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INTEGRAL REPRESENTATIONS AND COMPLETE MONOTONICITY OF VARIOUS QUOTIENTS OF BESSEL FUNCTIONS

MOURAD E. H. ISMAIL

1. Introduction. Complete monotonicity of functions, Definition 3.1, is often proved by showing that their inverse Laplace transforms are non-negative. There are relatively few simple functions whose inverse Laplace transforms can be expressed in terms of standard higher transcendental functions. Inverting a Laplace transform involves integrating a complex-valued function over a vertical line, and establishing the positivity of the resulting integral can be tricky. Sometimes asymptotic methods are helpful, see for example Fields and Ismail [**6**]. The Stieltjes transform

$$\mathscr{G}(d\mu, z) = \int_{0+}^{\infty} \frac{d\mu(t)}{z+t}, \quad |\arg z| < \pi, \quad \int_{0}^{\infty} |d\mu| < \infty,$$

is nothing but, at least formally, a two-fold Laplace transform, since

$$\mathscr{G}(d\mu, z) = \int_{0+}^{\infty} \int_{0+}^{\infty} e^{-xu} e^{-ut} du d\mu(t).$$

The Stieltjes transform has a very simple inversion formula, Lemma 3.6, which is a great advantage over the Laplace transform. The function $x^{-1/2} \cdot K_\nu(\sqrt{x})/K_{\nu+1}(\sqrt{x})$ illustrates this approach because its inverse Laplace transform, even in the simple case $\nu = 1/2, 3/2, 5/2, \ldots$ when $K_\nu(x)$ is a polynomial in x, is too complicated to be useful while its inverse Stieltjes transform, for $\nu > -1$, is

$$\frac{2t^{-1}}{\pi^2} \{J_{\nu+1}{}^2(\sqrt{t}) + Y_{\nu+1}{}^2(\sqrt{t})\}^{-1} dt,$$

see Grosswald [**7**] and Ismail [**11**]. The non-negativity of the inverse Stieltjes transform proves that both $x^{-1/2}K_\nu(\sqrt{x})/K_{\nu+1}(\sqrt{x})$ and its inverse Laplace transforms are completely monotonic for $\nu > -1$. The complete monotonicity of $x^{-1/2}K_\nu(\sqrt{x})/K_{\nu+1}(\sqrt{x})$ does imply the infinite divisibility of the student t-distribution of $2\nu + 2$ degrees of freedom. Infinitely divisible distri-

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butions are defined on page 173 of Feller [**5**]. For connections with the Bessel polynomials we refer the interested reader to Ismail and Kelker [**12**]. The present author [**11**] used the Stieltjes transform to show that $x^{-1/2}K_\nu(\sqrt{x})/K_{\nu+1}(\sqrt{x})$ is completely monotonic for all real ν. In another work, in preparation, the author and D. Kelker used the same method to prove the infinite divisibility of certain F distributions. That result is equivalent to the complete monotonicity of a quotient of two parabolic cylinder functions. Completely monotonic functions are encountered in several branches of mathematics beside probability theory. Askey's monograph [**1**] contains several problems on the (C, α) summability of Jacobi series and harmonic analysis whose solution reduces to showing that certain functions are either non-negative or completely monotonic.

The present work starts by evaluating the inverse Stieltjes transform of several quotients of Bessel functions. This is done in Section 4. These formulas, which I could not find in the standard tables, may be considered as integral representations for these various quotients of Bessel functions. In Section 5 we use these representations to establish the complete monotonicity of several functions. For example, it is shown that $K_\mu(\sqrt{x})/K_\nu(\sqrt{x})$, for $\mu \geqq |\nu|$, is a completely monotonic function of x. The only previously known result about this quotient is that it is decreasing. This was proved by L. Lorch [**14**] and independently by Hartman and Watson [**9**]. It might be of interest to mention that Hartman and Watson encountered this quotient in their study of normal distribution functions on spheres while Ismail and Muldoon encountered it in attempting to show that the eigenvalues of certain eigenvalue problems in n-dimensions increase with the dimension, a problem which came from genetics, see [**13**] for details. Ismail and Muldoon also encountered the monotonicity of $K_{\alpha+\nu}(x)/K_{\nu+\beta}(x)$ as a function of ν and $x^{-\beta}K_\nu(x)/K_{\nu+\beta}(x)$ as a function of x, where $x > 0$, $\beta \geqq \alpha \geqq 0$, $\nu + \beta > 0$. They proved that both functions are decreasing. In Section 5 we show that the former is a completely monotonic function of ν^2, when $\beta - \alpha$ is a positive integer while the latter is a completely monotonic function of x^2 for $\beta > 0$ and ν real. Related results have been obtained recently by Hartman [**8**]. For example he showed, using differential equations methods, that both $1/K_\mu(t)$ and $K_\mu(\tau)/K_\mu(t)$ are completely monotonic functions of μ^2 for $0 < t < \tau < \infty$. He also proved that $t^\mu/\Gamma(1 + \mu)$, for $t > 0$, is a completely monotonic function of μ^2 if and only if $0 < t \leqq e^{-\gamma}$, where γ is the Euler constant.

Our new integral representations are stated in Section 2 and proved in Section 4. Formula (2.3) was proved in the special case $\nu = 0$ by Carslaw and Jaeger [**2**] in their study of heat conduction problems. In Section 3, we include all the preliminary results needed in the proofs. Most of this preliminary material are standard results in mathematical analysis and were included for the sake of completeness. As we mentioned earlier, Section 5 is devoted to applying the integral representations of Section 2 to complete monotonicity problems.

2. Integral representations. This section contains the statements of several integral representations for quotients of Bessel functions. These formulas are actually new entries for the Stieltjes transforms tables. Throughout this section it is assumed that $|\arg z| < \pi$. The sought integral representations are the following

$$z^{-\beta/2}\frac{K_\nu(\sqrt{z})}{K_{\nu+\beta}(\sqrt{z})} = \frac{1}{\pi}\int_0^\infty \frac{J_{\nu+\beta}(\sqrt{t})Y_\nu(\sqrt{t}) - J_\nu(\sqrt{t})Y_{\nu+\beta}(\sqrt{t})}{(z+t)t^{\beta/2}\{J_{\nu+\beta}{}^2(\sqrt{t}) + Y_{\nu+\beta}{}^2(\sqrt{t})\}}dt, \tag{2.1}$$

where $\beta > 0$ and

(i) $\beta < 1$, if $\nu + \beta \leqq 0$, and (ii) $\nu > -1$, if $0 > \nu \geqq -\beta$.

$$\frac{K_{\nu+\beta}(\sqrt{z})}{K_\nu(\sqrt{z})} = 1 + \frac{1}{\pi}\left\{\sin\left(\frac{\beta\pi}{2}\right)\int_0^\infty \frac{\Lambda_1(t)dt}{z+t} + \cos\left(\frac{\beta\pi}{2}\right)\int_0^\infty \frac{\Lambda_2(t)dt}{z+t}\right\}, \tag{2.2}$$

with $\beta > 0$ when $-\beta < \nu < \min(0, 1 - \beta/2)$ and β is restricted to $\beta \in (0, 2)$ if $\nu \geqq 0$, where

$$\Lambda_1(t) = \{J_\nu(\sqrt{t})J_{\nu+\beta}(\sqrt{t}) + Y_\nu(\sqrt{t})Y_{\nu+\beta}(\sqrt{t})\}/\{J_\nu{}^2(\sqrt{t}) + Y_\nu{}^2(\sqrt{t})\},\quad t > 0$$

$$\Lambda_2(t) = \{J_\nu(\sqrt{t})Y_{\nu+\beta}(\sqrt{t}) - J_{\nu+\beta}(\sqrt{t})Y_\nu(\sqrt{t})\}/\{J_\nu{}^2(\sqrt{t}) + Y_\nu{}^2(\sqrt{t})\},\quad t > 0.$$

$$\frac{K_\nu(a\sqrt{z})}{zK_\nu(b\sqrt{z})} - \frac{1}{z}\left(\frac{b}{a}\right)^\nu = \frac{1}{\pi}\int_0^\infty \frac{J_\nu(a\sqrt{t})Y_\nu(b\sqrt{t}) - J_\nu(b\sqrt{t})Y_\nu(a\sqrt{t})}{(z+t)\{J_\nu{}^2(b\sqrt{t}) + Y_\nu{}^2(b\sqrt{t})\}t}dt, \tag{2.3}$$

where $a > b > 0$ and $\nu \geqq 0$.

$$\begin{aligned} z^{\nu/2}K_{\nu+\beta}(\sqrt{z}) &= \frac{1}{2}\int_0^\infty \frac{t^{\nu/2}}{z+t} \\ &\quad \times \left\{\cos\left(\frac{\pi\beta}{2}\right)J_{\nu+\beta}(\sqrt{t}) + \sin\left(\frac{\pi\beta}{2}\right)Y_{\nu+\beta}(\sqrt{t})\right\}dt, \end{aligned} \tag{2.4}$$

for $\beta \in [0, 2)$, $\nu + \beta \geqq 0$.

$$\frac{K_{a+\sqrt{\nu}}(x)}{K_{b+\sqrt{\nu}}(x)} = \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im}\{K_{a-i\sqrt{t}}(x)\,K_{b+i\sqrt{t}}(x)\}dt}{(\nu+t)|K_{b+i\sqrt{t}}(x)|^2}, \tag{2.5}$$

where $\beta > \alpha \geqq 0, x > 0$ and $|\arg \nu| < \pi$.

Formula (2.1) was proved by the author [**11**] in the special case $\beta = 1$ and $\nu + 1 \geqq 0$. For another related formula, see [**11**]. Carslaw and Jaeger proved (2.3) when $\nu = 0$. The relationship (2.4) when $\beta = 0$ appears in [**4**, (58) p. 96] and in Whittaker and Watson [**18**, problem 36, p. 383] when $\nu = \beta = 0$.

3. Preliminaries. The present section contains standard results on completely monotonic functions, Stieltjes transforms and Bessel functions.

Definition 3.1. A function $f(x)$ defined on $(0, \infty)$ and having continuous derivatives of all orders there is *completely monotonic* if $(-1)^n d^n f(x)/dx^n \geqq 0$ for $x > 0$.

LEMMA 3.2. (Bernstein's Theorem). *A function $f(x)$ is completely monotonic if and only if f has the representation*

$$f(x) = \int_{0+}^{\infty} e^{-xt} d\alpha(t)$$

where $\alpha(t)$ is nondecreasing and the integral converges for $x \in (0, \infty)$.

Proof. See Widder [**19**, p. 161].

Definition 3.3. A function $\alpha(t)$ of bounded variation on $[0, \infty)$ is called *normalized* if $\alpha(0) = 0, \alpha(t) = \frac{1}{2}\{\alpha(t^+) + \alpha(t^-)\}$.

LEMMA 3.4. (Uniqueness theorem for Laplace transforms). *Let $\alpha_1(t)$ and $\alpha_2(t)$ be two normalized functions. If*

$$\int_0^{\infty} e^{-xt} d\alpha_1(t) = \int_0^{\infty} e^{-xt} d\alpha_2(t) \quad \textit{for } x \geqq \sigma,$$

for some finite σ, then $\alpha_1(t) = \alpha_2(t)$.

Proof. See Widder [**19**, p. 63].

LEMMA 3.5. (Representation theorem for Stieltjes transforms). *In order for a function $F(z)$ to be a Stieltjes transform*

$$(3.1) \quad F(z) = \int_0^{\infty} \frac{d\mu(t)}{z + t}$$

it is sufficient that

(i) *$F(z)$ is analytic in $|\arg z| < \pi$.*

(ii) *$F(z) = o(1)$ as $|z| \to \infty$ and $F(z) = o(|z|^{-1})$ as $|z| \to 0$, uniformly in every sector $|\arg z| \leqq \pi - \epsilon, \epsilon > 0$.*

Proof. See Hirschman and Widder [**10**, pp. 235 and 238].

LEMMA 3.6. (Inversion of the Stieltjes transform). *If $F(z)$ satisfies* (3.1) *then the normalized μ is given by*

$$\mu(t_2) - \mu(t_1) = \lim_{\eta \to 0^+} \frac{1}{2\pi i} \int_{t_1}^{t_2} \{F(-t - i\eta) - F(-t + i\eta)\} dt.$$

Proof. See Stone [**16**].

Remark 3.7. If

$$f(x) = \int_0^{\infty} e^{-xt} d\alpha(t), \quad \int_0^{\infty} |d\alpha| < \infty, \quad \text{and} \quad \int_0^{\infty} f(x) dx$$

exist, then the Laplace transform of $f(x)$ is the Stieltjes transform of $d\alpha$, that is

$$\int_0^\infty e^{-xu} f(u)du = \int_0^\infty \frac{d\alpha(t)}{x+t}.$$

This well known observation follows from Fubini's theorem.

LEMMA 3.8. *On every sector* $|\arg z| \leqq 2\pi - 2\epsilon$, $\epsilon > 0$, *we have*

$$(3.2) \quad K_\mu(z) \sim (\pi/2z)^{1/2} e^{-z} \quad \text{as } |z| \to \infty,$$

$$(3.3) \quad K_\mu(z) \sim C_1 |z|^{-|\mu|} \quad \text{as } |z| \to 0,\ \mu \neq 0, \quad \text{and}$$

$$(3.4) \quad K_0(z) \sim C_2 \ln |z| \quad \text{as } |z| \to 0,$$

uniformly, where C_1 *and* C_2 *are nonzero constants.*

Proof. This follows from (4) p. 24, (12) and (13) p. 5, and, (37) and (38) p. 9 of Erdelyi et al. [**4**].

LEMMA 3.9. *For* $x > 0$ *and as* $\nu \to \infty$ *in the sector* $|\arg \nu| \leqq \pi/2 - \delta$, δ *is arbitrary*, $0 < \delta < \pi/2$, *we have*

$$(3.5) \quad K_\nu(x) \sim 2^\nu \nu^\nu e^{-\nu} x^{-\nu} \sqrt{\frac{\pi}{2\nu}}.$$

Proof. We apply Watson's lemma (see Theorem 3.3, p. 113 in Olver [**15**]) to the integral representation [**4**, p. 83]

$$K_\nu(x) = \pi^{-1/2}(2x)^\nu \Gamma(\nu + \tfrac{1}{2}) \int_0^\infty (t^2 + x^2)^{-\nu-1/2} \cos t\, dt, \quad \operatorname{Re} \nu > -\tfrac{1}{2},$$

by writing $(t^2 + x^2)^{-\nu-1/2}$ as $x^{-2\nu-1}(1 + t^2/x^2)^{-1/2} \exp\{-\nu \ln (1 + t^2/x^2)\}$ and change the variable t to $u = \ln (1 + t^2/x^2)$. Straightforward manipulation will establish (3.5) upon using the asymptotic expansion of $\Gamma(\nu + \frac{1}{2})$, see for example Erdelyi et al. [**3**, p. 47].

Remark 3.10. We believe that Lemma 3.9 must be known, but we could only find the special case $\nu \to +\infty$ in the standard books and tables.

4. Proofs of the integral representations. The current section contains only proofs of (2.1) – (2.5). The reader who is interested in the consequences of (2.1) – (2.5) is advised to proceed to the next section.

Proof of (2.1). Let

$$(4.1) \quad F(\nu, \beta, z) = z^{-\beta/2} K_\nu(\sqrt{z})/K_{\nu+\beta}(\sqrt{z}).$$

We first show that the above function is a Stieltjes transform, then use the Stieltjes inversion formula to calculate its inverse. Condition (i) of Lemma 2.5 is satisfied since $K_\mu(z)$ has no zeros in $|\arg z| \leqq \pi/2$ (see Watson [**17**, p. 511]). As $|z| \to \infty$, formula (3.2) implies that $F(\nu, \beta, z)$ is $O(|z|^{-\beta/2})$, hence is $o(1)$.

Now we come to the behaviour of F near $z = 0$. If $\nu \geqq 0$, then $F(\nu, \beta, z) = O(\ln z)$, by (3.3) and (3.4). Thus $F(\nu, \beta, z) = o(|z|^{-1})$ for $\nu \geqq 0$. If $\beta + \nu < 0$, then $\nu < 0$ and the relationships (3.3) and (3.4) combined with

(4.2) $\quad K_{-\mu}(z) = K_\mu(z)$

(see Watson [**17**, (8) p. 79]), yield $F(\nu, \beta, z) = O(|z|^{-\beta})$, hence is $o(|z|^{-1})$. When $\beta + \nu = 0$, $F(-\beta, \beta, z) = O(|z|^{-\beta}/\ln |z|)$, so that $F(-\beta, \beta, z) = o(|z|^{-1})$ for $\beta \leqq 1$. Finally we consider the case $0 > \nu > -\beta$. In this case, by (3.3) and (3.4), we obtain $F(\nu, \beta, z) = O(|z|^\nu)$, and is $o(|z|^{-1})$ if $\nu > -1$. This shows that $F(\nu, \beta, z)$ is a Stieltjes transform under the restrictions $\beta < 1\,(\beta \leqq 1)$ if $\nu + \beta < 0\,(\nu + \beta = 0)$ or $\nu > -1$ when $0 > \nu > -\beta$. Next we use (3.6) to find the inverse Stieltjes transform of $F(\nu, \beta, z)$. For $t > 0$, $\eta > 0$ we have

(4.3) $\quad K_\mu(\sqrt{-t + i\eta}) = K_\mu(e^{i\pi/2}\sqrt{t - i\eta}) = -\dfrac{i\pi}{2} e^{-i\mu\pi/2} H_\mu^{(2)}(\sqrt{t - i\eta}),$

and

(4.4) $\quad K_\mu(\sqrt{t - i\eta}) = K_\mu(e^{-i\pi/2}\sqrt{t + i\eta}) = \dfrac{i\pi}{2} e^{i\mu\pi/2} H_\mu^{(1)}(\sqrt{t + i\eta}),$

from Erdelyi et al. [**4**, (16), p. 6 and (17) p. 7]. Now let

$$F(\nu, \beta, z) = \int_0^\infty \frac{d\mu(t)}{z + t}, \quad \mu(t) \text{ normalized.}$$

By Lemma 2.6 and formulas (3.3) and (3.4) it is easy to see that μ is absolutely continuous and

(4.5) $\quad \mu'(t) = \dfrac{t^{-\beta/2}}{2\pi i} \dfrac{H_\nu^{(1)}(\sqrt{t})H_{\nu+\beta}^{(2)}(\sqrt{t}) - H_\nu^{(2)}(\sqrt{t})H_{\nu+\beta}^{(1)}(\sqrt{t})}{H_{\nu+\beta}^{(1)}(\sqrt{t})H_{\nu+\beta}^{(2)}(\sqrt{t})}.$

The Hankel functions $H_\mu^{(1)}$ and $H_\mu^{(2)}$ are related to the Bessel functions J_μ and Y_μ by, Erdelyi et al. [**4**, p. 4],

(4.6) $\quad H_\mu^{(1)}(z) = J_\mu(z) + i\, Y_\mu(z) \quad$ and $\quad H_\mu^{(2)}(z) = J_\mu(z) - i\, Y_\mu(z).$

Thus we get

$$\mu'(t^2) = \frac{t^{-\beta}}{\pi} \frac{J_{\nu+\beta}(t)Y_\nu(t) - J_\nu(t)Y_{\nu+\beta}(t)}{J_{\nu+\beta}^2(t) + Y_{\nu+\beta}^2(t)}, \quad t > 0,$$

which completes the proof of (2.1).

Proof of (2.2). Let

(4.7) $\quad G(\nu, \beta, z) = K_{\nu+\beta}(\sqrt{z})/K_\nu(\sqrt{z}) - 1.$

As in the proof of (2.1) we use (3.2), (3.3) and (3.4) to determine the asymptotic behaviour of $G(\nu, \beta, z)$ near $z = 0$ and $z = \infty$.

The analyticity of $G(\nu, \beta, z)$ follows from the analyticity of $K_\nu(z)$ and the fact that the half plane $|\arg z| \leqq \pi/2$ is free of zeros of $K_\nu(z)$ for ν real. We then

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use the inversion formula, Lemma 3.6, to evaluate the inverse Stieltjes transform. Finally, using (4.3), (4.4) and (4.6) we reduce the inverse Stieltjes transform to

$$\frac{1}{\pi}\left\{\sin\left(\frac{\beta\pi}{2}\right)\Lambda_1(t)+\cos\left(\frac{\beta\pi}{2}\right)\Lambda_2(t)\right\}.$$

The details will be left out because they are very similar to those used in proving (2.1).

The relationships (2.3) and (2.4) may similarly be proved. Their respective left hand sides satisfy the assumptions of the representation theorem, Lemma 3.5, and the inversion formula, Lemma 3.6, can then be used. The argument is almost identical with our proof of (2.1) and all the details will be omitted.

Proof of (2.5). As $\nu \to 0$ and for any δ, $0 < \delta < \pi$ the left hand side of equation (2.5) obviously remains bounded when $|\arg z| \leqq \pi - \delta$. As $\nu \to \infty$, the above mentioned left hand side is $O(\nu^{(b-a)/2})$ uniformly when $|\arg \nu| \leqq \pi - \delta$ as can be easily seen from Lemma 3.9. Both the numerator and denominator of that left hand side are analytic functions of ν, $|\arg \nu| < \pi$, Re $x > 0$ (see for example formula (21) on page 82 of Erdelyi et al. [**4**]). The function $K_\mu(x)$ as a function of μ for x fixed and positive vanishes only when μ is purely imaginary (see Erdelyi et al. [**4**, page 63]). Therefore $K_{\alpha+\sqrt{\nu}}(x)/K_{b+\sqrt{\nu}}(x)$ is an analytic functions of ν when $x > 0$ and $|\arg \nu| < \pi$. So the function $K_{a+\sqrt{\nu}}(x)/K_{b+\sqrt{\nu}}(x)$ is the Stieltjes transform of a unique signed measure $d\mu$, by Lemma 3.5. Now using the inversion formula for the Stieltjes transform, Lemma 3.6, it is easy to see that $d\mu/dt$ exists and is given by

$$\frac{d\mu}{dt}=\frac{1}{2\pi i}\left\{\frac{K_{a-i\sqrt{t}}(x)}{K_{b-i\sqrt{t}}(x)}-\frac{K_{a+i\sqrt{t}}(x)}{K_{b+i\sqrt{t}}(x)}\right\},\quad t>0.$$

5. Some completely monotonic functions. As we said earlier in the current section we use the integral representations of Section 2 to establish the complete monotonicity of quotients of Bessel functions. From Definition 3.1 it is clear that the product of completely monotonic functions is completely monotonic. This fact will be used repeatedly throughout the present section without mentioning it every time.

Our first result is

THEOREM 5.1. *The function* $K_{\nu+b}(\sqrt{x})/K_\nu(\sqrt{z})$ *is a completely monotonic function of* x *when* $\nu \geqq 0, \beta > 0$.

Proof. We first consider the case $0 < \beta < 1$. We appeal to the formulas

$$J_\mu(x)Y_\nu(x) - J_\nu(x)Y_\mu(x)$$

$$= \frac{4}{\pi^2}\sin\left[(\mu-\nu)\pi\right]\int_0^\infty K_{\nu-\mu}(2x\sinh t)\,e^{-(\mu+\nu)t}dt, \tag{5.1}$$

$x > 0$, $|\text{Re}\,(\mu - \nu)| < 1$ (see Erdelyi et al. [**4**, (67), p. 97]),

$$J_\mu(x)J_\nu(x) + Y_\mu(x)Y_\nu(x) \tag{5.2}$$
$$= \frac{4}{\pi^2}\int_0^\infty K_{\nu-\mu}(2x\sinh t)[e^{(\mu+\nu)t} + e^{-(\mu+\nu)t}\cos(\mu-\nu)t]dt,$$

$x > 0$, $|\text{Re}\,(\mu - \nu)| < 1$, and (2.2) to obtain

$$\frac{K_{\nu+\beta}(\sqrt{x})}{K_\nu(\sqrt{x})} = 1 + \frac{4\sin\left(\frac{\pi\beta}{2}\right)}{\pi^3}\int_0^\infty \frac{h(t)}{x+t}\{J_\nu^2(\sqrt{t}) + Y_\nu^2(\sqrt{t})\}^{-1}dt, \tag{5.3}$$

where $h(t)$ is given by

$$h(t) = \int_0^\infty K_\beta(2t\sinh y)\,e^{-(2\nu+\beta)y}\left\{e^{(4\nu+2\beta)y} + \cos\beta y - 2\cos^2\left(\frac{\beta\pi}{2}\right)\right\}dy \tag{5.4}$$

The integrand in $h(t)$ is positive because $K_\beta(u)$ is positive for $u > 0$ and

$$\frac{d}{dy}\left\{e^{(4\nu+2\beta)y} + \cos\beta y - 2\cos^2\frac{\beta\pi}{2}\right\} \geqq \beta\{e^{2\beta y} - \sin\beta y\} > 0$$

while at $y = 0$ the function $e^{4\nu+\beta y} + \cos\beta y - 2\cos^2(\beta\pi/2)$ is positive. Hence $h(t)$ is positive for $t > 0$ and the result follows. The result for all $\beta > 0$ follows from the above argument and the obvious identity

$$\frac{K_{\nu+\beta}(\sqrt{x})}{K_\nu(\sqrt{x})} = \frac{K_{\nu+\beta}(\sqrt{x})}{K_{\nu+[\beta]}(\sqrt{x})}\prod_{j=1}^{2[\beta]}\frac{K_{\nu+j/2}(\sqrt{x})}{K_{\nu+(j-1)/2}(\sqrt{x})}.$$

Remark 5.2. It is necessary to assume $\nu \geqq 0$ since $K_\nu(x) = K_{-\nu}(x)$ explains the necessity of $\nu + \beta \geqq |\nu|$.

THEOREM 5.3. *The function* $x^{-\beta/2}K_\nu(\sqrt{x})/K_{\nu+\beta}(\sqrt{x})$ *is a completely monotonic function of* x *for all real* ν, $\beta > 0$.

Proof. If $|\nu| \geqq |\nu + \beta|$ the result follows from Theorem 5.1 and the complete monotonicity of $x^{-\beta/2}$. So we may assume $\nu \geqq 0$ or $-\beta/2 < \nu < 0$. When $\beta > 0$ is sufficiently small we see from (5.1) that the integrand in (2.1) is non-negative. This proves the theorem for sufficiently small β. The theorem follows for general β by expressing $x^{-\beta/2}K_\nu(\sqrt{x})/K_{\nu+\beta}(\sqrt{x})$ as a product of functions of the same type where the indices of the K's differ by a number small enough to reduce it to the previous case.

We conclude by proving the following theorem.

THEOREM 5.4. *For* $n = 1, 2, 3, \ldots$, $\alpha \geqq 0$ *and* $x > 0$, *the function* $K_{\alpha+\sqrt{\nu}}(x)/K_{\alpha+\eta+\sqrt{\nu}}(x)$ *is a completely monotonic function of* ν.

Proof. It suffices to prove the theorem for $n = 1$. The integral representation (2.5) shows that we need only to prove the positivity of

$$f(x) = \text{Im}\,\{K_{\alpha-it}(x)\,K_{\alpha+1+it}(x)\},\quad t > 0.$$

Using the differential recurrence relation (see [**4**, p. 79])

$$K_{\nu-1}(x) = \frac{-2\nu}{x} K_\nu(x) - 2K_\nu'(x), \quad K_\nu'(x) = \frac{d}{dx} K_\nu(x),$$

we can write $f(x)$ in the form

$$(5.5) \quad f(x) = \frac{2t}{x} |K_{\nu+1+it}(x)|^2 + g(x),$$

with

$$2ig(x) = K_{\alpha+1-it}(x)\, K_{\alpha+1+it}'(x) - K_{\alpha+1+it}(x)\, K_{\alpha+1-it}'(x).$$

Using the differential equation satisfied by K_ν we derive the following differential equation for $g(x)$

$$(5.6) \quad x^2 g'(x) + xg(t) = 2(\alpha + 1)t\, |K_{\alpha+1+it}(x)|^2,$$

Now eliminate $|K_{\alpha+1+it}(x)|^2$ between (5.5) and (5.6) to get

$$(\alpha + 1)xf(x) = x^2 g'(x) + (\alpha + 2)xg(x),$$

that is

$$(5.7) \quad (\alpha + 1)x^\alpha f(x) = \frac{d}{xdx} \{x^{\alpha+2} g(x)\}.$$

On the other hand (5.6) shows that $xg(x)$ is a strictly increasing for $x > 0$, hence $x^{\alpha+2}g(x)$ is also strictly increasing, for $x > 0$, and the positivity of f follows from 5.7.

Added in proof. While this paper was in press, we discovered that formula (5.2) as stated in the paper and as stated in Erdelyi et al [**4**, (63), p. 96] contained a misprint which unfortunately has not been noticed yet. Formula (5.2) should read

$$(5.2) \quad J_\mu(x)J_\nu(x) + Y_\mu(x)\, Y_\nu(x) = \frac{4}{\pi^2}\int_0^\infty K_{\nu-\mu}(2x \sinh t)[e^{(\mu+\nu)t} + e^{-(\mu+\nu)t} \cdot \cos(\mu - \nu)\pi]dt,$$

hence (5.4) must be replaced by

$$(5.4) \quad h(t) = \int_0^\infty K_\beta(2t \sinh y)\, \{e^{(2\nu+\beta)y} - e^{-(2\nu+\beta)y}\}dy.$$

The positivity of $h(t)$ is now obvious and the rest of the proof goes the same way.

References

1. R. Askey, *Orthogonal polynomials and special functions*, CBMS, Volume 21 (SIAM, Philadelphia, 1975).

2. H. S. Carslaw and J. C. Jaeger, *Some problems in the mathematical theory of the conduction of heat*, Philosophical Magazine *26* (1938), 473–495.

3. A. Erdelyi, W. Magnus, F. Oberhettinger and G. F. Tricomi, *Higher transcendental functions*, Volume 1 (McGraw-Hill, New York, 1953).

4. ——— *Higher transcendental functions*, Volume 2 (McGraw-Hill, New York, 1953).

5. W. Feller, *An introduction to probability theory and its applications*, Volume 2 (John Wiley, New York, 1966).

6. J. L. Fields and M. E. H. Ismail, *On the positivity of some* ${}_1F_2$'s, SIAM J. Math. Anal. (1975), 551–559.

7. E. Grosswald, *The student t-distribution of any degrees of freedom is infinitely divisible*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete *36* (1976), 103–109.

8. P. Hartman, *Completely monotone families of solutions of* n^{th} *order linear differential equations and infinitely divisible distributions*, Ann. Scoula Norm. Sup. Pisa (4) *3* (1976), 267–287.

9. P. Hartman and G. Watson, *"Normal" distribution functions on spheres*, Ann. Prob. *2* (1974), 593–607.

10. I. I. Hirschman and D. V. Widder, *The convolution transform* (Princeton University Press, Princeton, 1955).

11. M. E. H. Ismail, *Bessel functions and the infinite divisibility of the student t-distribution*, Ann. Prob. *5* (1977), 582–585.

12. M. E. H. Ismail and D. H. Kelker, *The Bessel polynomials and the student t-distribution*, SIAM J. Math. Anal. *7* (1976), 82–91.

13. M. E. H. Ismail and M. E. Muldoon, *Monotonicity of the zeros of a cross-product of Bessel functions*, SIAM. J. Math. Anal. *9* (1978), to appear.

14. L. Lorch, *Inequalities for some Whittaker functions*, Arch. Math. (Brno) *3* (1967), 1–9.

15. F. W. J. Olver, *Asymptotics and special functions* (Academic Press, New York, 1974).

16. M. H. Stone, *Linear transformations in hilbert space and their applications to analysis*, American Mathematical Society Colloquium Publications *15* (New York, 1932).

17. G. N. Watson, *A treatise on the theory of bessel functions*, second edition (Cambridge University Press, Cambridge, 1944).

18. E. T. Whittaker and G. N. Watson, *Modern analysis*, fourth edition (Cambridge University Press, Cambridge, 1927).

19. D. V. Widder, *The Laplace transform* (Princeton University Press, Princeton, 1941).

McMaster University,
Hamilton, Ontario