|_\infty = \sup_g |T_{\delta_P}{}^{-1}\Phi(g)|.$$

Now, one can apply Lemma 2.2 to the function

$$g \mapsto q^{\langle \bar{H}_M(\gamma g), \zeta_0 + \delta_P\rangle}$$

so, from (2.2.1)

$$|M(w, \zeta)\Phi(g_i)| \leqq \|\Phi\|_\infty \cdot c' \cdot \prod_j \frac{q^{\langle \alpha_j^*, \zeta_0 - \delta_P\rangle N r_j}}{1 - q^{\langle \alpha_j^*, \zeta_0 - \delta_P\rangle r_j}}$$

cf. (2.2.1).

Let Φ run through a basis of the finite dimensional space $\mathscr{C}_0(P, K', \omega \cdot \delta_P)$, then the following result has been proved.

LEMMA. *There are constants $c > 0$, and N, such that*

$$\|M(w, \zeta)\| \leqq c \prod_j \frac{q^{\langle \alpha_j^*, \zeta_0 - \delta_P \rangle N r_j}}{1 - q^{\langle \alpha_j^*, \zeta_0 - \delta_P \rangle r_j}}, \quad \operatorname{Re} \zeta = \zeta_0.$$

3.9. It will be convenient to make use of some remarks concerning Fourier transforms and approximations. As these arguments (and simple variations) will occur many times in later sections, we shall give the technique in some detail, and forego repeating it.

Let $r > 1$, and set

$$X_M(\mathbf{R})_r = \text{interior}\,\{\text{convex linear hull}\,\{w \cdot \delta_P \cdot r | w \in W(M, M)\}\}$$
$$X_M(\mathbf{C})_r = \{\zeta \in X_M(\mathbf{C}) | \operatorname{Re} \zeta \in X_M(\mathbf{R})_r\}.$$

Then $X_M(\mathbf{C})_r$ gives rise to an open complex submanifold $D_M(\xi)_r$ of $D_M(\xi)$, and $X_M(\mathbf{C})_r \cap (\delta_P + C_P)$ is a non empty open set.

Let $\varphi \in \mathscr{C}_0(P, K', \xi)$, then the inner product formula and 3.8 tell us that if ζ_0 belongs to a compact subset of $iL_{Z_M} \backslash X_M(\mathbf{C})$

$$\|\varphi^\wedge\|^2 = \sum_{W(M,M)} \int_{\operatorname{Re} \zeta = \zeta_0} \langle \Phi(-w\bar{\zeta}), M(w, \zeta)\Phi(\zeta) \rangle d\zeta$$
$$\leqq c_0 \sum \left(\int_{\operatorname{Re} \zeta = \zeta_0} |\Phi(-w\bar{\zeta})|^2 \right)^{1/2} \left(\int_{\operatorname{Re} \zeta = \zeta_0} |\Phi(\zeta)|^2 \right)^{1/2}.$$

Here we are writing $\Phi(\zeta) = \bigoplus_{\{\omega\}} T_\zeta^{-1} \Phi_{\omega \cdot \zeta \delta_P}$ where $\Phi_{\omega \cdot \delta_P}$ is the Fourier transform of φ, so that $\Phi(\zeta)$ is a function

$$iL_{Z_M} \backslash X_M(\mathbf{C}) \rightarrow \bigoplus_{\{\omega\}} \mathscr{C}_0(P, K', \omega \cdot \delta_P).$$

Suppose on the other hand, that one is given a holomorphic function

$$\Psi : iL_{Z_M} \backslash X_M(\mathbf{C})_r \rightarrow \bigoplus_{\{\omega\}} \mathscr{C}_0(P, K', \omega \cdot \delta_P).$$

Set $\Psi_\zeta = T_\zeta \Psi(\zeta)$. The integral $\int_{\operatorname{Re} \zeta = \zeta_0} \Psi_\zeta(g) d\zeta$ is independent of $\zeta_0 X_M(\mathbf{R})_r$, by analyticity; call the resulting function $\psi(g)$. From the Plancherel theorem,

$$\int_{Z_M(F) Z(\mathbf{A}) \backslash Z_M(\mathbf{A})} |\psi(zg)|^2 e(zg, -\zeta_0 - 2\zeta_0 - 2\delta_P) dz$$
$$= \int_{\operatorname{Re} \zeta = \zeta_0} |\Psi_\zeta(g)|^2 e(g, -2\zeta_0 - 2\delta_P)$$

so that formally, since both sides are functions on $M(F) Z_M(\mathbf{A}) \backslash M(\mathbf{A})$

$$\int_{M(F) Z_M(\mathbf{A}) \backslash M(\mathbf{A})} \int_{Z_M(F) Z(\mathbf{A}) \backslash Z_M(\mathbf{A})} = \int_{M(F) Z_M(\mathbf{A}) \backslash M(\mathbf{A})} \int_{\operatorname{Re} \zeta = \zeta_0}.$$

If one interchanges the last two integrals, one obtains a convergent integral; this follows from the definition of $\Psi(g, \zeta)$, and from the fact that $\int_{\zeta = \zeta_0}$ is over a finite number of circles for which the new integrand is a continuous function. Fubini's theorem implies that the first two integrals exist. Thus

$$\int_{Z(\mathbf{A})N(\mathbf{A})P(F)\backslash G(\mathbf{A})} |\psi(g)|^2 e(g, -2\zeta_0)dg < \infty$$

and the argument implies that the expression below is finite

$$(3.9.1) \quad \sum_{W(M,M)} \int_{Z(\mathbf{A})N(\mathbf{A})P(F)\backslash G(\mathbf{A})} |\psi(g)|^2 \{e(g, -2w\zeta_0) + e(g, 2w\zeta_0)\}.$$

Let $\mathscr{C}_{\zeta_0}(P, K', \xi)$ be the space of functions on $N(\mathbf{A})P(F)\backslash G(\mathbf{A})$ satisfying (i), (iii) of 1.6 and (3.9.1), this last making it into a Hilbert space. We write $\|\varphi\|_{\zeta_0}$ for the norm on this space. If $\varphi \in \mathscr{C}_0(P, K', \xi)$, then the remarks concerning the inner product formula imply that

$$\|\varphi^\wedge\|^2 \leqq C\|\varphi\|_{\zeta_0}{}^2$$

and this means that the map $\Theta : \varphi \mapsto \varphi^\wedge$ can be extended to the space $\mathscr{C}_0(P, K', \xi)$. Thus if $\mathscr{H}(P, K', r)$ denotes the space of holomorphic functions

$$\Psi(\zeta) : \mathrm{i}L_{Z_M}\backslash X_M(\mathbf{C})_r \rightarrow \bigoplus_{\{\omega\}} \mathscr{C}_0(P, K', \omega \cdot \delta_P)$$

then we have shown that the map Θ gives rise to a map

$$\mathscr{H}(P, K', r) \rightarrow \mathscr{L}(\{P\}, \xi)$$

which we also denote by Θ. The inner product formula still holds for this class of functions, as follows by an easy continuity argument.

In the sequel, we shall refer to arguments of the above type as *approximation arguments*.

4. The principle of the constant term.

4.1. In this section we study more closely the relationship between the various constant terms of a function on $G(F)\backslash G(\mathbf{A})$ and the function itself. The main results are based on a variant of an observation of Harder's (cf. lemma 1.6.7 of [**10**]), and are somewhat simpler to prove than in the number field case (cf. § 5 of [**13**], and Chapter III of [**11**]). First some notation.

Let f be a continuous function on $G(F)\backslash G(\mathbf{A})$, transforming by ξ. Given $P = NM$ a parabolic, we write $f^P \sim 0$ if the cuspidal component of f^P is zero (2.9).

We record the next lemma for reference; it follows from an argument similar to that of Lemma 3.1.

LEMMA. *Let $\Phi_\zeta \in \mathscr{C}_0(P, \omega \cdot \zeta\delta_P)$, and $E(g, \Phi_\zeta)$ the associated Eisenstein series (under the assumptions of* 2.6). *Then $E^{P'}(g, \Phi) \sim 0$, unless $P' \in \{P\}$.*

4.2. Fix constants c_1, c_2 as in 1.4.1, 1.4.2; we shall in fact suppose that c_2 has also been chosen as in 1.5.5 so that the results of 1.5 are in force (especially 1.5.9). Write D for the image of $G(F)P_0\,(c_1, c_2)K'$ in $G(F)\backslash G(\mathbf{A})$; if $m \in P_0(c_1, c_2)$, let δ_m be the characteristic function of the double coset $G(F)mK'$ (as usual, K' is an open compact subgroup in $G(\mathbf{A})$). Thus δ_m has compact support mod $Z(\mathbf{A})$ in $G(F)\backslash G(\mathbf{A})$, and we define a new function Δ_m, by

$$\Delta_m(g) = \int_{Z(F)\backslash Z(\mathbf{A})} \delta_m(zg)\xi^{-1}(z)dz.$$

In particular Δ_m has compact support mod $Z(\mathbf{A})$ in $G(F)\backslash G(\mathbf{A})$, and $\Delta_m \in \mathscr{L}(\xi)$.

LEMMA. ([**10**], Lemma 1.6.7). *The function Δ_m can be written in the form*

$$\Delta_m = \psi + \Theta + R$$

where ψ, Θ, R all have compact support mod $Z(\mathbf{A})G(F)$, *such that $\psi \in \mathscr{L}(\{G\}, \xi)$, Θ is a finite sum of Θ-functions, and R has support outside of D.*

Proof. Let C be a subset of $G(\mathbf{A})$ compact modulo $G(F)Z(A)$, so that supp Δ_m is contained in C modulo $G(F)$; we can suppose that $C \supseteq K'$. In particular, the image of C in $G(F)\backslash G(\mathbf{A})/K'$ is a finite set modulo $Z(\mathbf{A})$. Let V_C be the space of functions on $G(F)\backslash G(\mathbf{A})/K'$ which have support in C, and which transform according to ξ; then V_C is a finite dimensional, hence closed, subspace of $\mathscr{L}(\xi)$. From this, and 2.8 we can argue as Harder does to obtain the result.

4.3. Now let $P_1, \ldots, P_t$ be the standard parabolics of rank r, where r is fixed, and $r \geqq 1$.

THEOREM. *Let $\{\phi_n\}$ be a sequence of functions on $G(F)\backslash G(\mathbf{A})$ satisfying the following conditions:*

(i) *ϕ_n transforms by ξ, and is right invariant by K'.*

(ii) *If P is a standard parabolic (possibly G) not of rank r then the cuspidal component of ${\phi_n}^P$ is zero.*

(iii) *If $P = P_i$ is one of $P_1, \ldots, P_t$, then ${\phi_n}^P \rightarrow \Phi_i$ in the finite dimensional inner product space $\mathscr{C}_0\,(P, K', \omega\delta_P)$ (where ω is unitary).*

Then $\phi_n \rightarrow \phi$ uniformly on compact subsets of $G(F)\backslash G(\mathbf{A})$ where ϕ is right invariant by K' and transforms by ξ. Moreover $\phi^{P_i} = \Phi_i$ for $1 \leqq i \leqq t$, and the cuspidal component of ϕ^P is zero otherwise.

Proof. First, observe that convergence in the mean implies pointwise (uniform) convergence for sequences in $\mathscr{C}_0\,(P, K', \omega\delta_P)$. Indeed, one can choose a subset $C \subseteq M(\mathbf{A})$ which is compact modulo $Z_M(\mathbf{A})M(F)$ such that all elements of $\mathscr{C}_0\,(P, K', \omega\delta_P)$, viewed as functions on $M(F)\backslash M(\mathbf{A})$, have support in C. Thus $Z_M(\mathbf{A})M(F)\backslash C/K'$ is a finite set (this is why the vector space in question is finite dimensional) and the result follows directly from this.

(i) Suppose that $m \in P_0(c_1)$ does not belong to $P_0(c_1, c_2)$. This means that there is a simple root α such that $|\alpha(m)| \geqq q^{c_2}$. We then know from 1.5 that $\phi_n(m) = \phi_n{}^P(m)$ if P is the maximal parabolic corresponding to α, and the remark above implies that

$$\phi_n(m) \to \begin{cases} \Phi_i(m) & \text{if } P = P_i, \quad i \in \{1, \dots, t\} \\ \phi^P(m) & \text{defined by induction, if not.} \end{cases}$$

If both $|\alpha(m)| \geqq q^{c_2}$, $|\beta(m)| \geqq q^{c_2}$ for $\alpha \neq \beta$, then uniqueness of limits implies that it is irrelevant which parabolic one chooses. If $g = \gamma mk$, $\gamma \in G(F)$, m as above, $k \in K'$, then $\phi_n(\gamma mk) = \phi_n(m)$ and we can argue as before.

(ii) Suppose that $m_0 \in P_0(c_1, c_2)$, then by Lemma 4.2,

$$\text{volume } (G(F)\backslash G(F)m_0K')\phi_n(m_0) = \int \phi_n(g)\,\{\bar{\Psi}(g) + \bar{\Theta}(g) + \bar{R}_n(g)\}dg.$$

By assumption, the first integral is zero. The second can be rewritten as a sum of integrals of the form

$$\int \phi_n{}^P(g)\psi(g), \quad \Theta = \psi^\wedge;$$

in particular we can apply the remark at the beginning of the proof to see that such a sequence of integrals converge (note that ϕ has compact support modulo M^0, when viewed as a function on $M(\mathbf{A})$). As for (iii), we know that $R(g)$ has its support disjoint from $G(F)P_0(c_1, c_2)K$, and we can apply (i) and uniform convergence (the ϕ_n's are right invariant by K') to see that the third integral also converges. We can extend the convergence to points of the form $g = \gamma mk$ with $\gamma \in G(F)$, $m \in P_0(c_1, c_2)$, and $k \in K'$, in the same way as we did in (i).

To summarize, we have shown that the ϕ_n converge uniformly to a function ϕ on $G(F)P_0(c_1)K'$. To go from this set to $G(F)P_0(c_1)K$ is easy: one simply repeats the arguments above for sets of the form $G(F)P_0(c_1)k_iK'$, where $\{k_i\}$ is a (finite) set of coset representatives for K/K'. The final assertions in the statement of the theorem are straightforward limit arguments, using the definitions, and the fact that $N(F)\backslash N(\mathbf{A})$ is compact.

4.4. We shall make some refinements of Theorem 4.3, necessary for the applications.

COROLLARY. *Suppose that* $\{\phi_n\}$ *is as in the statement of Theorem* 4.3 *subject to the following modifications:*

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(ii)′ *If* $P = P_i$ *is one of* $P_1, \ldots, P_t$, *then*

$$\phi_n^P \in \mathscr{C}_0(P, K', \omega\zeta_n\delta_P) \quad \textit{and} \quad e(\cdot, -\zeta_n)\phi_n^P \to \Phi_i \in \mathscr{C}_0(P, K', \omega\zeta_0\delta_P)$$

where $\zeta_n \to \zeta_0$ *in* $D_M(\xi)$. *Then the conclusion of Theorem* 4.3 *holds, except that now* $\phi^P = e(\cdot, \zeta_0)\Phi_i$ *if* $P = P_i$ *is one of* $P_1, \ldots, P_t$.

The proof of this is clear, using Theorem 4.3, and the definition of $e(\cdot, \zeta)$ (1.6).

4.5. The next assertion is not so much a corollary, as an addendum, to Theorem 4.3.

Lemma. *Suppose in* 4.4 *that for each* P, *it is true that*

$$2(\delta_{P_0} - \delta_P - \operatorname{Re}\zeta_0) \in C_{P_0}{}^* \ (\textit{dual chamber}).$$

Then the function ϕ *in Theorem* 4.4 *is square integrable on* $Z(\mathbf{A})G(F)\backslash G(\mathbf{A})$, *i.e., it is an element of* $\mathscr{L}(\xi)$.

Proof. In view of what we know, we only have to check that ϕ^P is square integrable on a set of the form $L_0{}^*(c)$ where P is a maximal parabolic, $L_0{}^*$ is the lattice $L^*_{Z_0}$ and c is a suitable constant. This is left to the reader. We remark in passing that such a computation is essentially the same as the corresponding (known) one in the number field case: indeed one is computing a sum over a lattice which is truncated below, and this sum is comparable to the integral over the corresponding real vector space which is truncated below.

4.6. The following variant of 4.4 is the one that occurs for Eisenstein series.

Theorem. *We suppose* $\{\phi_n\}$ *is as in* 4.4, *with condition* (iii) *replaced by the following*

(iii) $\phi_n^P = \sum_{j=1}^{t} \Phi_{n,j}$

where $\Phi_{n,j} \in \mathscr{C}_0(P, K', \omega_j\zeta_{n,j}\delta_P)$. *If*

$$e(\cdot, -\zeta_{n,j})\Phi_{n,j} \to \Phi_j \textit{ in } \mathscr{C}_0(P, K', \omega_j\delta_P) \quad \textit{and} \quad \zeta_{n,j} \to \zeta_{0,j},$$

then $\phi_n \to \phi$ *uniformly on* $G(F)\backslash G(\mathbf{A})$.

We remark that there is, of course, a corresponding assertion to that of 4.3–4.4 for the constant terms of the function ϕ.

4.7. There is one piece of notation which will be useful now, and later. If $\Phi \in \mathscr{C}_0(P, K', \omega\delta_P)$ then we have seen how to construct a function $\Phi_\zeta \in \mathscr{C}_0(P, K', \omega\zeta\delta_P)$ (via $e(\cdot, \zeta)$) and thus $E(g, \Phi_\zeta)$ (if it makes sense).

We shall often write $E(g, \Phi_\zeta)$ in the form $E(g, \Phi, \zeta)$ to emphasize the function Φ.

4.8. The next result can be proved in several ways, but we refer the reader to [**10**], Theorem 1.6.6 for a proof that is in the spirit of this section. In what follows the unexplained notation is that of Sections 2–3 of this chapter.

THEOREM (i) *Suppose that the terms $M(w, \zeta)$, $w \in \bigcup_{P' \in \{P\}} W(M, M')$ associated to the Eisenstein series $E(g, \Phi, \zeta)$ can be analytically continued over a region D. Then $E(g, \Phi, \zeta)$ can itself be analytically continued over this region, and if $P' \in \{P\}$ then $E^{P'}(g, \Phi, \zeta)$ is given by* (3.3.2).

(ii) *Suppose that $M(w, \zeta)$ satisfies the functional equation*

$$M(w_1, \zeta)M(w_2, \zeta) = M(w_1w_2, \zeta)$$

over the region D'.

Then $E(g, \Phi, \zeta)$ satisfies the functional equation

$$E(g, M(w, \zeta)\Phi, {}^w\zeta) = E(g, \Phi, \zeta)$$

over the same region.

(iii) *Suppose finally that the operator valued function $M(w, \zeta)$ is rational* (*see* 3.5 *for a definition*). *Then $E(g, \Phi, \zeta)$ is also rational as a function of ζ.*

Of course in (iii) it is supposed that $M(w, \zeta)$ has been analytically continued over all of $D_M(\xi)$.

4.9. The final task of this section is to make some estimates on the partition constructed in 1.7. These estimates will be used in just one place (3.3.2), in an argument similar to one used by Langlands ([**13**], p. 131); perhaps the reader should skip what follows until it is necessary to refer to it.

Let $\{P\}$ be an associate class of parabolics of rank 1. If $z \in \mathbf{C}^*$, let $\mathcal{L}(z) \subseteq \mathcal{L}(\{P\}, K', \xi)$ be the space of functions ϕ such that

(a) the cuspidal component of ϕ^P is zero if $P \notin \{P\}$

(b) $\phi^{P_i}(g) = T_{(-\zeta^i(z))}\Psi^{(i)}(g)$,

for some $\Psi = (\Psi_1, \ldots, \Psi_s) \in \mathcal{C}$ if $P_i \in \{P\}$ (notation as in 3.3.2).

It will be shown in loc. cit. that $\mathcal{L}(z)$ is finite dimensional. We shall suppose that $0 \neq \phi_n \in \mathcal{L}(z_n)$ is a sequence with $z_n \in \mathbf{R}^+$ and $z_n \to 1$.

It is easily seen, by the kind of arguments employed in 1.7.2, that given $P_0(c_1, c_2)$, we can find t_1, t_2 such that

$$\begin{aligned} P_0(c_1, c_2) \subseteq \{g \in \mathfrak{S} | q^{t_1} \leqq |\alpha_i(p_i)| \leqq q^{t_2}, g = P_ik \in P^{\alpha_i}(\mathbf{A})K \\ \text{for each } \alpha_i \in \Delta\} \leqq \text{complement } (\cup \mathfrak{S}_{\alpha_i}(t_2)), \mathfrak{S}_\alpha(t) \end{aligned}$$

as in 1.7.2. We can always enlarge t_2 to ensure that the conditions of Lemma 7.4 are satisfied. Observe that the image of $P_0(c_1, c_2)K$ in $G(F)\backslash G(\mathbf{A})$ is compact modulo $Z(\mathbf{A})$.

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The spaces $\mathscr{L}(z_n)$ are orthogonal to each other for different z. We shall suppose therefore that $\{\phi_n\}$ is an orthonormal sequence. Let C denote the image of $P_0(c_1, c_2)K$ in $G(F)Z(\mathbf{A})\backslash G(\mathbf{A})$, then if F_i is the image of $\mathfrak{S}_{\alpha_i}(t_2)$

$$1 = \int_{\hat{F}} \phi_n\bar{\phi}_n + \sum_i \int_{F_i} \phi_n\bar{\phi}_n, \quad \tilde{F} = \text{complement } (\cup F_i)$$

$$\geqq \int_C |\phi_n|^2 + \sum_i \int_{F_i} \phi_n\bar{\phi}_n.$$

We can suppose that c_2 is chosen so that if $|\alpha_i(p_0)| \geqq q^{c_2}$ (for $g = p_0k$, $p \in P_0(c_1)$) then $\phi_n(g) = \phi_n^{P_i}(g)$. Thus

$$1 \geqq c_n + \sum_i \int_{F_i} |\phi_n^{P_i}|^2.$$

From this it follows that $c_n \leqq 1$. On the other hand if we write F_i' for the image of $\mathfrak{S}_{\alpha_i}(t_2)$ in $P(F)Z(\mathbf{A})\backslash G(\mathbf{A})$ then

$$\int_{F_i} |\phi_n^{P_i}(g)|^2 = \int_{F_i'} |\phi_n^{P_2}(g)|^2 \quad \text{(by (1.7.4))}.$$

This latter integral is at least

$$\frac{q^{-2s_nc_2}}{1 - q^{-2s_n}} a_n |\Psi_n^{(i)}|^2$$

where $z_n = q^{s_n}$ (this is calculated as in 2.2, for example). Here

$$a_n = \inf_{\mathfrak{l}_i} \{e(-2\zeta^{(i)}(z_n), H_M(\mathfrak{l}_i))\}$$

where $\mathfrak{l}_i$ runs through a set of coset representatives of $L_{Z_M}{}^*\backslash L_M{}^*$, so that $a_n \to 1$ as $s_n \to 0$. Summing, we find

$$1 \geqq c_n + \frac{q^{-2s_nc_2}}{1 - q^{-2s_n}} \cdot a_n \cdot |\Psi_n|^2.$$

This implies that $\Psi_n \to 0$, and hence, by 4.6 that $\phi_n \to 0$. Since this is true for arbitrary sequences ϕ_n, and since the spaces $\mathscr{L}(z_n)$ are finite dimensional, it follows that there can only be finitely many such $\mathscr{L}(z_n) \neq \{0\}$, with z_n real, $|z_n| > 1$ (we assumed that $z_n > 1$, but the same argument works if $|z_n| > 1$). We formulate this as a lemma.

LEMMA. *There are only finitely many of the spaces $\mathscr{L}(z) \neq \{0\}$ for $|z| > 1$ and real.*

3. Some Functional Equations.

1. Applications of some functional analysis.

1.1. Let r be a real number, $r > 1$. If $P = NM \in \{P\}$, write

$$X_M(\mathbf{R})_r = \text{interior } \{\text{convex linear hull } \{w \cdot \delta_{P'} \cdot r | \; w \in W(M', M), P' \in \{P\}\}\}$$

$$X_M(\mathbf{C})_r = \{\zeta \in X_M(\mathbf{C}) | \operatorname{Re} \zeta \in X_M(\mathbf{R})_r\}.$$

Then $X_M(\mathbf{C})_r$ gives rise to an open complex submanifold $D_M(\xi)_r$ of $D_M(\xi)$, and $X_M(\mathbf{C})_r \cap (\delta_P + C_P)$ is non empty, cf. 2.3.9. Let

$$f : \mathrm{i}L_{Z_M} \backslash X_M(\mathbf{C})_r \rightarrow \mathbf{C}$$

be holomorphic. Then f can be viewed as an element of $\Gamma(D_M(\xi)_r, \vartheta_M)$. We define $\mathscr{D}(P, r)$ to be the set of these f which are bounded, and set

$$\mathscr{D}(\{P\}, r) = \bigoplus_{P \in \{P\}} \mathscr{D}(P, r).$$

Suppose $P_1, \ldots, P_s \in \{P\}$ are the distinct elements of $\{P\}$. If $f = (f_1, \ldots, f_s) \in \mathscr{D}(\{P\}, r)$, we write

$$\|f\|_\infty = \max_i \sup_{\zeta \in X_{M_i}(\mathbf{C})} |f_i(\zeta)|.$$

As in 2.3.9, let $\mathscr{H}(P, K', r)$ be the space of holomorphic functions

$$\Phi(\zeta) : \mathrm{i}L_{Z_M} \backslash X_M(\mathbf{C})_r \rightarrow \bigoplus_{\{\omega\}} \mathscr{C}_0(P, K', \omega\delta_P).$$

Put

$$\mathscr{H}(\{P\}, K', r) = \bigoplus_{i=1}^{s} \mathscr{H}(P, K', r).$$

We write $\mathscr{H}(r)$ if there is no confusion. As in 2.3.9 one can extend the map

$$\Theta : \mathscr{C}_0(P, K', \xi) \rightarrow \mathscr{L}(\{P\}, \xi)$$
$$\varphi \rightarrow \varphi^\wedge$$

to a map

$$\Theta : \mathscr{H}(P, K', r) \rightarrow \mathscr{L}(\{P\}, \xi).$$

If $\Phi = (\Phi_1, \ldots, \Phi_s) \in \mathscr{H}(r)$, set

$$\Phi^\wedge = \Phi_1^\wedge + \ldots + \Phi_s^\wedge \in \mathscr{L}(\{P\}, \xi).$$

The inner product formula for $(\Phi^\wedge, \Psi^\wedge)$ then reads

$$(\Phi^\wedge, \Psi^\wedge) = \sum_{i,j} \sum_{w \in W(M_j, M_i)} \int_{\operatorname{Re} \zeta_i = \zeta_{i,0}} \langle M(w, \zeta_i)\Phi_i(\zeta_i), \Psi_j(-w\zeta_i)\rangle d\zeta_i$$

as follows from a simple continuity argument, using the functions $\varphi^\wedge$, $\varphi \in \mathscr{C}_0(P, K', \xi)$. We shall also write this as (Φ, Ψ).

1.2. The next result is proved as in [**12**] Lemma 4.

LEMMA. (i) *Let $f \in \mathscr{D}(\{P\}, r)$ be invariant by all $W(M, M')$ for P, $P' \in \{P\}$. Then there exists a unique bounded linear operator A_f on $\mathscr{L}(\{P\}, \xi)$.*

(ii) *For f as in* (i) *set $f^*(\chi) = f(-\chi)$. Then $f = f^*$ implies that A_f is self-adjoint.*

(iii) *The spectrum of A_f lies in the closure of*

$$\mathit{image}\ (f) := \bigcup_i \mathit{image}\ (f_i)$$

where $f = (f_1, \ldots, f_s)$.

Proof. Let $\Phi \in \mathscr{H}(r)$. If $\Phi = (\Phi_1, \ldots, \Phi_s)$, then

$$f \cdot \Phi = (f_1\Phi_1, \ldots, f_s\Phi_s) \in \mathscr{H}(r)$$

also. Applying the Θ map above then gives existence. To prove boundedness, choose $N > \|f\|_\infty$. The inner product formula implies that $(f\Phi, \Psi) = (\Phi, f^*\Psi)$. For each $1 \leqq i \leqq s$, let $h_i = N^2 - f_i^* f_i$. The definition of N readily implies that $(h_i)^{1/2} = g_i$ exists as a holomorphic, bounded function on the domain of f_i, and $g_i = g_i^*$. Furthermore one can always arrange the square root so that $g = (g_1, \ldots, g_s)$ is invariant by all $W(M, M')$. Thus g satisfies the same conditions as f. Therefore, as

$$\begin{aligned} 0 &\leqq (A_f\varphi^\wedge, A_f\varphi^\wedge) = (f\Phi, f\Phi) = (f^*f\Phi, \Phi) \\ &= (N^2\Phi, \Phi) - (g\Phi, g\Phi) \leqq N^2(\Phi, \Phi) = N^2\|\varphi^\wedge\|^2 \end{aligned}$$

we see that A_f defines a bounded operator. Finally, the last part follows by applying the preceding to the function

$$f - \lambda = (f_1 - \lambda, \ldots, f_s - \lambda) \quad \text{for } \lambda \notin \bigcup \text{image } (f_i).$$

1.3. Until further notice we shall confine our attention to a class $\{P\}$ of associate parabolic subgroups of rank 1: if $P \in \{P\}$ then rank $H_M(Z_M(\mathbf{A})) = 1$. Proposition 1.3.3 implies that $\{P\}$ consists of at most 2 elements. If $\{P\}$ consists of 1 element, then $W(M, M)$ consists of 2 elements; we shall refer to this as case (i). If $\{P\}$ consists of 2 elements P, P' then $W(M, M')$ and $W(M, M)$ each have 1 element; we shall refer to this as case (ii).

Choose $z_0 \in Z_M(\mathbf{A})$ generating the lattice $L^*_{Z_M}$. Given $z \in \mathbf{C}^*$ let $\zeta(z)$ be that element of $X_M(\mathbf{C})$ such that

$$z = q^{\langle H_M(z_0), \zeta(z)\rangle}.$$

In this way we obtain an isomorphism of groups

$$\mathbf{C}^* \xrightarrow{\sim} iL_{Z_M}\backslash X_M(\mathbf{C}).$$

We can always suppose, after changing z_0 if necessary, that

$$\delta_P(z_0) = q^{\rho_P'} > 1, \quad \rho_P' \in \mathbf{R}.$$

Write $\rho_P = \rho_P'/\mu$ where $\mu > 0$ satisfies

$$\langle \delta_P, \epsilon_P\rangle = \mu\langle \delta_P, \alpha^*\rangle.$$

Here ϵ_P is the element dual to the root α corresponding to P. With the same convention, one defines

$$\zeta'(z) \in iL_{Z_{M'}}\backslash X_{M'}(\mathbf{C}),$$

and then 1.1.8 implies $w\zeta(z) = -\zeta'(z)$ for $w \in W(M, M')$, for w non trivial (because

$$\langle H_{M'}(z_0'), {}^w\alpha\rangle = -\langle H_M(z_0), \alpha\rangle < 0,$$

by our assumption on $\delta_P(z_0)$ above. Here α is the unique simple root corresponding to P).

1.4. In case (i) set $\mathscr{C} = \bigoplus_{\{\omega\}} \mathscr{C}_0(P, K', \omega\delta_P)$, where as usual the sum on the right is finite. One can suppose that if $\omega \in \{\omega\}$, then ${}^w\omega \in \{\omega\}$ for $w \in W(M, M)$. Let $M(z) = M(w, \zeta(z))$.

In case (ii) set

$$\mathscr{C}_0 = \bigoplus_{\{\omega\}} \mathscr{C}_0(P, K', \omega\delta_P),\ \mathscr{C}_0' = \bigoplus_{\{\omega'\}} \mathscr{C}_0(P', K', \omega'\delta_{P'}).$$

Of course these sums are finite and we suppose that $W(M, M')\{\omega\} \subset \{\omega'\}$. Set

$$\mathscr{C} = \mathscr{C}_0 \oplus \mathscr{C}_0'$$

and

$$M(z) = \begin{pmatrix} 0 & M(w^{-1}, \zeta'(z)) \\ M(w, \zeta(z)) & 0 \end{pmatrix}.$$

We write $\Phi(z)$ for an element $(\Phi_1(z), \Phi_2(z)) \in \mathscr{C}$. If $\Phi, \Psi \in \mathscr{C}$, then we define

$$E(g, \Phi, z) = \sum_i E(g, \Phi_i, \zeta^{(i)}(z)), \quad \Phi = (\Phi_1, \ldots, \Phi_s), s = 1 \text{ or } 2$$

$$E^P(g, \Phi, z) = \sum_i E^P(g, \Phi_i, z), \quad P \in \{P\}$$

and

$$\langle \Phi, \Psi \rangle = \sum_i \langle \Phi_i, \Psi_i \rangle.$$

Let

$$\varphi = \sum \varphi_i \in \mathscr{C}_0(\{P\}, K,' \xi), \psi \in \mathscr{C}_0(\{P\}, K', \xi).$$

Then the inner product formula (2.3.3.2) becomes

$$(\varphi^\wedge, \psi^\wedge) = \int_{|z|=c_0} \{\langle \Phi(z), \Psi(z^{-1}) \rangle + \langle M(z)\Phi(z), \Psi(z) \rangle\}$$

where $\Phi(z) = (\Phi_1(\zeta^{(1)}(z)), \ldots \Phi_s(\zeta^{(s)}(z)))$ etc.

Under these identifications, $M(z)$ is an analytic function for $|z| > \max(q^{\rho_P}, q^{\rho_{P'}}) = q^\rho$, say.

The domain of definition for the elements of $\mathscr{H}(r)$ etc. is simply a tube whose axis lies along the real axis in $\mathbf{C}$, and which has width ρ on either side of the imaginary axis.

1.5. For case (i) define an analytic function

$$h : \mathrm{i}L_{Z_M} \backslash X_M(\mathbf{C}) \to \mathbf{C}$$

via

$$h(z) = z + z^{-1}.$$

The remark above implies that this really defines an analytic function on $iL_{z_M}\backslash X_M(\mathbf{C})$, which we also denote by h.

For case (ii) set $h = (h_1, h_2)$, where h_1, h_2 are as for case (i).

In both cases $h = h^*$, and satisfies the conditions of Lemma 1.2. Thus we obtain a bounded self-adjoint operator acting on $\mathscr{L}(\{P\}, \xi)$.

LEMMA. *The spectrum of A_h is contained in the interval*

$$[-(q^\rho + q^{-\rho}), (q^\rho + q^{-\rho})].$$

Proof. Set $X + iY = w = z + z^{-1}$. Then Lemma 1.2 implies that the spectrum of A_h lies inside the ellipse

$$\frac{X^2}{(q^{\rho r} + q^{-\rho r})^2} + \frac{Y^2}{(q^{\rho r} - q^{-\rho r})^2} = 1$$

which is the image of $\{z : q^{-\rho r} < |z| < q^{\rho r}\}$ under the map $z \to z + z^{-1}$ where $r > 1$. Since A_h is self-adjoint one need only consider the intersection of the real axis with this ellipse, and this is just the closed interval

$$[-(q^{\rho r} + q^{\rho r}), (q^{\rho r} + q^{-\rho r})].$$

Since $r > 1$ is arbitrary, the result follows.

1.6. Next, we consider the resolvent $R(\lambda, A_h)$ of A_h. Fix $\lambda = q^t$ so that $|\lambda| > q^{\rho r} > q^\rho$, and put $\mu = \lambda + \lambda^{-1}$. Then we can apply Lemma 1.2 to the functions

$$(\mu - h(z))^{-1} \quad \text{in case (i)}$$

and

$$((\mu - h_1(z))^{-1}, (\mu - h_2(z))^{-1}) \quad \text{in case (ii)}.$$

In each case we see that these functions correspond to the resolvent $R(\lambda + \lambda^{-1}, A_h)$. Choose $q^{\rho r_1} > |\lambda|$. Set $w(z) = z + z^{-1}$; according to the inner product formula,

$$(R(\mu, A_h)\phi^\wedge, \psi^\wedge) =$$

$$\int_{|z|=q^{\rho r}} \frac{1}{\mu - w(z)} \{\langle \Phi(z), \Psi(\bar{z}^{-1})\rangle + \langle M(z)\Phi(z), \Psi(\bar{z})\rangle\} dz.$$

Of course $r > 1$ in this expression. If we use our remarks on approximation, we may replace $\varphi^\wedge$ by $\Phi^\wedge$, $\Phi \in \mathscr{H}(\{P\}, K', r)$; similarly for $\Psi^\wedge$. In particular take $\Phi(z)$ to be the function

$$z\Phi, \ \Phi \in \mathscr{C}$$

and similarly for $\Psi(z)$. The formula above becomes

$$\int_{|z|=q^{\rho r}} \frac{1}{\mu - w(z)} \{\langle \Phi, \Psi\rangle + \langle M(z)\Phi, \Psi\rangle\} dz.$$

Now $(R(\lambda + \lambda^{-1}, A_h)\varphi^\wedge, \Psi^\wedge)$ is an analytic function of λ provided

$$\lambda \notin \{\mu| \quad |\mu| \leqq 1\} \cup \{\mu| \operatorname{Im} \mu = 0, 1 \leqq \operatorname{Re} \mu \leqq q^\rho \text{ or } -q^\rho \leqq \operatorname{Re} \mu \leqq -1\}.$$

This follows from our knowledge of the spectrum of A_h, and the properties of the conformal mapping $w = \mu + \mu^{-1}$.

On the other hand, one may shift the contour in the integral above, and use the calculus of residues. Indeed, let $r_1 > r$ and write Γ for the circle $|z| = q^{\rho r_1}$. If $q^{\rho r_1} > |\lambda| > q^{\rho r}$, then the inner product formula and the calculus of residues yield

$$(1.6.1) \quad (R(\lambda + \lambda^{-1}, A_h)\varphi^\wedge, \psi^\wedge)$$
$$= \int_\Gamma \frac{1}{(\lambda + \lambda^{-1}) - (z + z^{-1})} \{\langle \Phi, \Psi \rangle + \langle M(z)\Phi, \Psi\rangle dx$$
$$+ \{\langle \Phi, \Psi \rangle + \langle M(\lambda)\Phi, \Psi\rangle\} \cdot g(\lambda)$$

where

$$g(\lambda) = \lim_{z \to \lambda} \left\{ \frac{1}{(\lambda + \lambda^{-1}) - (z + z^{-1})} \cdot (z - \lambda) \right\} = (1 - (\lambda^2)^{-1})^{-1}.$$

Now $(R(\lambda + \lambda^{-1}, A_h)\varphi^\wedge, \Psi^\wedge)$ is holomorphic whenever

$$\lambda \notin \{\mu| \, |\mu| \leqq 1\} \cup \{\mu| \operatorname{Im} \mu = 0, 1 \leqq \operatorname{Re} \mu \leqq q^\rho \text{ or } -q^\rho \leqq \operatorname{Re} \mu \leqq -1\} = B,$$

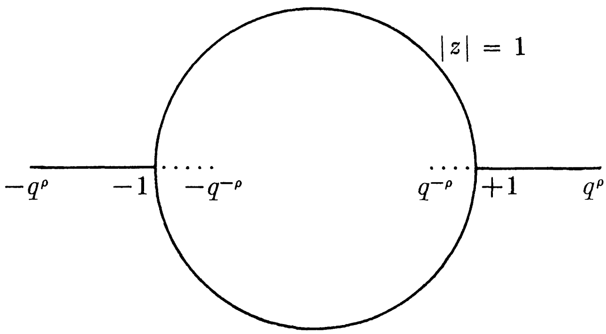
say, as remarked earlier. Moreover the integral over Γ is holomorphic as a function of λ, and so is the function $g(\lambda)$. From this we see that $M(\lambda)$ can be analytically continued over the complement of B.

Consequently we obtain the following result.

PROPOSITION. *The operator valued function $M(z)$ can be analytically continued as a holomorphic function outside the region*

$$\{z| \, |z| \leqq 1\} \cup \{z| \operatorname{Im} z = 0, 1 \leqq \operatorname{Re} z \leqq q^\rho \text{ or } -q^\rho \leqq \operatorname{Re} z \leqq -1\}.$$

1.7. To extend the results obtained on $M(z)$ to the function $E(g, \Phi, z)$ one applies 2.4.8. Thus $E(g, \Phi, z) = E(g, \phi, \zeta(z))$ can be continued to an analytic function over the region above.



2. Truncated Eisenstein series.

2.1. In Section 1 we constructed the generator $H_M(z_0)$ for $L^*_{Z_M}$. Since $L^*_{Z_M} \otimes \mathbf{R} = X_M^*(\mathbf{R})$ there is a unique element of $X_M(\mathbf{R})$, which we denote by κ, such that

$$\langle H_M(z_0), \kappa \rangle = 1.$$

From these remarks, we construct a function β_N, for each integer N, as follows:

$$\beta_N(g) = \begin{cases} 1 & \text{if } \langle \kappa, \bar{H}_M(g) \rangle \leqq N \\ 0 & \text{otherwise.} \end{cases}$$

Thus β_N: $P(F)N(\mathbf{A})\backslash G(\mathbf{A}) \rightarrow \mathbf{R}$ is a locally constant function.

2.2. Now let $\Phi = \oplus \Phi_i \in \mathscr{C}$ (notation as in Section 1). We write

$$F(g, \Phi_i, z) = q^{\langle \zeta^i(z), \bar{H}_i(g) \rangle} \Phi_i(g) \quad \text{where } \bar{H}_i(g) = \bar{H}_{M_i}(g).$$

Define the function F' by

$$F'(g, \Phi_i, z) = \beta_N(g) F(g, \Phi_i, z)$$

and the function F'' by

$$F''(g, \Phi_i, z) = (1 - \beta_N(g)) F(g, (M(z)\Phi)_i, z^{-1}).$$

Let $\chi_i \in D_{M_i}(\xi)$; the first thing we want to do is compute the Fourier transform of F', F'' evaluated at $\chi_i \delta_{P_i}$. For the moment we consider F'. By definition, if $\chi_i = (\omega_i, \lambda_i) \in D_{M_i}(\xi)$, the Fourier transform is just

$$\int_{Z_M(F)Z(\mathbf{A})\backslash Z_M(\mathbf{A})} F'(xg, \Phi_i, z)(\omega_i \lambda_i \delta_{P_i})^{-1}(x) dx.$$

Write $\Phi_i = \oplus_{\{\omega\}} \Phi_{i\omega}$ as usual, then the integral becomes

$$\bigoplus_{\{\omega\}} \int q^{\langle \zeta^i(z), \bar{H}_i(xg) \rangle} \Phi_{i\omega}(xg) \beta_N(xg)(\omega_i \lambda_i \delta_{P_i})^{-1}(x) dx$$
$$= \bigoplus_{\{\omega\}} \int \left(\int \omega(x) \omega_i(x)^{-1} dx \right) \omega \omega_i^{-1}(\dot{x}) \beta_N(\dot{x}g) e$$
$$\times \langle z, \bar{H}_i(\dot{x}g) \rangle (\lambda_i \delta_{P_i})^{-1}(\dot{x}) \Phi_{i\omega}(g) d\dot{x}$$

(inner integral taken over $Z_{M_i}(F)Z(\mathbf{A})\backslash Z_{M_i}{}^0$ and

$$e\langle z, H_i(\dot{x}g) \rangle = q^{\langle \zeta^i(z), \bar{H}_i(\dot{x}g) \rangle}).$$

The integral in braces is zero unless $\omega \omega_i^{-1}|_{Z_M{}^0} = 1$, and then it is equal to one. Moreover in this case, $\omega = \omega_i$, by our choice of orbit representatives (cf. 2.1.2). Thus the integral reduces to

$$\int_{L_{Z_M}{}^*} \beta_N(xg) q^{\langle \zeta^i(z) - \lambda_i, H_i(\dot{x}) \rangle} q^{\langle \zeta^i(z), \bar{H}_i(g) \rangle} \Phi_{i\omega_i}(g) d\dot{x}.$$

Set $H_i(\dot{x}) = nH_i(z_0)$ to obtain

$$(2.2.1) \quad \sum_{n=-\infty}^{N-\langle\kappa,\bar{H}_i(g)\rangle} q^{\langle\zeta^i(z)-\lambda_i,H_i(z_0)\rangle n} q^{\langle\zeta^i(z),\bar{H}_i(g)\rangle}\Phi_{i\omega_i}(g)$$

$$= q^{\langle\zeta^i(z),\bar{H}_i(g)\rangle}\Phi_{i\omega_i}(g)\frac{q^{N\langle\zeta^i(z)-\lambda_i,H_i(z_0)\rangle}}{1-q^{-\langle\zeta^i(z)-\lambda,H_i(z_0)\rangle}} q^{-\langle\kappa,\bar{H}_i(g)\rangle\langle\zeta^i(z)-\lambda_i,H_i(z_0)\rangle}$$

$$= q^{\langle\zeta^i(z),\bar{H}_i(g)\rangle}\Phi_{i\omega_i}(g)\frac{q^{N\langle\zeta^i(z)-\lambda_i,H_i(z_0)\rangle}}{1-q^{-\langle\zeta^i(z)-\lambda_i,H_i(z_0)\rangle}} q^{+\langle-\zeta^i(z)+\lambda_i,\bar{H}_i(g)\rangle}$$

$$= \frac{q^{N\langle\zeta^i(z)-\lambda_i,\bar{H}_i(z_0)\rangle}}{1-q^{-\langle\zeta^i(z)-\lambda_i,\bar{H}_i(z_0)\rangle}}\cdot q^{\langle\lambda_i,H_i(g)\rangle}\Phi_{i\omega_i}(g).$$

The operations performed above are valid provided that

$$\operatorname{Re}\langle\zeta^i(z)-\lambda_i, H_{M_i}(z_0^{(i)})\rangle > 0$$

which follows from assuming

$$\langle H_{M_i}(z_0^{(i)}), \operatorname{Re}\zeta^i(z)\rangle > r\rho \quad \rho, r > 1 \text{ as in Section 1}$$

$$(2.2.2) \quad |\langle H_{M_i}(z_0^{(i)}), \operatorname{Re}\lambda_i\rangle| < r\rho.$$

In particular, if we set $\Psi_1^{(i)}(\lambda_i, g)$ equal to the expression (2.2.1) it follows that the function

$$\bigoplus_{\{\omega\}} T_{\lambda_i}{}^{-1}\Psi_1^{(i)}(\lambda_i, g)$$

(cf. 2.2.6) is an element of the space $\mathscr{H}(P_i, K', r)$ (1.1).

The Fourier transform of $F''(g, \Phi_i, z)$ is readily obtained by means of similar manipulations; the result is the expression below, denoted by $\Psi_2(\lambda, g)$, where we have dropped the subscript i as much as possible:

$$(2.2.3) \quad \Psi_2(\lambda, g) = \frac{q^{-(N+1)\langle\zeta(z)+\lambda,H_M(z_0)\rangle}}{1-q^{-\langle\zeta(z)+\lambda,H_M(z_0)\rangle}} q^{\langle\lambda,H_M(g)\rangle}(M(z)\Phi)_{i\omega_i}(g).$$

The operations that one performs are certainly valid if one has

$$\operatorname{Re}\langle\zeta(z)+\lambda, H_M(z_0)\rangle > 0$$

which follows from assuming (2.2.2) above.

In particular, it follows that

$$\Psi_1 = \bigoplus_i T_{\lambda_i}{}^{-1}\Psi_1^{(i)}(\lambda_i, g) \quad \text{and} \quad \Psi_2 = \bigoplus T_{\lambda_i}{}^{-1}\Psi_2^{(i)}$$

are elements of $\mathscr{H}(\{P\}, K', r)$ so that the functions $\Psi_1{}^\wedge - \Psi_2{}^\wedge$ (2.3.9) are defined.

There is another expression for $\Psi_1{}^\wedge - \Psi_2{}^\wedge$ which we shall find useful. Let φ be a function on $G(F)\backslash G(\mathbf{A})$ such that if $G \neq P$ is not a maximal parabolic then $\varphi^P \equiv 0$. Given N as above, define

$$\Lambda^N\varphi(g) = \varphi(g) - \sum_{P \text{ maximal}}\sum_{P(F)\backslash G(F)}\varphi^P(\gamma g)(1-\beta_N(\gamma g)).$$

An argument similar to that used in 2.2.4 implies that each inner sum converges uniformly on compact sets, and, for a given g is a finite sum in fact. Applying the operator Λ^N to $E(g, \Phi, \zeta(z))$ one sees by a small computation that

$$\Lambda^N E(g, \Phi, \zeta(z)) = \Psi_1^\wedge - \Psi_2^\wedge.$$

2.3. The next step is to compute

$$(\Lambda^N E(g, \Phi, z_1), \Lambda^N E(g, \Psi, z_2))$$

using the inner product formula 2.3.3.2. To do this one shifts the contour of integration and computes residues. Since the method is carried out in some detail in [**13**] p. 134 we give only the final result: In the following, $z = q^\lambda$, $z_2 = q^\mu$.

$$(2.3.1) \quad (\Lambda^N E(g, \Phi, z_1), \Lambda^N E(g, \Psi, z_2))$$
$$= \frac{q^{N(\lambda+\bar{\mu})}}{1 - q^{-(\lambda+\bar{\mu})}} \langle \Phi, \Psi \rangle - \frac{q^{-(N+1)(\lambda+\bar{\mu})}}{1 - q^{-(\lambda+\bar{\mu})}} \langle M(\lambda)\Phi, M(\mu)\Psi \rangle$$
$$+ \frac{q^{N(\bar{\mu}-\lambda)}}{1 - q^{(\lambda-\bar{\mu})}} \langle M(\lambda)\Phi, \Psi \rangle - \frac{q^{(N+1)(\lambda-\bar{\mu})}}{1 - q^{(\lambda-\bar{\mu})}} \langle \Phi, M(\mu)\Psi \rangle.$$

We observe in passing, the occurrence of $(N + 1)$ in the exponent. Set $\lambda = \mu = \sigma + i\tau$, and choose $\Phi = \Psi$ so that $\langle \Phi, \Phi \rangle = 1$ and $\|M(\lambda)\Phi\| = \|M(\lambda)\|$. Then (2.3.1) reduces to

$$\left\{\frac{q^{2N\sigma}}{1 - q^{-2\sigma}} - \frac{q^{-2(N+1)\sigma}}{1 - q^{-2\sigma}} \|M(\lambda)\|^2\right\}$$
$$+ \left\{\frac{q^{-2iN\tau}}{1 - q^{2i\tau}} \langle M(\lambda)\Phi, \Phi \rangle - \frac{q^{2i(N+1)\tau}}{1 - q^{2i\tau}} \langle \Phi, M(\lambda)\Phi \rangle\right\}.$$

The second expression in braces may be expressed as

$$\frac{1}{1 - q^{2i\tau}} \{\langle q^{-2iN\tau} M(\lambda)\Phi, \Phi \rangle - \overline{q^{2i\tau} \langle q^{-2iN\tau} M(\lambda)\Phi, \Phi \rangle}\}.$$

A moment's reflection and the Cauchy-Schwartz inequality shows that the modulus of this is at most

$$\frac{2}{|1 - q^{2i\tau}|} |\langle M(\lambda)\Phi, \Phi \rangle| \leqq 2 \frac{\|M(\lambda)\|}{|1 - q^{2i\tau}|}.$$

Since (2.3.1) is evidently non negative, one deduces that

$$\frac{q^{2N\sigma}}{1 - q^{-2\sigma}} - \frac{q^{-2(N+1)\sigma}}{1 - q^{-2\sigma}} \|M(\lambda)\|^2 + 2 \frac{\|M(\lambda)\|}{|1 - q^{2i\tau}|} \geqq 0.$$

Thus

$$(2.3.2)\quad \|M(\lambda)\| \leqq \left\{\frac{2}{|1-q^{2i\tau}|} + \sqrt{\frac{4}{|1-q^{2i\tau}|^2} + \frac{4q^{-2\sigma}}{(1-q^{-2\sigma})^2}}\right\} \frac{1-q^{-2\sigma}}{2q^{-2(N+1)\sigma}}$$
$$\leqq \frac{1-q^{-2\sigma}}{|1-q^{2i\tau}|}\cdot q^{2(N+1)\sigma}\left\{2 + q^{-\sigma}\frac{|1-q^{2i\tau}|}{1-q^{-2\sigma}}\right\}$$
$$= \left\{2q^{2(N+1)\sigma}\frac{1-q^{-2\sigma}}{|1-q^{2i\tau}|} + q^{(2N+1)\sigma}\right\}.$$

(In deriving this, one uses the inequality $\sqrt{(x^2+y^2)} \leqq x + y$ for $x, y \geqq 0$.)

2.4. The expressions (2.3.1), (2.3.2) were derived assuming the conditions (2.2.2). On the other hand, if we write

$$E(g, \Phi, z) = E(g, \Phi, \lambda) \text{ for } z = q^{\lambda},$$

we have

$$\Lambda^N E(g, \Phi, \lambda) = \sum_{n=1}^{\infty} \frac{\Lambda^N E_n(g, \Phi, \lambda_0)}{n!}(\lambda - \lambda_0)^n$$

where

$$\Lambda^N E_n(g, \Phi, \lambda_0) = \frac{\partial^n}{\partial\lambda^n}\Lambda^N E(g, \Phi, \lambda)|_{\lambda=\lambda_0}$$

if $\Lambda^N E(g, \Phi, \lambda)$ is analytic at $\lambda = \lambda_0$. Hence

$$\|\Lambda^N E(g, \Phi, \lambda)\|^2 \leqq \sum \frac{\|\Lambda^N E_n(g, \Phi, \lambda_0)\|^2}{n!}|\lambda - \lambda_0|^{2n}.$$

Denote the expression (2.3.1) by $\omega(\lambda, \bar{\mu}; \Phi, \Psi)$; if we expand $(\Lambda^N E(g, \Phi, \lambda), \Lambda^N E(g, \Phi, \lambda))$ at $\lambda = \lambda_0$, we find

$$\|\Lambda^N E_n(g, \Phi, \lambda_0)\|^2 = \frac{\partial^{2n}}{\partial\lambda^n\partial\bar{\mu}^n}\omega(\lambda, \bar{\mu}; \Phi, \Phi)|_{\mu=\lambda_0}^{\lambda=\lambda_0}.$$

From this we see that $\Lambda^N E(g, \Phi, \lambda)$ can be analytically continued about λ_0 whenever that for $\omega(\lambda, \lambda; \Phi, \Phi)$ is analytic at (λ_0, λ_0). From the expression (2.3.1) and the region in which $M(\lambda)$ has been extended in Section 1, we see that (2.3.1), and hence $\Lambda^N E(g, \Phi, \lambda)$, can be extended as an analytic function over the same region. Consequently the formula (2.3.1) will persist in the same region as well. Setting $\lambda = \sigma + i\tau$, $\tau \neq 0$ and using (2.3.2) we find

$$\limsup_{\sigma \downarrow 0} \|M(\sigma + i\tau)\| \leqq 1.$$

It follows from 2.4.8 that $|E(g, \Phi, \lambda)|$ also remains uniformly bounded as $\sigma \downarrow 0$, for $g \in C$ a compact subset of $\mathfrak{S}$. The definition of $\Lambda^N E(g, \Phi, \lambda)$ then implies that it also remains uniformly bounded.

If we return to the expression (2.3.2) and multiply it by $(1 - q^{-2\sigma})/q^{2N\sigma}$, we see that

$$\lim_{\substack{\sigma\downarrow 0\\ \tau=\tau_0\neq 0}} M^*(\lambda)M(\lambda) = \mathrm{id} = \lim M(\bar{\lambda})M(\lambda)$$

since $M^*(\lambda) = M(\bar{\lambda})$ (2.3.7, and analytic continuation). In particular $M(i\tau)$, $\tau \neq 0$, is unitary, hence invertible. Arguing now as in [**11**], p. 139, we define $M(\lambda)$, $\operatorname{Re}\lambda < 0$ by $M(\lambda) = M^{-1}(-\lambda)$.

This is a meromorphic function, since $\det M(\lambda) \not\equiv 0$ ($\operatorname{Re}\lambda \geqq 0$), so long as we keep away from the region depicted in 1.7, and its reflection with respect to the unit circle ($z = q^\lambda$ in that figure).

The results of 2.4.8 readily imply that $E(g, \Phi, z)$ can be continued analytically to the same region, and that its behaviour is at least as good as that of $M(z)$.

We summarize our results in the following

PROPOSITION. (i) *The operator valued function $M(z)$ can be analytically continued to a meromorphic function over the complement of the region*

$$[-q^\rho, -q^{-\rho}] \cup [q^{-\rho}, q^\rho]$$

and satisfies the functional equation

$$M(z)M(z^{-1}) = \mathrm{id}.$$

(ii) *The function $E(g, \Phi, z)$ can be analytically continued as a meromorphic function over the same region as in* (i), *and satisfies the functional equation*

$$E(g, M(z)\Phi, z^{-1}) = E(g, \Phi, z).$$

The last assertion follows from 2.4.8, and the fact that $M(z)$ satisfies the functional equation $M(z)M(z^{-1}) = \mathrm{id}$.

3. Some applications of Stone's formula.

3.1. The results of the preceding sections have enabled us to continue $M(z)$, and hence $E(g, \Phi, z)$, over the complement of the set

$$D = [-q^\rho, -q^{-\rho}] \cup [q^{-\rho}, q^\rho].$$

In this section we shall prove that $M(z)$ can be analytically continued over $D\backslash\{\pm 1\}$, and that on $[-q^\rho, -1) \cup (1, q^\rho]$ it has at worst a finite number of simple poles. In the next it will be shown that $M(z)$ is analytic at $z = \pm 1$. Since the behaviour of $E(g, \Phi, z)$ is no worse than that of $M(z)$ (2.4), the same results hold true for it as well. The first step is to prove that there is a finite number of points in $[-q^\rho, -1) \cup (1, q^\rho]$ so that $M(z)$ is analytic for $|z| > 1$, except perhaps at these points. Then we use Stone's formula from spectral theory to see that these points can be at worst simple poles for $M(z)$. Finally we use Stone's formula and elementary analysis to see that $M(z)$ is analytic at $z = \pm 1$; it is defined on $(-1, -q^{-\rho}] \cup [q^{-\rho}, 1)$ by reflection.

3.2. LEMMA. *There is a finite number of points $z_1, \ldots, z_n$ in $R = [-q^{\rho}, -1) \cup (1, q^{\rho}]$ such that $M(z)$ is analytic for $|z| > 1$, except possibly at these points.*

Proof. We need only investigate the behaviour of $M(z)$ as z approaches an element of the set mentioned in the statement of the lemma.

To begin, suppose $z_0 \in R$, and suppose either

(i) there is a sequence $z_n \to z_0$, and a sequence $\{\Phi_n\}$, $\Phi_n \in \mathscr{C}$, $\|\Phi_n\| = 1$ such that $\{|M(z_n)\Phi_n|\}$ is unbounded, or

(ii) there are sequences $\{z_n\}$, $\{z_m'\}$ converging to z_0, and an element $\Phi \in \mathscr{C}$, such that

$$\lim_{z_n \to z_0} M(z_n)\Phi \neq \lim_{z_m' \to z_0} M(z_m')\Phi.$$

If (i) holds, set $\nu_n = \|M(z_n)\Phi_n\|$ and choose a subsequence $\nu_n^{-1}M(z_n)\Phi_n$ which converges to Φ_0 in the (normed) space $\mathscr{C}$. Then we can apply the results of 2.4.6 to $E(g, \nu_n^{-1}\Phi_n, z_n)$ to deduce that it converges to a function ϕ_0 say. Moreover, $\phi_0 \in \mathscr{L}(\xi)$ (cf. 2.4.5).

In case (ii) we consider

$$M(z_n)\Phi - M(z_n')\Phi \to \Phi_0,$$

to get

$$E(g, \Phi, z_n) - E(g, \Phi, z_n') \to \phi_0 \in \mathscr{L}(\xi).$$

In either case write $\Phi_0 = (\Phi_0^{(1)}, \ldots, \Phi_0^{(s)})$, $s = 1$ or 2. Then

(a) $\phi_0^{\tilde{P}}(g) \equiv 0 \quad \text{if } \tilde{P} \notin \{P\}$

(b) $\phi_0^{P_i}(g) = T_{(-\zeta^i(z))}\Phi^{(i)}(g) \quad \text{if } P_i \in \{P\}$

which follows by a well trodden path.

From this we see that ϕ_0 must be an element of $\mathscr{L}(\{P\}, K', \xi)$ in fact. In particular $\phi_0 \in \mathscr{L}(z)$, where $\mathscr{L}(z) \subset \mathscr{L}(\{P\}, K', \xi)$ is the space consisting of functions ψ such that

(a) $\psi^P(g) \equiv 0 \quad \text{if } \tilde{P} \notin \{P\}$

(b) $\psi^{P_i}(g) = T_{(-\zeta^i(z))}\Psi^{(i)}(g)$, some $\Psi \in \mathscr{C}$, if $P_i \in \{P\}$.

Since $\psi \to (T_{\zeta^i(z)}\psi^{P_i})$ is injective, we see that $\mathscr{L}(z)$ is finite dimensional. Moreover if z is real, we find

$$(\varphi^{\wedge}, \psi) = (\Phi(z), \Psi), \quad \text{any } \varphi \in \mathscr{C}_0(p, K', \xi).$$

An easy approximation argument implies that if A is the operator in

Section 1, then

$$(\varphi^\wedge, A\psi) = (A\varphi^\wedge, \psi) = (z + z^{-1})(\Phi(z), \Psi) = (z + z^{-1})(\varphi^\wedge, \psi)$$

if $\Phi(z) \in \mathscr{H}(r)$ (cf. Section 1). Thus if z is an element of R in the statement of the lemma, then $\mathscr{L}(z)$ is an eigenspace for A, and $\mathscr{L}(z_1)$ is orthogonal to $\mathscr{L}(z_2)$ if $z_1 \neq z_2$. One now argues (2.4.9) as in Langlands p. 131 to deduce that the set of points in R for which $\mathscr{L}(z) \neq 0$ is a discrete set. If z does not belong to this set, then $M(s)$ is bounded (on $\mathbf{C}^*\backslash\mathbf{R}^*$) in a neighbourhood of z, and $\lim_{s\to z} M(s)$ exists.

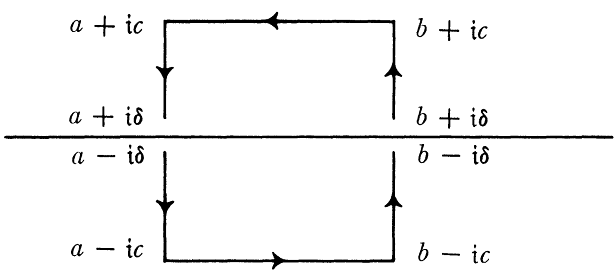
Case (ii) above shows that $M(z)$ thus defined is continous, whence analytic by Riemann's theorem on removable singularities.

We shall be done if we show that the set of $z \in R$ for which $\mathscr{L}(z) \neq 0$ does not have ± 1 as limit point; but this is Lemma 2.4.9.

3.3. For the next stage of the argument we shall recall Stone's formula, as applied to the operator A. It tells us that if $E(x)$ is the resolution of the identity for the (bounded) self-adjoint operator A, and that if $b > a$, c positive, then

$$\begin{aligned}(3.3.1)\quad & \tfrac{1}{2}\{(E(b)\varphi^\wedge, \psi^\wedge) + (E(b-0)\varphi^\wedge, \psi^\wedge)\} \\ & - \tfrac{1}{2}((E(a)\varphi^\wedge, \psi^\wedge) + (E(a-0)\varphi^\wedge, \psi^\wedge)) \\ & = -\lim_{\delta\downarrow 0} \frac{1}{2\pi i}\int_{C(a,b,c,\delta)} (R(\lambda, A)\varphi^\wedge, \psi^\wedge)d\lambda\end{aligned}$$

where $C(a, b, c, \delta)$ is the contour depicted below



For a proof of this fact, we refer the reader to [**15**], Theorem 5.10 (cf. also Reed and Simon: Functional Analysis vol. I p. 237).

3.4. Now we return to the points $z_1, \ldots, z_n$ in Lemma 3.2. Choose small closed discs of radius $\epsilon > 0$ around z_i and ± 1 which do not overlap and let C_i be the circle of radius ϵ about z_i, traversed in the anticlockwise sense, and let C be the contour which consists of the unit circle traversed in the anticlockwise sense, indented at ± 1 by those parts of the small circles described above with centres ± 1 which lie in $|z| \geqq 1$.

In view of what we now know about $M(z)$, we may replace the contour in the inner product formula for $(\varphi^\wedge, \psi^\wedge)$ to get

$$(3.4.1)\quad (\varphi^\wedge, \psi^\wedge) = \frac{1}{2\pi i}\int_C \langle \Phi(z), \Psi(\bar{z}^{-1})\rangle + \langle M(z)\Phi(z), \Psi(\bar{z})\rangle dz + \sum_{i=1}^{n} \frac{1}{2\pi i}\int_{C_i} \langle \Phi(z), \Psi(\bar{z}^{-1})\rangle + \langle M(z)\Phi(z), \Psi(\bar{z})\rangle dz.$$

3.5. LEMMA. *The points $z_1, \ldots, z_n$ are simple poles for $M(z)$.*

Proof. We shall first show that $z_i + z_i^{-1}$, $i = 1, \ldots, n$, is an isolated point in the spectrum of A, and then that z_i is a simple pole for $R(\lambda + \lambda^{-1}, A)$. An argument like that of 1.6 then implies that z_i must be a simple pole for $M(z)$.

First, choose $a < b$ so close that just one of the numbers $z_i + z_i^{-1}$ lies in the interval $[a, b]$. If we substitute (3.4.1) into (3.3.1) using this a and b, we can interchange the integrals and then use the calculus of residues, provided ϵ is suitably small, and c is suitably large (one requires the image of C_i under $z \to z + z^{-1}$ to be wholly inside the interior of the contour $C(a, b, c, \delta)$). The result is that (3.3.1) is equal to

$$(3.4.2)\quad \frac{1}{2\pi i}\int_{C_i} \langle M(z)\Phi(z), \Psi(\bar{z})\rangle dz.$$

This is true for any such a' and b'; if we let a', b' tend to a, b from below we see that $E(b) - E(a)$ must be equal to (3.4.2). It follows that $z_i + z_i^{-1}$ must be an isolated point for the spectrum of A.

On the other hand, if we compare the estimates for $\|M(z)\|$ given by (2.3.2) with the formula (1.6.1), we see that $R(\lambda + \lambda^{-1}, A)$ is of order $|\,|\lambda|^2/|\lambda^2| - \lambda^2|$. Thus the points z_i must be simple poles for $R(\lambda + \lambda^{-1}, A)$. Applying (1.6.1) again, we see that they must also be simple poles for $M(\lambda)$.

3.6. *Remark.* The above proof indicates the close relation between $M(z)$ and $R(z + z^{-1}, A)$, for $|z| > 1$. It shows that the points z_i, which are precisely the points for which $\mathscr{L}(z) \neq 0$ in $|z| > 1$ (so $\mathscr{L}(z)$ is an eigenspace for A), are exactly the points for which $M(z)$ is singular, with a simple pole, in $|z| > 1$. If we had applied (2.3.2) directly, the exactness part of this statement would have been lost, and we would know only that they are at worst simple poles.

3.7. The next step is to examine the behaviour of $M(z)$ as $x \to \pm 1$. Since the treatment in each case is parallel, we shall consider only the case $z \to +1$.

In what follows, $E(x)$ is as in 3.3; recall that the spectrum of the operator A lies between $-(q^\rho + q^{-\rho})$ and $(q^\rho + q^{-\rho})$.

LEMMA. (i) *If* $x \notin \{z_1 + z_1^{-1}, \ldots, z_n + z_n^{-1}, +2, -2\}$, *then*

$$(E(x)\varphi^\wedge, \psi^\wedge) - (E(x-0)\varphi^\wedge, \psi^\wedge) = 0,$$

and $(E(\cdot)\varphi^\wedge, \psi^\wedge)$ *is continuous at* x.

(ii) $(E(2)\varphi^\wedge, \psi^\wedge) - (E(2-0)\varphi^\wedge, \psi^\wedge)$

$$= \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{C(\epsilon)} \langle M(z)\Phi(z), \Psi(\bar{z})\rangle dz$$

where $C(\epsilon)$ *is that part of* $|z-1| = \epsilon$ *which lies in* $|z| \geqq 1$, *traversed in the anticlockwise sense.*

Proof. In both cases we shall use Stone's formula (3.3.1), and the expression (3.4.1).

Proof of (i). Suppose first that $x \in \mathbf{R}$, and $|x| > 2$. Then it is clear that $(E(\cdot)\varphi^\wedge, \psi^\wedge)$ is continuous at x, provided x is not one of the points listed in the statement of the lemma. Indeed, one can compute directly using Stone's formula and (3.4.1): for $y < x$ and sufficiently close, one has

$$((\tfrac{1}{2}(E(x) + E(x-0)) - \tfrac{1}{2}(E(y) + E(y-0))\varphi^\wedge, \psi^\wedge) = 0$$

("sufficiently close" means here that $[y, x]$ does not meet the images of the small circles C_i under the mapping $z \to z + z^{-1}$). This follows as in 3.5. Letting y approach x, we find

$$(E(x) - E(x-0)\varphi^\wedge, \psi^\wedge) = 0.$$

Since $(E(x)\varphi^\wedge, \psi^\wedge)$ is always right continuous, the result follows.

Next, suppose that $|x| < 2$ and x real; let y be as before. We must compute

$$\lim_{\delta \downarrow 0} \frac{-1}{2\pi i} \int_{C(y,x,c,\delta)} (R(\lambda, A)\phi^\wedge, \psi^\wedge) d\lambda.$$

If we regard (3.4.1) for a moment, we see that we can easily dispose of the terms coming from $\int_{C_i}$ by using previous arguments, so we limit our attention to the limit, as $\delta \downarrow 0$, of

$$\text{(3.7.1)} \quad \frac{-1}{2\pi i} \int_{C(y,x,c,\delta)} \int_C \frac{1}{\lambda - (z + z^{-1})} \langle \Phi(z), \Psi(\bar{z}^{-1})\rangle$$
$$+ \langle M(z)\Phi(z), \Psi(\bar{z})\rangle dz d\lambda.$$

For a given $\delta > 0$, we can certainly interchange the integrals, and then the first thing to do is to compute

$$\int_{C(y,x,c,\delta)} \frac{1}{\lambda - (z + z^{-1})} d\lambda.$$

We first observe that we need only compute this integral for those z such

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that $z + z^{-1} = w \in [a, b] \supset [y, z]$, say. Indeed, if we consider (3.7.1) with $\int_C$ replaced by $\int_{\{z \in \mathbf{C} | w \notin [a,b]\}}$ we see by previous techniques that the contribution is zero; of course we are supposing $a < y < x < b$. Write $C(a, b)$ for the set of z mentioned above. In general,

$$\int_{C(y,x,c,\delta)} \frac{1}{\lambda - w} d\lambda, \quad w = z + z^{-1}$$
$$= \int_y^x \frac{1}{\mu + i\delta - w} - \int_y^x \frac{1}{\mu - i\delta - w} d\mu$$
$$= \tan^{-1}(\delta/(\mu - w))|_y^x - \tan^{-1}(-\delta/(\mu - w))|_y^x,$$
$$w = z + z^{-1} \text{ is real,}$$

$= 2(\phi_x(w) - \phi_y(w))$, if $\phi_x(w) = \tan^{-1}(\delta/x - w)$, etc. As $\delta \downarrow 0$ we see that

$$\begin{cases} (\phi_x(w) - \phi_y(w)) \to 0 \text{ if } w \notin [y, x] \\ (\phi_x(w) - \phi_y(w)) \to \pi \text{ if } w \in (y, x) \\ \phi_x(w) - \phi_y(w) \to \pi/2 \text{ if } w = x \text{ or } y. \end{cases}$$

From this we see that (3.7.1) is equal to

$$\frac{1}{2\pi i} \int 2\pi\{\langle \Phi(z), \Psi(\bar{z}^{-1})\rangle + \langle M(z)\Phi(z), \Psi(\bar{z})\rangle\} dz$$

(integral over $\{z \in \mathbf{C} | w(z) \notin [y, x]\}$). If we let $y \uparrow x$ we see as before that $E(x) = E(x - 0)$.

(ii) Finally, we must consider the point $1 + 1^{-1} = 2$. Since the computations are similar to several made before, and we are tired, we shall be brief. Let $y < 2 < x$, and $|x - y|$ small. Our previous results imply that Stone's formula reduces to $(E(x) - E(y)\varphi^\wedge, \psi^\wedge)$ (by continuity). A (what is by now) straight forward computation shows that

$$\left(\lim_{x \downarrow 2} E(x) - E(y)\varphi^\wedge, \psi^\wedge\right)$$
$$= \frac{1}{2\pi i} \int_{C(\epsilon)} \langle \Phi(z), \Psi(\bar{z}^{-1})\rangle + \langle M(z)\Phi(z), \Psi(\bar{z})\rangle$$

with

$$C(\epsilon) = \{z | |z - 1| = \epsilon\} \cap \{z | |z| > 1\},$$

and $\epsilon \downarrow 0$ as $y \uparrow 2$. The result then follows by letting $y \uparrow 2$, and observing that

$$\lim_{x \downarrow 2} (E(x)\varphi^\wedge, \psi^\wedge) = (E(2)\varphi^\wedge, \psi^\wedge)$$

by right continuity.

3.8. Lemma.

$$(E(2)\varphi^\wedge, \psi^\wedge) - (E(2-0)\varphi^\wedge, \psi^\wedge) = \langle M\Phi(1), \Psi(1)\rangle$$
$$= \frac{1}{2\pi i}\lim_{\epsilon\downarrow 0}\int_{C(\epsilon)} \langle M(z)\Phi(1), \Psi(1)\rangle dz = 0.$$

Proof. We know that

$$(E(2)\varphi^\wedge, \psi^\wedge) - (E(2-0)\varphi^\wedge, \psi^\wedge)$$
$$= \lim_{\epsilon\downarrow 0}\frac{1}{2\pi i}\int_{C(\epsilon)} \langle M(z)\Phi(z), \Psi(\bar z)\rangle dz,$$

by Lemma 3.7. In view of the definition of $E(2) - E(2-0)$, the right side defines a positive definite Hermitian symmetric form on $\mathscr{H}(r)$. It is however defined when $\Phi(z)$, $\Psi(z)$ are defined and analytic only in a neighbourhood of $z = 1$, and an approximation argument shows that it is still positive definite on this new space of functions. In particular let $h(z)$ be scalar valued, analytic near $z = 1$; then

$$\text{(3.8.1)}\quad \lim_{\epsilon\downarrow 0}\int_{C(\epsilon)} h(z)\overline{h(\bar z)}\langle M(z)\Phi(z), \Phi(\bar z)\rangle \geqq 0.$$

We can take $h(z) = (\delta \pm (z-1))^{1/2}$, so that $h(z)\overline{h(\bar z)}$ is just $\delta \pm (z-1)$, where $\delta > 0$. Since Stone's formula implies

$$\int_{C(\epsilon)} \langle M(z)\Phi(z), \Phi(\bar z)\rangle dz = O(1) \quad \text{as } \epsilon \to 0,$$

so for $\delta \geqq \epsilon$ we infer

$$\frac{1}{2\pi}\int_{C(\epsilon)} (z-1)\langle M(z)\Phi(z), \Phi(z)\rangle dz = O(\epsilon).$$

Applying this to Ψ, then $\Phi + \Psi$, we see that

$$\text{(3.8.2)}\quad \lim_{\epsilon\downarrow 0}\frac{1}{2\pi i}\int_{C(\epsilon)} (z-1)\langle M(z)\Phi(z), \Psi(\underline{z})\rangle dz = 0.$$

Applying the result just obtained to $(\Phi(z) - \Phi(1))/(z-1)$, we arrive at the first equality of the lemma.

To obtain the second, let $\varphi' = (E(2) - E(2-0))\varphi^\wedge$. Noting that $\chi_K{}'*(\varphi^\wedge) = (\chi_K{}'*\varphi)^\wedge$ where $\chi_K{}'$ is the characteristic function for K', and $*$ denotes convolution, we see that if $\varphi^\wedge$ is locally constant then so is $A\varphi^\wedge$, and $E(x)\varphi^\wedge$, all x. Consequently $(E(2) - E(2-0))\varphi^\wedge$ is continuous. One can now show that $\phi' \equiv 0$. Namely the constant term $\phi'^{\tilde P} \equiv 0$ unless $\tilde P \in \{P\}$, in which case it is given by

(3.8.3) $\quad (M\Phi(1))_i$

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where $\tilde{P} = P_i \in \{P\}$ $(i = 1$ or $2)$. The expression (3.8.3) must be square integrable on $\mathfrak{S}$ (cf. 1.5.9). On the other hand, a short computation shows that this is impossible because $\delta_{P_i}(g)$ is not L^2 on $\mathfrak{S}$, (cf. [**13**] p. 144).

4. Behaviour at $z = \pm 1$.

4.1. Let C' be the unit circle $|z| = 1$, taken in the clockwise direction. Let Γ be a circle of radius $1 + \epsilon$, centre the origin, with ϵ such that none of the points $z_1, \ldots, z_n$ lie in the closed disc bounded by Γ; we shall take Γ in the anticlockwise direction.

LEMMA. *For ζ inside the region bounded by Γ and C', one has*

$$M(\zeta) = \frac{1}{2\pi i}\int_{C'} M(z)\frac{|\zeta|^2 - |z|^2}{z|z-\zeta|^2}\,dz + \frac{1}{2\pi i}\int_{\Gamma} M(z)\left\{\frac{1}{z-\zeta} - \frac{1}{z-\bar{\zeta}^{-1}}\right\}dz.$$

Proof. According to the theory of residues

$$M(\zeta) = \frac{1}{2\pi i}\int_{C}\frac{M(z)dz}{z-\zeta} + \frac{1}{2\pi i}\int_{\Gamma}\frac{M(z)dz}{z-\zeta}$$

where C is just the contour of 3.4 taken clockwise.

C

Because of 3.8 however, we may replace C by C'. Moreover, $\bar{\zeta}^{-1}$ lies inside the unit circle, so that

$$0 = \frac{1}{2\pi i}\int_{C'}\frac{M(z)}{z-\bar{\zeta}^{-1}}\,dz + \frac{1}{2\pi i}\int_{\Gamma}\frac{M(z)}{z-\bar{\zeta}^{-1}}\,dz.$$

Thus

$$M(\zeta) = \frac{1}{2\pi i}\int_{C'} M(z)\left\{\frac{1}{z-\zeta} - \frac{1}{z-\bar{\zeta}^{-1}}\right\}dz + \int_{\Gamma}.$$

The result follows since

$$\frac{1}{z-\bar{\zeta}^{-1}} = \frac{1}{z - z\bar{z}/\bar{\zeta}} \quad \text{for } |z| = 1.$$

4.2. Set $z = q^{t+i\Theta}$, $\zeta = q^{\sigma+i\tau}$, $r = q^{\sigma}$, $\alpha = \log q$, $\omega = 2\pi/\alpha$. Then, from Lemma 4.1, one can write

$$(4.2.1)\quad M(\zeta) = \frac{1}{\omega}\int_0^{\omega}\frac{(r^2-1)M(q^{i\Theta})d\Theta}{1 - 2r\cos\alpha(\tau-\Theta) + r^2} + \frac{1}{2\pi i}\int_{\Gamma} M(z)(\ldots)dz.$$

LEMMA. *Suppose* $\lim_{\tau\to 0} M(q^{i\tau}) = M(1)$ *exists. Then*

$$\lim_{\substack{\sigma\downarrow 0\\ \tau\to 0}} M(q^{\sigma+i\tau}) = M(1).$$

Proof. Set

$$P_r(\Theta) = \frac{r^2 - 1}{1 - 2r\cos\alpha\Theta + r^2}.$$

This is a Poisson kernel, so that

$$\frac{1}{\omega}\int_0^\omega P_r(\Theta)d\Theta = 1.$$

Thus

$$M(\zeta) - M(1) = M(\cdot)*P_r(\Theta) - M(1)*P_r(\Theta) + \int_\Gamma$$

$$= (M(\cdot) - M(1))*P_r(\Theta) + \int_\Gamma,$$

using obvious notation. As ζ approaches 1, the integral over Γ approaches zero, and the result follows.

4.3. We pause to make two remarks. Suppose $q^{i\Theta}$ is an eigenvalue of multiplicity m for $M(q^{i\tau})$; it is a known fact which follows from the theory of algebraic functions that if $|\tau - \tau'|$ is sufficiently small, there are m eigenvalues (counted according to multiplicities) of $M(q^{i\tau'})$ close to $q^{i\Theta}$. For this we refer the reader to e.g. Reed and Simon, Functional Analysis vol. IV Chapter XII.1.

Secondly, we know that $\|E(\,.\,,\Phi, q^{i\tau})\|$ is bounded for $0 < \tau \leqq T$, if T is given; this follows from the remarks made in 2.4. Now take $N = 0$, and let $\sigma \downarrow 0$ in the formula 2.3.1. The result is readily computed to be

$$\begin{aligned}(4.3.1)\quad &(\Lambda^0 E(\cdot\,,\Phi, q^{i\tau}), \Lambda^0 E(\cdot\,,\Psi, q^{i\tau}))\\ &= \langle\Phi,\Psi\rangle - \frac{q^{i\tau}}{2}\langle M^{-1}(q^{i\tau})M'(q^{i\tau})\Phi,\Psi\rangle\\ &\quad + \frac{1}{1-q^{2i\tau}}\{\langle M(q^{i\tau})\Phi,\Psi\rangle - q^{2i\tau}\langle M^{-1}(q^{i\tau})\Phi,\Psi\rangle\}.\end{aligned}$$

In particular, if we take $B(q^{i\tau})$ to be equal to

$$I - \frac{q^{i\tau}}{2}M^{-1}(q^{i\tau})M'(q^{i\tau}) + \frac{1}{1-q^{2i\tau}}\{M(q^{i\tau}) - q^{2i\tau}M^{-1}(q^{i\tau})\}$$

then we know that $\|B(q^{i\tau})\|$ is bounded for $T \geqq \tau > 0$, T given.

4.4. The next stage of the argument is to prove that $\lim_{\tau\to 0} M(q^{i\tau}) = M(1)$ described in Lemma 4.2 does actually exist. For this, we shall study the behaviour of the eigenvalues of $M(q^{i\tau})$ as τ varies.

To this end, let Φ_j ($j = 1, \ldots, n$) be an orthonormal basis of eigenvectors corresponding to eigenvalues $q^{i\Theta_j}$ ($j = 1, \ldots, n$) for the unitary transformation $M(q^{i\tau})$. Let $\Phi \in \mathscr{C}$, $\|\Phi\| = 1$; then $\Phi = \sum \alpha_j \Phi_j$ and $\sum |\alpha_j|^2 = 1$. We can write

$$\langle M(q^{i\tau})\Phi, \Phi\rangle = \sum_{j=i}^{n} q^{i\Theta_j}|\alpha_j|^2$$

and, from the definition of B,

$$(4.4.1)\quad \langle M'(q^{i\tau})\Phi, \Phi\rangle = 2q^{-i\tau}\left\{\sum_{j=1}^{n} |\alpha_j|^2\left(q^{i\Theta_j} + \frac{1}{1-q^{2i\tau}}(q^{2i\Theta_j} - q^{2i\tau}) - \langle M(q^{i\tau})B(q^{i\tau})\Phi, \Phi\rangle\right)\right\}$$

$$= \sum_{j=1}^{n} 2q^{-i\tau}|\alpha_j|^2\left\{q^{i\Theta_j} - \frac{q^{i\Theta_j}}{\sin\alpha\tau}\sin\alpha(\Theta_j - \tau) - \langle M(q^{i\tau})B(q^{i\tau})\Phi, \Phi\rangle\right\}$$

with $\alpha = \log q$.

Write

$$M(q^{i\tau'}) = M(q^{i\tau}) + i\alpha\int_{\tau}^{\tau'} q^{iy}M'(q^{iy})dy$$

and suppose $|B(q^{i\tau})| \leqq b$ for $0 < \tau \leqq T$. If $|\tau' - \tau|$ is so small that

$$|M'(q^{i\tau'}) - M'(q^{i\tau})| \leqq b$$

then (4.4.1) implies

$$\langle M(q^{i\tau'})\Phi, \Phi\rangle = \sum_{j=1}^{n} |\alpha_j|^2 q^{i\Theta_j} \times\left\{1 + 2(q^{it} - 1)\left(1 - \frac{\sin\alpha(\Theta_j - \tau)}{\sin\alpha\tau}\right) + \beta(t)\right\}$$

where $t = \tau' - \tau$ and $|\beta(t)| \leqq 2b\,|q^{it} - 1|$. This expression is equal to

$$\sum_{j=1}^{n} |\alpha_j|^2 q^{i\Theta_j}\{1 + 2(q^{it} - 1)(-\sin\alpha\Theta_j\cot\alpha\tau) + 2(q^{it} - 1)(1 + \cos\alpha\Theta_j) + \beta(t)\}.$$

If we set

$$\delta_1(t) = 2(q^{it} - 1)(1 + \cos\alpha\Theta_j) + \beta(t)$$

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then

$$|\delta_1(t)| \leqq (4 + 2b)|q^{it} - 1| = 2c|q^{it} - 1|, \; c = 2 + b.$$

Let

$$\nu_j(t) = 2e^{i\nu(t)}\{|q^{it} - 1|(\pm \sin \alpha\Theta_j \cot \alpha\tau - c) \\ + (c|q^{it} - 1| \mp \delta(t))\}$$

and

$$u_j(t) = (1 \mp \nu_j(t))q^{i\Theta_j}$$

where

$$q^{it} - 1 = e^{i\nu(t)}|q^{it} - 1|$$

and

$$\delta(t) = 1/2e^{-i\nu(t)}\delta_1(t).$$

The upper or lower sign is taken according as $\sin \alpha\Theta_j \geqq 0$ or $\sin \alpha\Theta_j \leqq 0$. We note that for t small and negative, the angle $\nu(t)$ lies in the third quadrant, and $\nu(t) \to 3\pi/2$ as $t \to 0$.

If

$$|\sin \alpha\Theta_j| \cot \alpha\tau \geqq c + 1/2$$

then, in the expression for $v_j(t)$, we see that the first expression in parentheses is positive, while the second lies in the sector

$$\{z|\;|\arg z| \leqq \pi/4\}.$$

LEMMA. *Suppose $q^{i\Theta}$ is an eigenvalue of $M(q^{i\tau})$ of multiplicity m, with $|\sin \alpha\Theta| \cot \alpha\tau \geqq c + 1/2$. If $t = \tau' - \tau$ is negative and sufficiently small, then the m eigenvalues of $M(q^{i\tau})$ which are close to $q^{i\Theta}$ all lie in*

$$X(t) = \{q^{i\Theta}(1 \mp 2e^{i\nu(t)}|q^{it} - 1|(\pm \sin \alpha\Theta \cot \alpha\tau - c + z))| \\ |\arg z| \leqq \pi/3\}.$$

Proof. First, a careful look at the diagrams below shows that if $x, y \notin X(t)$, then $ax + (1 - a)y \notin X(t)$, provided $x, y \in \{z|\;|z| = 1\}$.

Suppose $q^{i\Theta}$ is an eigenvalue of multiplicity m. Write $q^{i\Theta_1} = q^{i\Theta_2} = \ldots = q^{i\Theta_m} = q^{i\Theta}$. If

$$\Phi = \sum_{j=1}^{m} \alpha_j\Phi_j$$

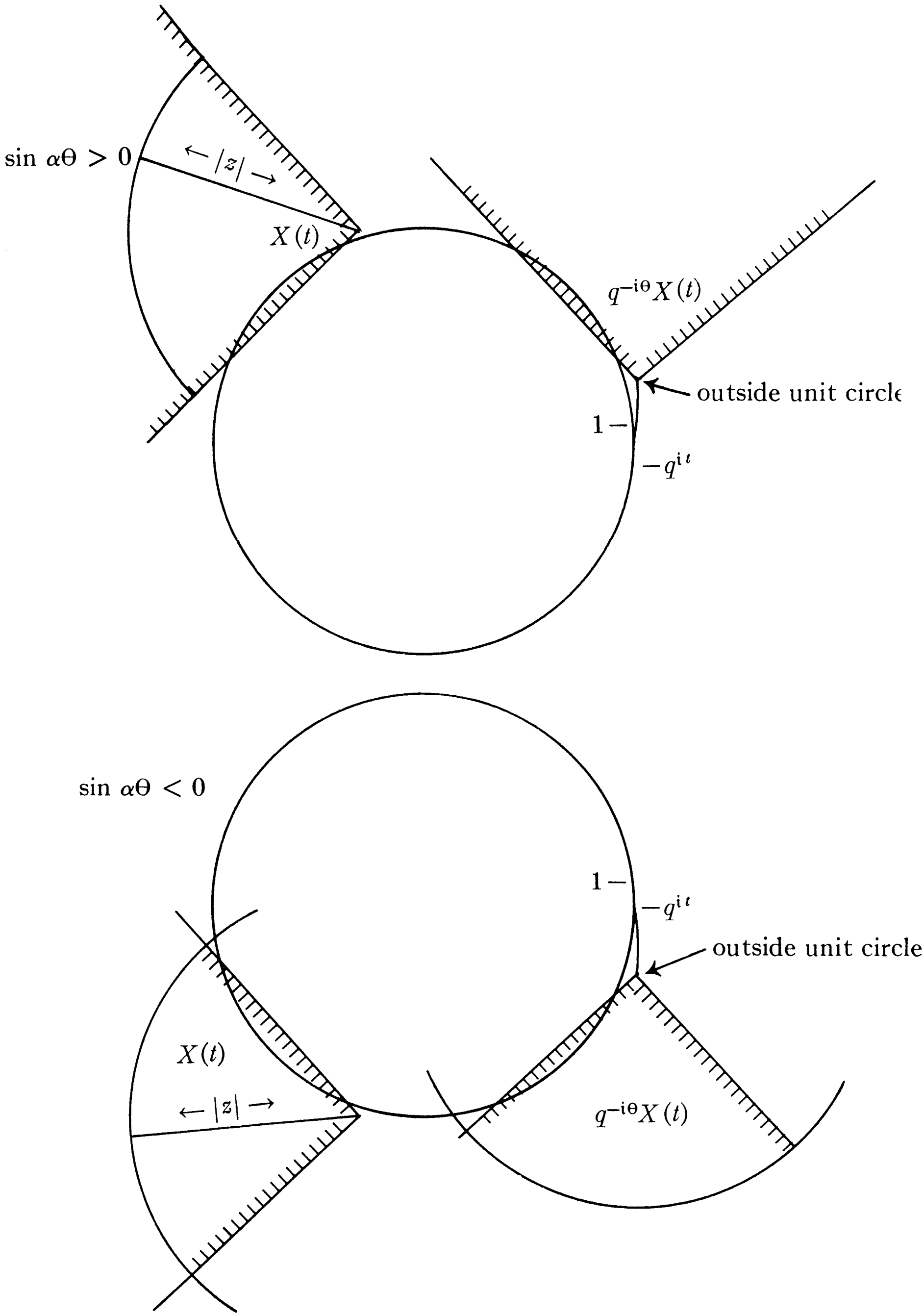
satisfies

$$|\Phi|^2 = \sum |\alpha_j|^2 = 1,$$

then

$$\langle M(q^{i(t+\tau)})\Phi, \Phi\rangle = \sum_{j=1}^{m} u_j(t)|\alpha_j|^2.$$

The remarks before the lemma imply that $u_j(t) \in X(t)$, if t is small enough (we are assuming $|M'(q^{i\tau'}) - M'(q^{i\tau})| \leqq b$);



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since $X(t)$ is convex it follows that $\langle M(q^{i(t+\tau)})\Phi, \Phi\rangle$ does also. If it were not the case that $X(t)$ contained m eigenvalues then by counting the dimensions of the eigenspaces involved, we see that we could choose Φ to be the linear combination of eigenvectors of $M(q^{i(\tau+t)})$ belonging to eigenvalues which lie in the complement of $X(t)$, and the remarks at the beginning of the proof would then imply that $\langle M(q^{i(\tau+t)})\Phi, \Phi\rangle$ must also lie in the complement of $X(t)$. This is a contradiction.

4.5. If $|\sin \alpha\Theta|\cot \alpha\tau \geqq c + 1/2$, then Lemma 4.4 implies that if $q^{i\Theta'}$ is one of the m eigenvalues of $M(q^{i\tau'})$ close to $q^{i\Theta}$ (an eigenvalue of multiplicity m of $M(q^{i\tau})$), then

$$q^{i(\Theta'-\Theta)} - 1 = \mp 2e^{i\nu(t)}|q^{it} - 1|(|\sin \alpha\Theta| \cot \alpha\tau - c + z)$$

with $|\arg z| \leqq \pi/3$. It also follows from contemplating the diagrams that in case

$$|\sin \alpha\Theta| \cot \alpha\tau \geqq c + 1/2,$$

then $\mp (\Theta - \Theta') \geqq 0$ where the sign is taken according as $\sin \alpha\Theta > 0$ or $\sin \alpha\Theta < 0$. Of course z depends on $\tau' - \tau$.

4.6. For the next lemma we must elaborate on the first remark made in 4.3. Let Ω be a region where $M(z)$ is analytic and $\zeta \in \Omega$; set

$$f(w, \zeta) = \det (M(\zeta) - wI) \quad \text{with } w \in \mathbf{C}.$$

Then

$$f(w, \zeta) = w^n + a_1(\zeta)w^{n-1} + \ldots + a_n(\zeta).$$

If $z = q^{i\tau}$ is the point in 4.4–4.5 and we suppose, as we may, that $z \in \Omega$, then $f(w, z) = 0$. It follows as in the theory of algebraic functions that the roots $w(\zeta)$ of the polynomial $f(w, \zeta)$ are branches of analytic functions with at worst algebraic singularities; the set of algebraic singularities constitutes a discrete set. It follows that the number of eigenvalues of $M(\zeta)$ is a constant, except at this discrete set of "exceptional" (or "critical") points; it is not the case however that this constant is necessarily equal to n. It also follows by taking Puiseux expansions that the branches of the analytic functions representing the roots have finite limits at the critical points.

4.7. Lemma. *Suppose $z = q^{i\tau}$ is not a critical point. Then in the notation of* 4.4,

$$|\sin \alpha\Theta| \leqq (c + 1/2) \tan \alpha\tau.$$

Proof. Suppose $|\sin \alpha\Theta| > (c + 1/2) \tan \alpha\tau$. Then each root $w(\zeta)$ above is differentiable at z. From 4.4 we have

$$q^{i(\Theta'-\Theta)} - 1 = \mp 2e^{i\nu(t)}|q^{it} - 1|(|\sin \alpha\Theta| \cot \alpha\tau - c + z(t))$$

where $z(t) = x(t) + iy(t)$, $x(t) \geqq 0$, and $w(\zeta) = q^{i\Theta'(\tau')}$ for $\zeta = q^{i\tau'}$. That is

$$q^{-i\Theta}q^{i\tau}(q^{i\Theta'} - q^{i\Theta}) = \mp 2(q^{i\tau'} - q^{i\tau})\{|\sin \alpha\Theta| \cot \alpha\tau - c + z(t)\}$$

or

$$q^{-i\Theta}q^{i\tau}\frac{\Delta w}{\Delta\zeta} = \mp 2\{|\sin \alpha\Theta| \cot \alpha\tau - c + z(t)\}.$$

Let $\Delta\zeta \to 0$, then

$$q^{-i\Theta}q^{i\tau}\frac{dw}{d\zeta}\bigg|_{\zeta=q^{i\tau}} = \mp 2\{|\sin \alpha\Theta| \cot \alpha\tau - c + x + iy\}$$

where $x + iy = \lim_{\zeta\to q^{i\tau}} z(t)$ exists because $dw/d\zeta$ exists at $\zeta = q^{i\tau}$. The left hand side is simply

$$\frac{d\Theta'}{d\tau'}\bigg|_{\tau'=\tau} = \mp 2A$$

where

$$A = |\sin \alpha\Theta| \cot \alpha\tau - c + x + iy.$$

Since everything else is real, we infer that $y = 0$. On the other hand $x \geqq 0$ because each $x(t) \geqq 0$. The equation

$$\frac{d\Theta'}{d\tau}\bigg|_{\tau'=\tau} = \mp 2A, \quad A > 0$$

now says that Θ' is a decreasing function if $\sin \alpha\Theta > 0$, and that it is an increasing function if $\sin \alpha\Theta < 0$. Referring to remarks in 4.5 concerning $\mp(\Theta - \Theta') \geqq 0$ we see that this is a contradiction.

4.8. We now deal with the critical points. If $z = q^{i\tau}$ is critical, then we may choose a neighbourhood Δ about z so that z is the only critical point in Δ. Let $w(\zeta)$, as above, correspond to $\Theta'(\tau')$. As remarked in 4.6, the function $w(\zeta)$ has a finite limit as $\zeta \to z$. On the other hand, 4.7 says

$$|\sin \alpha\Theta'| \leqq (c + 1/2) \tan \alpha\tau'$$

for all $z \neq \zeta = q^{i\tau'}$ in Δ; hence by continuity

$$|\sin \alpha\Theta| \leqq (c + 1/2) \tan \alpha\tau.$$

4.9. The preceding sequence of arguments has shown that if $q^{i\Theta}$ is an eigenvalue of $M(q^{i\tau})$ then

$$|\sin \alpha\Theta| = O\ (\tan \alpha\tau)$$

when $\tau > 0$. This is just what we want for the next lemma.

LEMMA. $\lim_{\tau\downarrow 0} M(q^{i\tau})$ *exists, and equals* $\lim_{\tau\downarrow 0} M^{-1}(q^{i\tau})$.

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Proof. Since

$$\left\langle \frac{(M(q^{i\tau}) - q^{2i\tau}M^{-1}(q^{i\tau}))}{1 - q^{2i\tau}} \Phi, \Phi \right\rangle = -\sum_j \frac{\sin \alpha(\Theta_j - \tau)}{\sin \alpha\tau} |\alpha_j|^2$$

we see that

$$\left\| \frac{M(q^{i\tau}) - 2q^{i\tau}M^{-1}(q^{i\tau})}{1 - q^{2i\tau}} \right\|$$

is bounded as $\tau \downarrow 0$, and referring to (4.4.1) we see that $\|M'(q^{i\tau})\|$ is as well. Using anew the formula

$$M(q^{i\tau'}) - M(q^{i\tau}) = i\alpha \int_\tau^{\tau'} M'(q^{i\tau})d\tau$$

we see that for $\tau \downarrow 0$, the limit of $M(q^{i\tau})$ exists. Differentiating the functional equation for $M(z)$ we find

$$M'(z^{-1}) = z^2M^{-1}(z)M'(z)M(z^{-1})$$

hence $M'(q^{i\tau})$ is bounded for $\tau < 0$. Repeating the argument above, we find that $\lim_{\tau\uparrow 0} M(q^{i\tau})$ exists. Since

$$|\sin \alpha\Theta| = O\ (\tan \alpha\tau) \quad \text{each } \Theta,$$

we see that

$$\|M(q^{i\tau}) - M^{-1}(q^{i\tau})\| \to 0 \quad \text{as } \tau \to 0$$

so that

$$\lim_{\tau\downarrow 0} M(q^{i\tau}) = \lim_{\tau\uparrow 0} M(q^{-i\tau}) = M(1),$$

and we are finished.

4.10. Proposition. *The points $z = \pm 1$ are removable singularities for $M(z)$.*

Proof. We know that

$$\lim_{\substack{\sigma\downarrow 0 \\ \tau\to 0}} M(q^{\sigma+i\tau}) = M(1) = \lim_{\tau\to 0} M(q^{i\tau}).$$

On the other hand $M(1)$ must be unitary, hence invertible, so from the functional equation

$$M(1) \lim_{\substack{\sigma\downarrow 0 \\ \tau\to 0}} M^{-1}(q^{\sigma+i\tau}) = I.$$

From this it follows from its definition for $|z| < 1$ that $M(z)$ is uniformly bounded in a neighbourhood of $z = 1$, and we are done. The argument for $z = -1$ is the same.

5. Rationality.

5.1. Let $\varphi(z) : \mathbf{C} \rightarrow V$ be a function with values in a finite dimensional complex vector space. We shall say that φ is *rational* if there is a polynomial $p(z)$ such that $p(z)\varphi(z)$ can be written in the form $a_0 + a_1 z + \ldots + a_n z^n$ where $a_i \in V$. If V comes equipped with an inner product, then it is equivalent to saying $\langle \phi(z), v \rangle$ is a rational function of z for each $v \in V$. In particular if $\varphi : \mathbf{C} \rightarrow \mathrm{End}_{\mathbf{C}}(W)$, W finite dimensional, then $\varphi(z)$ is rational if and only if all its matrix coefficients $\varphi_{ij}(z)$ (with respect to some basis of W) are rational functions of z.

Our preceding results tell us that $M(z)$ can be analytically continued to a meromorphic function on $\mathbf{C}^* = \mathbf{C}\backslash\{0\}$.

PROPOSITION. *The meromorphic function $M(z)$ defined on $\mathbf{C}^*$ can be continued to a rational function on $\mathbf{C}$.*

Proof. From 2.3.8 we know that $\|M(z)\|$ is bounded by a rational function of z as $|z| \rightarrow \infty$. From the functional equation for $M(z)$ this implies that $\|M^{-1}(z)\|$ is bounded by a rational function of z as $z \rightarrow 0$, hence that $M^{-1}(z)$ has at worst a pole at $z = 0$ (more precisely: $M^{-1}(z)$ can be extended to a function which has at worst a pole at $z = 0$). We can also infer from this that $\det M^{-1}(z)$ cannot vanish identically in a neighbourhood of $z = 0$ (because none of the matrix coefficients of $M^{-1}(z)$ do). Hence

$$M(z) = \mathrm{Adj}\, M^{-1}(z)/\det M^{-1}(z)$$

can also be extended to a function which has at worst a pole at $z = 0$. This means that $M(z)$ is a meromorphic function on all of $\mathbf{C}$, bounded by a rational function as $|z| \rightarrow \infty$, hence must be rational.

5.2. Combining the results of this chapter with 2.4.8 again, we have the following theorem.

THEOREM (i) *The endomorphism valued function $M(z)$, which is initially defined and analytic for $|z| > q^\rho$, can be analytically continued to a rational function in z which satisfies the functional equation*

$$M(z)M(z^{-1}) = I.$$

It is analytic on the set $[-q^\rho, -1) \cup (1, q^\rho]$ except possibly for simple poles, and it is analytic at $z = \pm 1$.

(ii) *The function $E(g, \Phi, z)$, which is initially defined and analytic for $|z| > q^\rho$, can be analytically continued to a rational function in z which satisfies the functional equation*

$$E(g, M(z)\Phi, z^{-1}) = E(g, \Phi, z).$$

It is analytic on the set $[-q^\rho, -1) \cup (1, q^\rho]$ except possibly for simple poles, and it is analytic at $z = \pm 1$.

4. The General Cuspidal Case.

In Chapter 3 we showed that the Eisenstein series defined in Chapter 2, together with their constant terms, could be analytically continued over all of $D_M(\xi)$ in case $P = NM$ was a maximal parabolic. In addition they were shown to satisfy certain functional equations. We shall use these results in this chapter to show that Eisenstein series of the most general type defined in Chapter 2 enjoy similar properties; of course similar results will be deduced for their constant terms. The results of this paper will be used in the next paper to show that more general Eisenstein series, not necessarily arising from cusp forms on M^0, also enjoy the requisite properties, and this will enable us to give a spectral decomposition for $\mathscr{L}(\xi)$.

The method used in this chapter is, modulo semantics, one due to Langlands (cf. [**13**], p. 151–160, and also [**12**] for an example). Since the details are sometimes the same, we shall be brief in places.

1. Some identifications.

1.1. Let $P = NM$ be a parabolic corresponding to $\Theta \subseteq \Delta$ (1.1). If α is one of the simple roots in $\Delta\backslash\Theta$, then there is a unique parabolic of rank one less than P and containing P which corresponds to the maximal split torus $\bigcap_{\beta\in\Theta} \ker \beta \cap \ker \alpha$; we shall denote it by P', or if necessary, by P_α. In this case $P \cap M'$ is then a parabolic subgroup of the group M', of rank 1, which we denote by ${}^{\psi}P$.

1.2. Suppose K is the maximal compact subgroup of $G(\mathbf{A})$, chosen long ago. If we consider the group $G(\mathbf{A}) \times K$ then it is apparent that the apparatus we have developed in earlier chapters can easily be adapted to this group. Here $G(F) \times \{1\}$ is the discrete group, and for "parabolics" we take subgroups of the form $P(\mathbf{A}) \times K$; all the relevant function spaces can be defined much as before. This however will become more evident as the discussion proceeds.

Let P' be as above, then $G(\mathbf{A}) = P'(\mathbf{A})K$. If ϕ is a function on $G(\mathbf{A})$ we can define a function $\tilde{\phi}$ on $P'(\mathbf{A}) \times K$ by

$$\tilde{\phi}(p', k) = \phi(pk^{-1}),$$

which of course defines a function $\tilde{\phi}$ on $M'(\mathbf{A}) \times K$; this function is right invariant by the group

$$K_0 = \{(k', k) | \, k \in P'(\mathbf{A}) \cap K, k' = \text{image of } k \text{ in } M'(\mathbf{A})\}.$$

Conversely a function $\tilde{\phi}$ invariant under right translations by K_0 gives rise to a function on $N'(\mathbf{A})\backslash G(\mathbf{A})$. Now suppose that

$$\phi \in \mathscr{C}_0(P, K', \omega\zeta\delta_P).$$

Then the function $\tilde{\phi}$ constructed above has the following properties.

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(i) $\tilde{\phi}$ is a measurable function on ${}^{\Psi}N(\mathbf{A}){}^{\Psi}P(F) \times \{1\}\backslash M'(\mathbf{A}) \times K$.

(ii) For each $(z, 1)z \in Z_{\Psi_M}(\mathbf{A})$,

$$\tilde{\phi}((m, k)(z, 1) = \omega\zeta\delta_P(z)\tilde{\phi}(m, k).$$

We observe here that ${}^{\Psi}M = M$.

(iii) If $k_1 \in \bigcap_{k \in K} kK'k^{-1} \cap M'(\mathbf{A})$, $k_2 \in \bigcap_{k \in K} kK'k^{-1}$ then

$$\tilde{\phi}(mk_1, kk_2) = \tilde{\phi}(m, k) \quad \text{each } (m, k) \in M'(\mathbf{A}) \times K.$$

We observe here that $\bigcap_{k \in K} kK'k^{-1}$ is a subgroup of finite index in K.

(iv) There is an analogous cusp form condition on $M^0 \times K$

(v) $$\int_K \int_{{}^{\Psi}N(\mathbf{A}){}^{\Psi}P(F)\, Z_M(\mathbf{A})\backslash M'(\mathbf{A})} |\tilde{\phi}(m', k)|^2 \delta_P^{-2}(m')|\zeta^{-2}|(m')dm'dk < \infty.$$

Conversely, such a function $\tilde{\phi}$ satisfying conditions (i)–(v) above gives rise to a function $\phi \in \mathscr{C}_0(P, K', \omega\zeta\delta_P)$.

2. The case of a reflection.

2.1. Let P, Θ be as above, and P' the parabolic subgroup corresponding to $\Theta \cup \{\alpha\} = \Psi$ as before. In 1.1.8 we constructed an element $s = w_0 = w_\Psi w_\Theta$; the proof of Proposition 1.1.8 implies that $s(\Theta) > 0$ but $s(\alpha) < 0$. Moreover in Proposition 1.3.3 we showed that if Θ' was arbitrary then any $w \in W(\Theta, \Theta')$ could be written (not necessarily uniquely) in the form $s_n s_{n-1} \ldots s_1$ where each s_i was a w_0 with respect to some Ψ. Let us call such s_i reflections, by analogy with the usual case. We are going to show in this section that if s is a reflection then $E(g, \Phi_\zeta)$ is meromorphic on a certain convex hull related to s and that on this same set

$$E(g, N(s, \zeta)\Phi_\zeta) = E(g, \Phi_\zeta). \tag{2.1.1}$$

Moreover we shall show that the constant term $E^Q(g, \Phi_\zeta)$ is given by the expression in (2.3.1) on this set, and that if $t \in W(\overline{\Theta}, \Theta')$ is any element, then

$$M(t, s\zeta)M(s, \zeta) = M(ts, \zeta).$$

The general case will be shown to follow easily from such reflections in Section 3.

2.2. We begin with some remarks of a general nature, which will be formalized in the sequel. Let P be a parabolic, $P = NM$, and let $P \subseteq Q = N_Q M_Q$; then $P \cap M_Q$ is a parabolic subgroup of the group M_Q. Let us write ${}^{\Psi}P$ for this last parabolic subgroup so that ${}^{\Psi}P = {}^{\Psi}N{}^{\Psi}M$. The group $X_M(\mathbf{R})$ can be identified with the homomorphisms

$$\chi : Z(\mathbf{A})Z_M(F)\backslash Z_M(\mathbf{A}) \rightarrow \mathbf{R}_+^*.$$

Similarly the group $X_{\Psi_M}(\mathbf{R})$ can be identified with the group of quasi-

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characters

$$\chi : Z_{M_Q}(\mathbf{A})Z_{\psi_M}(F)\backslash Z_{\psi_M}(\mathbf{A}) \to \mathbf{R}_+^*.$$

The group $X_{M_Q}(\mathbf{R}) = X_{M}'(\mathbf{R})$ can be identified in the same way as $X_M(\mathbf{R})$. By means of these identifications we get a projection

$$p : X_M(\mathbf{R}) \twoheadrightarrow X_{M}'(\mathbf{R}).$$

If $j : X_{M}'(\mathbf{R}) \times X_M(\mathbf{R})$ is the natural injection mentioned in 1.2.3, then $p \,.\, j = \mathrm{id}$, and the complement of $X_{M}'(\mathbf{R})$ in $X_M(\mathbf{R})$ can be identified with $X_{\psi_M}(\mathbf{R})$ in a natural way. We write $\zeta = {}^{\psi}\zeta + \zeta'$ to denote this splitting. Note that

$$\langle \chi, j^* H_{M'}(g) \rangle = \langle j\chi, H_{M'}(g) \rangle.$$

2.3. Now fix P, Θ, P', α, Ψ as in 2.1. We shall need a lemma which is a simple variant of 2.2.2. To avoid unnecessary notation we shall write $e(x)$ for the expression $\exp((\log q)x)$. We also write ${}^{\psi}W(P, Q)$ for the set of elements of $M'(F)$ which send $M_P = M_{\psi_P}$ to $M_Q = M_{\psi_Q}$; in particular an element of ${}^{\psi}W(P, Q)$ fixes $Z_{M'}(\mathbf{A})$ pointwise.

LEMMA. (i) *Let* $\phi : N'(\mathbf{A})P'(F)\backslash G(\mathbf{A}) \to \mathbf{C}$ *be a function which satisfies*

$$|\phi(m'k)| \leqq ce(\langle H_M(m'), {}^{\psi}\lambda \rangle)$$

whenever $m' \in {}^{\psi}\mathfrak{S}$, $k \in K$, *where* ${}^{\psi}\mathfrak{S}$ *is a Siegel domain for* $M'(\mathbf{A})$ *with respect to* ${}^{\psi}P_0 = P_0 \cap M'$. *Then the series*

$$\sum_{\gamma \in P'(F) \backslash G(F)} \phi(\gamma g) e(\langle H_{M'}((\gamma g), \lambda' \rangle)$$

converges absolutely provided that $\mathrm{Re}\, \lambda \in C_P + 2\delta_P$.

(ii) *Suppose that* ϕ *is as above except that*

$$|\phi(m'k)| \leqq c\{e(\langle H_M(m'), {}^{\psi}\lambda + {}^{\psi}\delta_P \rangle) + e(\langle H_{M_Q}(m'), s{}^{\psi}\lambda + {}^{\psi}\delta_Q \rangle)\}$$

$Q \in \{P\}$, $s \in {}^{\psi}W(P, Q)$. *Then*

$$\sum_{P'(F) \backslash G(F)} \phi(\gamma g) e(\langle H_{M'}(\gamma g), \lambda' + \delta_{P'} \rangle)$$

converges absolutely provided that $\mathrm{Re}\, \lambda$ *belongs to the convex hull of*

$$(s^{-1}(C_Q + 2\delta_Q) - \delta_P) \cap s^{-1}(C_Q + \delta_Q) \quad \textit{and}$$
$$(C_P + \delta_P) \cap (C_P + 2\delta_P - s^{-1}\delta_Q).$$

Proof. Of course we may suppose that the constant $c = 1$ for the purpose of argument, and that λ is real.

(i) If $m' \in {}^{\psi}\mathfrak{S}$, $k \in K$ then

$$|\phi(m'k)| \leqq e(\langle H_M(m'), {}^{\psi}\lambda \rangle)$$

implies that if $g = \gamma n'm'h$ with $\gamma \in M'(F)$, $n' \in N'(\mathbf{A})$, $m' \in {}^{\psi}\mathfrak{S}$, then

$$|\phi(g)| \leqq e(\langle H_M(g), {}^{\psi}\lambda\rangle)$$

because $H_M(g) = H_M(m')$. Now

$$e(\langle H_M(g), {}^{\psi}\lambda\rangle) \leqq \sum_{\gamma \in P(F)\backslash P'(F)} e(\langle H_M(\gamma g), {}^{\psi}\lambda\rangle)$$

so that

$$e(\langle H_M(g), \lambda' + {}^{\psi}\lambda\rangle) \leqq e(\langle H_M(g), \lambda'\rangle) \sum_{P(F)\backslash P'(F)} e(\langle H_M(\gamma g), {}^{\psi}\lambda\rangle).$$

It follows that

$$\sum_{\gamma \in P'(F)\backslash G(F)} |\phi(\gamma g)| e(\langle H_{M'}(\gamma g), \lambda'\rangle) \leqq \sum_{\delta \in P(F)\backslash G(F)} e(\langle H_M(\delta g), \lambda\rangle)$$

because

$$\langle H_{M'}(\gamma g), \lambda'\rangle = \langle H_M(\gamma g), \lambda'\rangle \quad \text{and} \quad \lambda = {}^{\psi}\lambda + \lambda'.$$

Since the right hand side converges by 2.2.2, the result follows.

(ii) Our framework implies that the inequality in the lemma can be written

$$|\phi(m'k)| \leqq e(\langle H_M(m'), {}^{\psi}\lambda + {}^{\psi}\delta_P\rangle) + e(\langle H_{M_Q}(m'), s^{\psi}\lambda + {}^{\psi}\delta_Q\rangle)$$

where Q is the parabolic subgroup of G such that $Q \cap M'$ corresponds to the conjugate of $P \cap M'$ (see 1.1.8 for the terminology); thus ${}^{\psi}W(P, Q) = \{s\}$, ${}^{\psi}W(P) = \{1\}$. Suppose first that

$$\lambda \in s^{-1}(C_Q + 2\delta_Q) - \delta_P.$$

Write

$$s\lambda = \nu + 2\delta_Q - s\delta_P;$$

the first term on the right hand side of the inequality can be written

$$e\langle H_{M_Q}(m'), {}^{\psi}\nu + 2\delta_{\psi_Q}\rangle$$

and summing twice over successive cosets we find

$$\sum_{\delta \in P'(F)\backslash G(F)} e(\langle H_{M'}(\delta g), \lambda' + \delta_{P'}\rangle) \sum_{Q(F)\backslash P'(F)} e(\langle H_{M_Q}(\gamma\delta g), {}^{\psi}\nu + 2\delta_{\psi_Q}\rangle)$$
$$= \sum_{Q(F)\backslash G(F)} e(\langle H_{M_Q}(\gamma g), \nu' + 2\delta_{P'} + {}^{\psi}\nu + 2\delta_{\psi_Q}\rangle), \quad (\lambda' = \nu' + \delta_{P'})$$

for the same reasons as in (i). The quasicharacter on the right hand side is just $\nu + 2\delta_Q$, so the series converges by 2.2.2 again. As for the second term on the right hand side of the inequality, if $\lambda \in s^{-1}(C_Q + \delta_Q)$ it follows easily as in (i) that proceeding as above we obtain a series which converges by 2.2.2. Thus if

$$\lambda \in (s^{-1}(C_Q + 2\delta_Q) - \delta_P) \cap s^{-1}(C_Q + \delta_Q) \neq \emptyset$$

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then the series in the statement of the lemma part (ii) will converge absolutely. Similarly if

$$\lambda \in (C_P + \delta_P) \cap ((C_P + 2\delta_P) - s^{-1}\delta_Q) \neq \emptyset$$

the statement in the lemma holds true, and using convexity of the exponential function the general case follows.

2.4. Let $\Phi \in \mathscr{C}_0(P, K', \omega\zeta\delta_P)$; we may write this in the form

$$e\langle H_{M'}(g), \zeta' + \delta_{P'}\rangle\Phi'(g)$$

with

$$\Phi'(g) = e\langle H_{M'}(g), -\zeta' - \delta_{P'}\rangle\Phi(g).$$

The function Φ' is again a function on $N'(\mathbf{A})P'(F)\backslash G(\mathbf{A})$ and it transforms on $Z_{M'}(\mathbf{A})$ via the character ω. The Eisenstein series $E(g, \Phi)$ can be written

$$\text{(2.4.1)} \quad E(g, \Phi) = \sum_{P'(F)\backslash G(F)} e\langle H_{M'}(\delta g), \zeta' + \delta_{P'}\rangle \sum_{P(F)\backslash P'(F)} \Phi'(\gamma\delta g).$$

The second sum on the right can be identified with an Eisenstein series $E(m', k), \tilde{\Phi}')$ arising from

$$\tilde{\Phi}' : ({}^{\psi}P(F)\,{}^{\psi}N(\mathbf{A})\backslash M'(\mathbf{A})) \times K \to \mathbf{C}$$

which transforms via the character ω on $Z_{M'}(\mathbf{A}) \times \{1\}$. We are going to apply 2.3 (ii) to the function $E(g, \Phi')$ on $N'(\mathbf{A})P'(F)\backslash G(\mathbf{A})$ which corresponds to $E((m', k), \tilde{\Phi}')$; we know that $\{{}^{\psi}P\}$ consists of just two elements ${}^{\psi}P$, ${}^{\psi}Q$, and ${}^{\psi}W(P, Q)$ consists of a unique element, namely s. We also know that on the Siegel domain ${}^{\psi}\mathfrak{S}$, any reasonable function becomes equal to the appropriate constant term if one goes far enough in the appropriate direction (1.5.9).

Let ${}^{\psi}\omega = \omega|_{Z_{\psi_M}(\mathbf{A})}$, and $\omega' = \omega|_{Z_{M'}(\mathbf{A})}$ so that ${}^{\psi}\omega$ is a character of $Z_{\psi_M}(\mathbf{A})$ which prolongs ω': there is a connected component of $D_{\psi_M}(\omega')$ in which ${}^{\psi}\omega$ lies. In view of what we know about the Eisenstein series in one variable, we can choose a polynomial $p({}^{\psi}\zeta)$ so that

$$p({}^{\psi}\zeta)E(m', k), \tilde{\Phi}') \quad \text{and}$$
$$p({}^{\psi}\zeta)E^{\psi Q}((m', k), \tilde{\Phi}'), p({}^{\psi}\zeta)E^{\psi P}((m', k), \tilde{\Phi}')$$

are holomorphic on this connected component of $D_{\psi_M}(\omega')$ (in fact, on each connected component of $D_{\psi_M}(\omega')$). Let U be a bounded subset of this component, then 2.4.8 and the results of 3.5.2 together with the above remarks imply that for ${}^{\psi}\zeta \in U$, $m' \in {}^{\psi}\mathfrak{S}$ and $k \in K$, we have

$$\text{(2.4.2)} \quad |p({}^{\psi}(\zeta))E((m', k), \tilde{\Phi}')| \leqq c(\chi_1 + e\langle H_M(m'), {}^{\psi}\zeta + {}^{\psi}\delta_P\rangle + e\langle H_{M_Q}(m'), s^{\psi}\zeta + {}^{\psi}\delta_Q\rangle$$

where c is some positive constant and χ_1 is the characteristic function of a set of the form $S \times K$ where $S \subseteq {}^{\psi}\mathfrak{S}$ is compact modulo $Z_{M'}(\mathbf{A})$. We can ignore χ_1 in the application of 2.3 (ii) because of this last property; applying the lemma we find that the expression (2.4.1) multiplied by $p({}^{\psi}\zeta)$ converges absolutely provided ζ lies in the convex hull alluded to in 2.3 (ii). This implies that $E(g, \Phi)$ is meromorphic on this set. In fact, we can do better: if ζ belongs to the convex hull of 2.3 (ii) and if $M(s, {}^{\psi}\zeta)$ is analytic at ${}^{\psi}\zeta$, then the expression on the right hand side of (2.4.1) will converge absolutely. This is simply because we shall obtain the estimate (2.4.2) without $p({}^{\psi}\zeta)$ in this case, by using the same arguments. It is a straightforward matter to verify that for $\operatorname{Re} \zeta \in C_P + \delta_P$ we have

$$N(s, \zeta)\Phi_\zeta = e\langle \bar{H}_{M_Q}(\quad), \zeta' + \delta_{P'}\rangle N(s, {}^{\psi}\zeta)\Phi_{\zeta}';$$

for this we refer the reader to e.g. [**11**] Lemma 108. This provides us with an analytic continuation of $M(s, \zeta)$. Moreover, on the convex hull of 2.3 (ii) we have the functional equation

$$E(g, N(s, \zeta)\Phi, {}^{s}\zeta) = E(g, \Phi_\zeta)$$

simply because this relation holds in the one variable case.

Finally, we note that by uniqueness of analytic continuation all our functions are invariant by the lattice iL_{Z_M}.

2.5. Let $\Phi \in \mathscr{C}_0(P, K', \omega\delta_P)$, and let R be any element of $\{P\}$; if $\operatorname{Re} \zeta \in C_P + \delta_P$, then $E^R(g, \Phi_\zeta)$ is equal to

$$\sum_{w \in W(M, M_R)} e\langle \bar{H}_{M_R}(g), w\zeta\rangle M(w, \zeta)\Phi(g) = \sum_{w \in W(M, M_R)} N(w, \zeta)\Phi_\zeta(g).$$

By analytic continuation this relation persists on the convex hull of 2.3 (ii). Substitute the functional equation that we have into $E^R(g, \Phi_\zeta)$ to obtain

$$\sum_{w \in W(Q, R)} N(w, s\zeta)N(s, \zeta)\Phi_\zeta = \sum_{w \in W(P, R)} N(ws, \zeta)\Phi_\zeta.$$

This means that $N(w, s\zeta)\ N(s, \zeta) = N(ws, \zeta)$, and we have now proved everything claimed in 2.1.

3. The general case.

3.1. Let w be any element of $W(P, R)$, $R \in \{P\}$. In 1.3.3 we showed that $w = w_n \ldots w_1$ where each w_i had the form given in the preceding section. Suppose $n = 1$ for the moment. In 2.4 we observed that

$$N(w, \zeta)\Phi_\zeta = e\langle \bar{H}_{M_Q}(\quad), \zeta' + \delta_{P'}\rangle N(w, {}^{\psi}\zeta)\Phi_{\zeta}';$$

consequently, $\mathrm{M}(w, \zeta)$ "depends" only on a single variable, and extends to a rational function. Moreover, we now know that on the convex hull in 2.3 (ii), that for $n > 1$,

$$M(w, \zeta) = M(w_n, \ldots, w_2, w_1\zeta)M(w_1, \zeta).$$

By induction, both expressions on the right are meromorphic functions on $iL_{Z_M}\backslash X_M(\mathbf{C})$, so that the left side admits an analytic continuation.

To obtain the functional equation, proceed by induction again:

$$M(tw, \zeta) = M(tw_n \dots w_1, \zeta) = M(tw_n \dots w_2, w_1\zeta)M(w_1, \zeta)$$

from Section 2. Now

$$M(tw_n \dots w_2, w_1\zeta) = M(t, w_n \dots w_2w_1\zeta)M(w_n \dots w_2, w_1\zeta),$$

by induction so that

$$M(tw, \zeta) = M(t, w\zeta)M(w_n \dots w_2, w_1\zeta)M(w_1, \zeta)$$
$$= M(t, w\zeta)M(w, \zeta),$$

by Section 2 again.

3.2. Let us agree to define a hyperplane to be the image in $iL_{Z_M}\backslash X_M(\mathbf{C})$ of a hyperplane in $X_M(\mathbf{C})$; we shall now show that the singularities of $M(w, \zeta)$ lie along a finite number of hyperplanes. If $w = w_n \dots w_1$ as above and $n = 1$, this is certainly true. In general,

$$M(w, \zeta) = M(w_n \dots w_2, w_1\zeta)M(w_1, \zeta)$$

and we conclude by induction.

3.3. Suppose that $f : iL_{Z_M}\backslash X_M(\mathbf{C}) \to V$ is a function with values in a finite dimensional complex inner product space, whose singularity set is contained in a finite set of hyperplanes. Let us say that f is *rational* if we can find coordinates $s_1, \dots, s_n$ for $X_M(\mathbf{C})$ such that

$$(s_1, \dots, s_n) \to (q^{s_1}, \dots, q^{s_n})$$

is a isomorphism of $L_{Z_M}\backslash X_M(\mathbf{C})$ onto $(\mathbf{C}^*)^n$, and a polynomial $p(z_1, \dots, z_n)$ where $z_i = q^{s_i}$ such that for each $v, w \in W$, the map

$$(z_1, \dots, z_n) \to p(z_1, \dots, z_n) \langle f(z_1, \dots, z_n)v, w\rangle$$

is a polynomial. It is evident that if f is rational for one basis of $X_M(\mathbf{C})$, it is rational for any other.

Write

$$M(w, \zeta) = M(w', w_1\zeta)M(w_1, \zeta)$$

where $w = w'w_1$ as above. Let us indicate how to choose coordinates so that $M(w, \zeta)$ is rational. Suppose $w = w_2w_1$, where the w_i are reflections. Suppose that w_1 corresponds to the root α, and w_2 to the root β, and write

$$M(w, \zeta) = M(w_2, w_1\zeta)M(w_1, \zeta).$$

We know that $w_1\zeta = \zeta' + w_1{}^{\psi}\zeta$ (using the notation of Section 2). From 3.1 we see that we can pick a variable s_1 so that ${}^{\psi}\zeta$ corresponds to s_1 and $w_1s_1 = -s_1$. From this we see that $M(w_2, w_1\zeta)$ is a rational func-

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tion for a suitable choice of coordinates, and that $M(w_1, \zeta)$ is a rational function for another choice of coordinates. The remark above concerning preservation of rationality under change of basis then implies $M(w, \zeta)$ is also a rational function. In general, if $w = w_n \ldots w_1$, we write

$$M(w, \zeta) = M(w_n, w_{n-1} \ldots w_2 w_1 \zeta) M(w_{n-1}, w_{n-2} \ldots w_1 \zeta) \ldots M(w_1, \zeta)$$

and argue as above to show that $M(w, \zeta)$ is rational.

3.4. Finally, we argue as Harder does in [**10**] Theorem 1.6.6 (see 2.4.8) to see that the analytic behaviour of $E(g, \Phi_\zeta)$ is no worse than that of $M(w, \zeta)$, that $E(g, \Phi_\zeta)$ can be analytically continued to a rational function on each component of $D_M(\xi)$, and that it satisfies the requisite functional equations.

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