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PRODUCTS OF DECOMPOSABLE POSITIVE OPERATORS

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ABSTRACT. In recent years there has been a growing interest in problems of factorization for bounded linear operators. We first show that many of these problems properly belong to the category of C^*-algebras. With this interpretation, it becomes evident that the problem is fundamental both to the structure of operator algebras and the elements therein. In this paper we consider the direct integral algebra $\int_X^{\oplus} \mathcal{B}(\mathcal{H})\, d\mu$ with $\mathcal{H}$ separable and infinite dimensional. We generalize a theorem of Wu (1988) and characterize those decomposable operators which are products of non-negative decomposable operators. We do this by first showing that various results on operator ranges may be generalized to "measurable fields of operator ranges".

Introduction. Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and U a unitary operator on $\mathcal{H}$. In 1958 [9] Halmos and Kakutani showed that U may be factored as a product of 4 symmetries. Later it was proven by Radjavi [17] than any normal operator is the product of 4 Hermitian operators. More recently, using non-trivial results on operator ranges [6], Wu was able to bring these theorems together while showing that the set of finite products of nonnegative operators coincides with both the set of products of normal operators and the set of products of Hermitian operators [21].

With regard to the finite-dimensional case, facts of this kind include that T is a product of 5 positive operators if and only if $\det T > 0$ [2]; and that a unitary operatory U is a product of four involutions if and only if $\det U = \pm 1$ [16].

Over the years there has been a growing interest in such factorization theorems. A good survey paper here is by Wu [22] and a good expository paper, "Bad Products of Good Matrices", is by Halmos [8]. In general the problems are twofold: with $\mathcal{S}$ a set of (distinguished) operators it is asked which operators T may be either factored as, or approximated by, a finite product of operators, with each factor belonging to $\mathcal{S}$.

As it turns out, of particular importance are the sets $\mathcal{P}_k = \{T \in \mathcal{B}(\mathcal{H}) : T$ is a finite product of k positive invertible operators$\}$, $k = 1, 2, 3, \ldots$, $\mathcal{P}_\infty = \bigcup_{k=1}^\infty \mathcal{P}_k$ and $\bar{\mathcal{P}}_k =$ the norm closure of $\mathcal{P}_k$.

So in addition to the factorization theorems mentioned thus far, certain approximation theorems have been obtained in [13]. For example, Theorem 1 of that paper characterizes those normal operators in $\bar{\mathcal{P}}_2$, while Theorem 3 gives the equality of the norm closure of the following five collections: $\mathcal{P}_5$, $\mathcal{P}_\infty$, the invertibles, Fredholm operators of index zero, and the collection of operators T with the property that $\dim\ker T = \dim\ker T^*$.

At this point we may now broaden our perspective. Problems of factorization are (topologico-) algebraic, while determining the norm closure of distinguished subsets

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(such as $\mathcal{P}_\infty$ or the invertibles) is (algebraico-) topological. Therefore, since positivity is essentially an operator-theoretic notion, we find that our questions properly belong to the category of C^*-algebras, and those specifically concerning the sets $\mathcal{P}_k$ should be rephrased as follows: Letting $\mathcal{A}$ be an C^*-algebra, which elements of $\mathcal{A}$ can be factored as, or approximated by, finite products of k positive operators, with each factor also from $\mathcal{A}$? (In this framework it is seen that Ballantine [2] and Wu [21] were considering factorization in the relatively large and well-structured C^*-algebra $\mathcal{B}(\mathcal{H})$, for $\mathcal{H}$ respectively finite and infinite dimensional.) We note that to answer this requires insight into both the algebra and the elements therein. Hence the question is fundamental, for it concretely intertwines single operator theory with the theory of operator algebras, and implicitly concerns deep structure theorems for both.

It has been the object of our current research to begin dealing with these matters. In doctoral work (to appear) we considered families of algebras obtained from finite-dimensional C^*-algebras by such constructions as direct integrals and direct limits. Thus we obtained our so-called "APN-algebras" along with, for example, a characterization of which of these have Q_4 dense (See discussion preceding Corollary 2.14). In our next two papers we pursue these questions of factorization and approximation by products of positive operators, but in a different class of C^*-algebras. As opposed to our thesis work, the underlying Hilbert spaces for this new class of algebras are all separable and infinite-dimensional. So with (X,μ) a σ-finite standard measure space and $\mathcal{H}$ separable and infinite-dimensional, a basic object in our present study is the von Neumann algebra of decomposable operators, otherwise known as the direct integral $\int_X^\oplus \mathcal{B}(\mathcal{H})\,d\mu$.

Our main theorem of this paper, Theorem B, generalizes the result of Wu [21] by chracterizing those decomposable operators which may be factored as a finite product of decomposable non-negative operators. Much as in Wu's theorem, this set coincides with finite products of decomposable self-adjoint operators.

Central to establishing Theorem B is a generalization of Dixmier's proof of a result of von Neumann. As stated by Dixmier, von Neumann's result is in terms of operator ranges, and can be found in [6] as Theorem 3.6. There it is proven that if $\mathcal{R}$ is a non-closed operator range in a separable Hilbert space $\mathcal{H}$, then there is a unitary operator U on $\mathcal{H}$ such that $\mathcal{R} \cap U\mathcal{R} = \{0\}$. Our generalization of this is Theorem A, which we now state.

THEOREM A. *Let $x \longmapsto T(x)$ be a measurable field of bounded operators with $\mathcal{R}(x) =$ range $T(x)$. If $\mathcal{R}(x)$ is not closed a.e. then there exists a measurable field of unitary operators $x \longmapsto U(x)$ such that $\mathcal{R}(x) \cap U(x)\mathcal{R}(x) = \{0\}$.*

The proof of this relies on certain facts concerning "measurable fields of operator ranges" (to be defined below). We are therefore brought in a natural way to consider "measurable fields of closed operators", a notion which generalizes measurable fields of operators $x \longmapsto T(x)$ by allowing the function values $T(x)$ to be closed but not necessarily bounded. Our paper thus divides into two sections. In Section 1 we treat "measurable fields of operator ranges" and provide generalizations of selected theorems from the paper by Fillmore and Williams [6]—the objective here is to establish Theorem A. In Section 2 we then use Theorem A in the proof of Theorem B, our factorization theorem. We point out that because our setting is the algebra of decomposable operators, we could interpret Theorems A and B by saying that von Neumann's result and Wu's theorem are both "decomposable".

1. **Decomposable operator ranges.** Let us begin by recalling a few facts from the theory of direct integrals. Our reference here is [19]. In what follows (X, μ), or sometimes just X, will stand for a standard Borel space with σ-finite measure μ. (We suppress notation for the underlying σ-field, or Borel structure.)

DEFINITION 1.1. A *measurable field* of Hilbert spaces over (X, μ) is a family $\{\mathcal{H}(x) : x \in X\}$ of Hilbert spaces indexed by x together with a subspace $\mathcal{S}$ of the product space $\prod_{x\in X} \mathcal{H}(x)$ with the following properties:

(i) For any $\xi \in \mathcal{S}$ the function $x \longmapsto \|\xi(x)\|$ is μ-measurable.

(ii) For any $\eta \in \prod_{x\in X} \mathcal{H}(x)$, if the function $x \longmapsto \langle \xi(x), \eta(x)\rangle$ is μ-measurable for every $\xi \in \mathcal{S}$, then $\eta \in \mathcal{S}$.

(iii) There exists a *fundamental sequence*, *i.e.* a countable subset $\{\xi_1, \xi_2, \ldots\}$ of $\mathcal{S}$ such that for almost every $x \in X$ the set $\{\xi_1(x), \xi_2(x), \ldots\}$ is total in $\mathcal{H}(x)$.

TERMINOLOGY. Members of $\mathcal{S}$ are called *measurable vector fields*.

Now let $\mathcal{H}'$ be the collection of measurable vector fields such that

$$\|\xi\|^2 = \int_X \|\xi(x)\|^2 \, d\mu < \infty.$$

With respect to the natural point-wise linear operations, $\mathcal{H}'$ is a vector space and the sesquilinear form

$$\langle \xi, \eta\rangle = \int_X \langle \xi(x), \eta(x)\rangle \, d\mu$$

gives a Hilbert space $\mathcal{H}$ in the usual way, that is, by identifying two fields ξ, η if $\xi(x) = \eta(x)$ a.e.

DEFINITION 1.2. We call this Hilbert space the *direct integral* of the measurable field of Hilbert spaces $\{\mathcal{H}(x) : x \in X\}$ and denote it by

$$\mathcal{H} = \int_X^{\oplus} \mathcal{H}(x) \, d\mu.$$

Each vector $\xi \in \mathcal{H}$ is written as

$$\xi = \int_X^{\oplus} \xi(x) \, d\mu$$

or sometimes just

$$x \longmapsto \xi(x)$$

where it is understood that this a representative for the equivalence class ξ.

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When $\mathcal{H}(x) = \mathcal{H}^\circ$ for some fixed Hilbert space $\mathcal{H}^\circ$, then the field $\{\mathcal{H}(x) : x \in X\}$ is called the *constant field* and we have a natural isomorphism

$$\int_X^{\oplus} \mathcal{H}^\circ \, d\mu \cong \mathcal{L}^2(X, \mu) \otimes \mathcal{H}^\circ$$

where $\mathcal{L}^2(X, \mu)$ is the usual Hilbert space of equivalence classes of L^2-functions.

DEFINITION 1.3. Given two measurable fields of Hilbert spaces $\{\mathcal{H}_1(x) : x \in X\}$ and $\{\mathcal{H}_2(x) : x \in X\}$, a field of bounded operators

$$x \longmapsto T(x) \in \mathcal{B}\big(H_1, (x), H_2(x)\big)$$

is called *measurable* if for any $\xi \in \mathcal{S}_1$, the vector field

$$x \longmapsto T(x)\xi(x) \in \mathcal{H}_2(x)$$

is measurable, *i.e.* belongs to $\mathcal{S}_2$.

If a measurable operator field is essentially bounded (in the sense that the function $x \longmapsto \|T(x)\|$ is essentially bounded) then for each

$$\xi = \int_X^{\oplus} \xi(x) \, d\mu \in \int_X^{\oplus} \mathcal{H}_1(x) \, d\mu$$

$$T\xi = \int_X^{\oplus} T(x)\xi(x) \, d\mu \in \int_X^{\oplus} \mathcal{H}_2(x) \, d\mu.$$

We write this operator as

$$T = \int_X^{\oplus} T(x) \, d\mu$$

and call it the *direct integral* of the (essentially bounded) measurable field $x \longmapsto T(x)$. The norm of T equals the essential supremum ess sup $\|T(x)\|$.

TERMINOLOGY. We call operators of this form *decomposable*.

NOTATION. (i) For any subset $\mathcal{S}$ of a Hilbert space $\mathcal{H}$, $[\mathcal{S}]$ will denote its norm closure.

(ii) For any operator A, $\ker A$ will denote the kernel of A and $\operatorname{rg} A$ will be the range of A.

We use the next lemma to state some basic facts on measurable fields of bounded operators.

LEMMA 1.4. *Let $x \longmapsto T(x)$ be a measurable field of bounded operators. Then*

(i) $x \longmapsto \ker T(x)$ *is a measurable field of Hilbert spaces.*

(ii) $x \longmapsto [\operatorname{rg} T(x)]$ *is a measurable field of Hilbert spaces.*

Moreover, if $x \longmapsto T(x) = U(x)|T(x)|$ is the canonical polar decomposition of $T(x)$, then

(iii) $x \longmapsto U(x)$ *is a measurable field of partial isometries.*

PROOF. For (ii), let $\{\xi_n\}_{n=1}^{\infty}$ be a fundamental sequence for the measurable field $x \longmapsto \mathcal{H}(x)$. Then $\{\eta_n = T\xi_n\}$ defines a fundamental sequence for the field $x \longmapsto [\operatorname{rg} T(x)]$, from which we conclude that this field is measurable.

For (i), let $\mathcal{K}(x) = [\operatorname{rg} T^*(x)]$. We then have that since $x \longmapsto T(x)$ is measurable, so is $x \longmapsto T^*(x)$. From (ii), $x \longmapsto \mathcal{K}(x)$ is measurable, which implies that $x \longmapsto P_{\mathcal{K}(x)}$ is measurable, where $P_{\mathcal{K}(x)}$ is the orthogonal projection onto $\mathcal{K}(x)$. But now $x \longmapsto I - P_{\mathcal{K}(x)} = P_{\ker T(x)}$ is measurable. So by (ii) again, it follows that $x \longmapsto \operatorname{rg}(P_{\ker T(x)}) = \ker T(x)$ is measurable.

Finally, to prove (iii), let

$$X_k = \{x \in X : \|T(x)\| \leq k\}, \quad k = 1, 2, 3, \ldots$$

and $T_k = T|_{X_k}$.

Now just use the usual theorem for direct integrals together with the fact that $X = \bigcup_{k=1}^{\infty} X_k$.

Now, to get to Theorem A requires the notion of a measurable field of operator ranges. From Theorem 1.1 of [6], this means we must have some definition of a measurable field of closed operators, for any operator range can be characterized as the range of some closed operator, or in fact as the domain of some closed operator. The key fact for our purposes is that, by definition, a closed operator A on $\mathcal{H}$ has a closed graph in $\mathcal{H} \oplus \mathcal{H}$. Hence the orthogonal projection P of $\mathcal{H} \oplus \mathcal{H}$ onto the graph of A, $\operatorname{gra} A$, is bounded. Since P acts on the direct sum $\mathcal{H} \oplus \mathcal{H}$, it can be expressed uniquely in terms of a 2×2 matrix (P_{ij}) of bounded linear operators on $\mathcal{H}$. Stone [18] called this the characteristic matrix of the operator A. Therefore, like Nussbaum in [14], we are led to the following:

DEFINITION 1.5. A field of closed operators $x \longmapsto T(x)$ is said to *measurable* if the field of characteristic matrices $x \longmapsto \big(P_{ij}(x)\big)$ is measurable.

This definition is justified by Proposition 6 of [14] where it is shown to agree with Definition 1.3 whenever it is the case that $T(x)$ is bounded a.e.

DEFINITION 1.6. A field $x \longmapsto T(x)$ of closed (not necessarily bounded) operators is said to be *weakly measurable* if for each measurable vector field $x \longmapsto f(x)$ such that $f(x) \in \operatorname{dom} T(x)$ a.e., the new vector field defined by $x \longmapsto T(x)f(x)$ is also measurable.

Corollary 2 of [14] states that a measurable field of closed operators is weakly measurable.

DEFINITION 1.7. Let $x \longmapsto T(x)$ be a measurable field of closed operators. Then $\operatorname{dom} T$ is the set of equivalence classes of all square integrable vector fields $x \longmapsto f(x)$ such that $f(x) \in \operatorname{dom} T(x)$ a.e. and $x \longmapsto T(x)f(x)$ is square integrable.

PROPOSITION 1.8. *The mapping $T: f \longmapsto g$ where $f \in \operatorname{dom} T$ and $g(x) = T(x)f(x)$ is a closed linear operator in $\mathcal{H} = \int_X^{\oplus} \mathcal{H}(x)\, d\mu$ denoted $T \sim T(x)$.*

PROOF. See [14], Proposition 7.

Implicit in the above result is the precise relation between a closed operator A and its characteristic matrix $P_A = (P_{ij})$. For the convenience of the reader we therefore pause to give a table of the basic equations, as also found in [20], p. 73:

$$P_{11} = (A^*A + 1)^{-1} \quad P_{12} = A^*(AA^* + 1)^{-1}$$
$$P_{21} = A(A^*A + 1)^{-1} \quad P_{22} = AA^*(AA^* + 1)^{-1}.$$

As well, we have that

$$\operatorname{dom} A = \operatorname{rg}\big((P_{11}, P_{12}) : \mathcal{H} \oplus \mathcal{H} \longrightarrow \mathcal{H}\big)$$

and

$$\operatorname{rg} A = \operatorname{rg}\big((P_{21}, P_{22}) : \mathcal{H} \oplus \mathcal{H} \longrightarrow \mathcal{H}\big).$$

LEMMA 1.9. *Suppose $x \longmapsto T(x)$ is an essentially bounded measurable field of operators, so that $T = \int_X^{\oplus} T(x)\,d\mu$ is a bounded decomposable operator. If* $\operatorname{rg} T(x)$ *is dense a.e., then* $\operatorname{rg} T$ *is dense.*

PROOF. Suppose $g = \int_X^{\oplus} g(x)\,d\mu$ and $\langle g, Tf\rangle = 0$ for all $f = \int_X^{\oplus} f(x)\,d\mu$. Then $\langle T^* g, f\rangle = 0$ for all $f = \int_X^{\oplus} f(x)\,d\mu$ which implies $T^*(x)g(x) = 0$ a.e. Therefore $g(x) \in \big(\operatorname{rg} T(x)\big)^{\perp} = \{0\}$ a.e. so that $g = 0$. It follows that $(\operatorname{rg} T)^{\perp} = \{0\}$ and hence that $\operatorname{rg} T$ is dense.

COROLLARY 1.10. (TO LEMMA 1.9). *Suppose $x \longmapsto T(x)$ is a measurable field of bounded operators such that* $\operatorname{rg} T(x)$ *is dense a.e. Then*

$$[\operatorname{rg} T] = \int_X^{\oplus} \mathcal{H}(x)\,d\mu.$$

PROOF. Since $x \longmapsto \|T(x)\|$ is measurable,

$$X_k = \{x \in X : \|T(x)\| \leq k\}$$

is measurable for each $k = 1, 2, 3, \ldots$.

Suppose now that

$$g = \int_X^{\oplus} g(x)\,d\mu \quad \text{and} \quad \langle g, Tf\rangle = 0$$

for all $f \in \operatorname{dom} T$. Then with

$$\mathcal{H}_k = \int_{X_k}^{\oplus} \mathcal{H}(x)\,d\mu, \quad g_k = \int_{X_k}^{\oplus} g(x)\,d\mu$$

and $T_k = \int_{X_k}^{\oplus} T(x)\,d\mu$ we have, for each fixed k, $\langle g_k, T_k f_k\rangle = 0$ for all $f_k \in \mathcal{H}_k$. Therefore, by the lemma, $g_k = 0$ for each k, from which it follows that $g = 0$.

COROLLARY 1.11. *Let $x \longmapsto T(x)$ be a measurable field of bounded operators. Then*

$$[\operatorname{rg} T] = \int_X^{\oplus} [\operatorname{rg} T(x)]\,d\mu.$$

EXAMPLE 1.12. Let X be $[0, 1]$ with Lebesgue measure. Define $T(x) = M_x =$ multiplication by x on $\mathbb{C}$, $x \in [0, 1]$. Then $\operatorname{rg} T(x) = \mathbb{C}$, $\operatorname{rg} T \underset{\neq}{\subset} \mathcal{L}^2[0, 1] = \int_X^{\oplus} \mathbb{C}\,d\mu$ but $[\operatorname{rg} T] = \mathcal{L}^2[0, 1] = \int_X^{\oplus} \operatorname{rg} T(x)\,d\mu$.

We now provide a generalization of a result by Douglas, found as Theorem 2.1 in [6].

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THEOREM 1.13. *Let $x \longmapsto A(x)$ and $x \longmapsto B(x)$ be measurable fields of bounded operators. Then the following conditions are equivalent:*

(1) $\mathrm{rg}\, A(x) \subseteq \mathrm{rg}\, B(x)$ *a.e.*

(2) $A(x)A^*(x) \leq \lambda^2(x)B(x)B^*(x)$ *a.e. for some positive measurable function* $x \longmapsto \lambda(x) > 0$ *a.e.*

(3) $A(x) = B(x)C(x)$ *for some measurable field of bounded operators* $x \longmapsto C(x)$.

Moreover, the measurable field $x \longmapsto C(x)$ can be chosen to be essentially bounded if and only if the function in (2) can be chosen to be essentially bounded.

PROOF. Suppose that (1) is true. Let

$$B_0(x) = B(x)|_{(\ker B(x))^\perp}.$$

By Lemma 1.4 this defines a measurable field of bounded operators, and by Theorem 3 of [14],

$$x \longmapsto B_0^{-1}(x)\colon \mathrm{rg}\, B(x) \longrightarrow \big(\ker B(x)\big)^\perp$$

is a measurable field of closed operators. This implies that

$$x \longmapsto B_0^{-1}(x)A(x) = C(x)$$

is also a measurable field of closed operators. By the closed graph theorem $C(x)$ is bounded a.e. and $A(x) = B(x)C(x)$ a.e.

That (3) implies (1) is obvious.

If (2) holds, then

$$||A^*(x)f(x)|| \leq \lambda(x)||B^*(x)f(x)|| \quad \text{for all } f(x) \in H(x) \text{ a.e.}$$

Therefore the linear maps

$$D(x)\colon \mathrm{rg}\, B^*(x) \longrightarrow \mathrm{rg}\, A^*(x)$$

defined by

$$D(x)B^*(x)f(x) = A^*(x)f(x)$$

are bounded by $\lambda(x)$ a.e. Therefore $D(x)$ extends by continuity to $[\mathrm{rg}\, B^*(x)]$; then put $D(x) = 0$ on $\ker B(x)$. As in [6]

$$A(x) = B(x)D^* \text{ a.e.}$$

so it's enough to show that

$$x \longmapsto C(x) = D^*(x)$$

is a measurable field of bounded operators, and since $D(x)$ is bounded a.e., it is sufficient to prove that the field is weakly measurable (See Proposition 6 of [14]). For this, suppose

$$f \in \int_X^{\oplus} [\mathrm{rg}\, B^*(x)]\, d\mu = \Big[\mathrm{rg}\big(B^* \sim B^*(x)\big)\Big]$$

is of the form $f = B^*g$, $g \in \mathcal{H}$. Then

$$D(x)f(x) = D(x)B^*(x)g(x) = A^*(x)g(x)$$

is measurable. Hence $x \longmapsto D(x)$ is a measurable field and (2) implies (3).

Suppose now that (3) is true. Let $f(x) \in \mathcal{H}(x)$. Then, just as in [6], $\lambda(x) = \|C^*(x)\|$. This completes the proof.

We now wish to generalize Theorem 1.1 of [6] by characterizing measurable fields of operator ranges arising from decomposable operators. We need the following definition:

DEFINITION 1.14. A field $x \longmapsto \mathcal{V}(x)$ of vector subspaces (not necessarily closed) is *measurable* if there exists a sequence $\{\xi_n\}_{n=1}^{\infty}$ in $\int_X^{\oplus} \mathcal{H}(x)\,d\mu$ such that

(i) $\xi_n(x) \in \mathcal{V}(x)$ for all $n = 1, 2, 3, \ldots$ a.e. and

(ii) $\{\xi_n(x)\}_{n=1}^{\infty}$ is total in $[\mathcal{V}(x)]$ a.e.

For completeness we include the following:

The *lower direct integral* is given by

$$\underline{\int_X^{\oplus}} \mathcal{V}(x)\,d\mu = \left\{ f \in \int_X^{\oplus} [\mathcal{V}(x)]\,d\mu : f(x) \in \mathcal{V}(x) \text{ a.e.} \right\};$$

the *upper direct integral* is given by

$$\overline{\int_X^{\oplus}} \mathcal{V}(x)\,d\mu = \int_X^{\oplus} [\mathcal{V}(x)]\,d\mu$$

Clearly, the lower integral is a subspace of the upper integral.

REMARK. Using the Lebesgue dominated convergence theorem it is straightforward to show that if

$$\mathcal{V} = \underline{\int_X^{\oplus}} \mathcal{V}(x)\,d\mu$$

then $[\mathcal{V}] = \overline{\int_X^{\oplus}} \mathcal{V}(x)\,d\mu = \int_X^{\oplus} [\mathcal{V}(x)]\,d\mu$. See also Corollary 1.11.

REMARK. If T is decomposable, it does not follow that

$$\operatorname{rg} T = \underline{\int_X^{\oplus}} \operatorname{rg} T(x)\,d\mu.$$

For example, let T be multiplication by x on $\mathcal{L}^2[0,1]$. In terms of direct integrals

$$T = \int_{[0,1]}^{\oplus} Mx\,d\mu, \quad \mathcal{H} = \int_{[0,1]}^{\oplus} \mathbb{C}\,d\mu \quad \text{and} \quad \operatorname{rg} T(x) = \mathbb{C} \text{ a.e.}$$

Hence, in this case,

$$\underline{\int_X^{\oplus}} \operatorname{rg} T(x)\,d\mu = \mathcal{H} \underset{\neq}{\supset} \operatorname{rg} T.$$

Before proceeding to our theorem, we present two elementary results concerning the spaces just defined.

PROPOSITION 1.15. *Let $T = \int_X^{\oplus} T(x)\, d\mu$ be a decomposable operator. Then the field of vector spaces $x \longmapsto \operatorname{rg} T(x)$ is measurable.*

For a proof argue as in Lemma 1.4(ii).

THEOREM 1.16. *Let $x \longmapsto \mathcal{R}(x) \subseteq \mathcal{H}(x)$ be a measurable field of vector subspaces (not necessarily closed). Then the following are equivalent:*

(1) There exists a measurable field of bounded operators $x \longmapsto T(x)$ such that

$$\mathcal{R}(x) = \operatorname{rg} T(x) \text{ a.e.}$$

(2) There exists a measurable field of closed operators $x \longmapsto T(x)$ such that

$$\mathcal{R}(x) = \operatorname{rg} T(x) \text{ a.e.}$$

(3) There exists a measurable field of closed operators $x \longmapsto T(x)$ such that

$$\mathcal{R}(x) = \operatorname{dom} T(x) \text{ a.e.}$$

(4) There exists a field of inner products $\langle\, , \rangle'_x$ on $\mathcal{R}(x)$ such that $x \longmapsto (\mathcal{R}(x), \langle\, , \rangle'_x)$ is a measurable field of Hilbert spaces such that for all $f(x) \in \mathcal{R}(x)$, $\|f(x)\|' \geq \|f(x)\|$ a.e.

(5) There is a sequence of measurable fields $x \longmapsto \mathcal{H}_n(x)$ of closed subspaces of $\mathcal{H}(x)$ such that

(i) the spaces $\mathcal{H}_n(x)$, $n = 0, 1, 2, \ldots$ are all mutually orthogonal a.e. and

(ii) $\mathcal{R}(x) = \{\sum_{n=0}^{\infty} f_n(x) : f_n(x) \in \mathcal{H}_n(x)$ and $\sum_{n=0}^{\infty}(2^n \|f_n(x)\|)^2 < \infty\}$.

PROOF. To begin, observe that by Proposition 6 [14], (1) implies (2). Assume now that $\mathcal{R}(x) = \operatorname{rg} T(x)$ for a measurable field of closed operators. Let $\mathcal{D}(x) = \operatorname{dom} T(x)$ and put $T_1(x) = T(x)|_{\mathcal{D}(x) \cap (\ker T(x))^{\perp}}$. Then $T_1(x)$ is closed, one-one and has range $\mathcal{R}(x)$ a.e., so that $T^{-1}(x)$ is closed with domain $\mathcal{R}(x)$. Moreover, since $x \longmapsto T(x)$ is measurable, it is easy to check that $x \longmapsto T_1(x)$ is also measurable. Hence, by Theorem 3 of [14], $x \longmapsto T_1^{-1}(x)$ is a measurable field of closed operators, so (2) implies (3).

If $\mathcal{R}(x) = \operatorname{dom} T(x)$ for a measurable field of closed operators, then $\mathcal{R}(x)$ is complete in the inner product defined by $\langle f(x), g(x) \rangle'_x = \langle f(x), g(x) \rangle_x + \langle T(x)f(x), T(x)g(x) \rangle_x$ for all $f(x), g(x) \in \mathcal{R}(x)$. That the field of Hilbert spaces is measurable follows by considering the sequence

$$\xi_n = T\eta_n = \int_X^{\oplus} T(x)\eta_n(x)\, d\mu, \quad n = 1, 2, 3, \ldots$$

where η_n is a fundamental sequence for $\mathcal{H} = \int_X^{\oplus} \mathcal{H}(x)\, d\mu$. Since the inequality is obvious, this completes the proof that (3) implies (4).

To show that (4) implies (1), note first, exactly as in Theorem 1.1 [6], that the inclusion maps

$$T(x)\colon \big(\mathcal{R}(x), \langle\, , \rangle'_x\big) \longrightarrow \mathcal{H}(x)$$

are bounded a.e. Therefore, by Proposition 6 [14] this field is measurable if it is weakly measurable. But this is straightforward since the underlying measure spaces are the same.

We therefore obtain, by Theorem 3 [14], a measurable field of bounded operators $x \longmapsto T^*(x)$. With $T^*(x) = U(x)|T(x)|$ the canonical polar decomposition, Lemma 1.4(iii) gives us a measurable field of partial isometries

$$x \longmapsto U(x): \mathcal{H}(x) \longrightarrow \big(\mathcal{R}(x), \langle\, ,\, \rangle'_x\big)$$

such that

$$[\operatorname{rg} T^*(x)] = \big(\ker T(x)\big)^{\perp} = \mathcal{R}(x).$$

As a field of operators from $\mathcal{H}(x)$ into $\mathcal{H}(x)$, $U(x)$ has range $\mathcal{R}(x)$ and is bounded a.e., since

$$\|U(x)f(x)\| \leq \|U(x)f(x)\|' \leq \|f(x)\|.$$

Next we show that (1) implies (5). Recall that for any bounded operator $T(x)$ on $\mathcal{H}(x)$, $T(x)$ and $A(x) = \big(T(x)T^*(x)\big)^{1/2}$ have the same range. (See Theorem 2.1 of [6]). Let $E(x)$ be the spectral measure of $A(x)$, $E_n(x) = E(x)\big(2^{-n-1}\|A(x)\|, 2^{-n}\|A(x)\|\big)$ and let $\mathcal{H}_n(x) = E_n(x)\mathcal{H}(x)$, $n \geq 0$. By Theorem 5.1 of [1] and Lemma 1.4, we obtain both a measurable field of projections and a measurable field of spaces. That (i) and (ii) are satisfied is the same as in [6].

To complete the proof, we show that (5) implies (1). With $P_n(x)$ the orthogonal projection onto $\mathcal{H}_n(x)$ and $D(x) = \sum_{n=0}^{\infty} 2^{-n} P_n(x)$, then $x \longmapsto D(x)$ defines a measurable field of bounded operators with $\operatorname{rg} D(x) = \mathcal{R}(x)$.

COROLLARY 1.17. *Let $T(x)$, $\mathcal{R}(x)$ and $\langle\, ,\, \rangle'_x$ be as above, and suppose that the field is essentially bounded. Let $T = \int_X^{\oplus} T(x)\, d\mu$ and $\mathcal{R} = \operatorname{rg} T$. Then the Hilbert space $(\mathcal{R}, \langle\, ,\, \rangle')$ [6] is isomorphic to $\int_X^{\oplus} \big(\mathcal{R}(x), \langle\, ,\, \rangle'_x\big)\, d\mu$ and $\mathcal{R} = \big\{\sum_{n=0}^{\infty} f_n : f_n \in \mathcal{H}_n = \int_X^{\oplus} \mathcal{H}_n(x)\, d\mu$ and $\sum_{n=0}^{\infty}\big(2^{2n}\int_X \|f_n(x)\|^2\, d\mu\big) < \infty\big\}$.*

We leave the proof to the reader.

QUESTION. If $x \longmapsto T(x)$ is a measurable field of operators, then $x \longmapsto \operatorname{rg} T(x)$ is a measurable field of operator ranges (vector spaces). If $x \longmapsto \operatorname{rg} T(x)$ is a measurable field of operator ranges, does there exist a measurable field of bounded operators $x \longmapsto S(x)$ such that $\operatorname{rg} S(x) = \operatorname{rg} T(x)$ a.e.?

DEFINITION 1.18. An operator range is of *type J_S* (Dixmier's notation) if it is dense and is determined, as in Theorem 1.1 [6] by an orthogonal sequence of infinite dimensional closed subspaces.

DEFINITION 1.19. Let $x \longmapsto \mathcal{R}(x)$ and $x \longmapsto \mathcal{S}(x)$ be measurable fields of vector spaces. They are called (decomposably) *unitarily equivalent* if there exists a measurable field of unitary operators $x \longmapsto U(x)$ such that $\mathcal{S}(x) = U(x)\mathcal{R}(x)$ a.e.

Generalizing Theorem 3.1 [6] it is possible to show that when $\mathcal{R}(x) = \operatorname{rg} A(x)$ for some measurable field of bounded operators, we may replace "unitary" by "invertible" in the above definition. We leave the details to the reader.

LEMMA 1.20. *Let $x \longmapsto T(x)$ and $x \longmapsto S(x)$ be measurable fields of operators such that* $\operatorname{rg} T(x)$ *and* $\operatorname{rg} S(x)$ *are of type J_S a.e. Then the fields of operator ranges $x \longmapsto \operatorname{rg} T(x)$ and $x \longmapsto \operatorname{rg} S(x)$ are unitarily equivalent.*

PROOF. Let $\mathcal{H}_n(x)$ and $\mathcal{K}_n(x)$ be the spaces corresponding to $T(x)$ and $S(x)$ respectively, as obtained in Theorem 1.1 [6]. Then for each $n = 0, 1, 2, \ldots$

$$\dim \mathcal{H}_n(x) = \dim \mathcal{K}_n(x) \text{ a.e.}$$

and $\dim\bigl(\sum_n^{\oplus} \mathcal{H}_n(x)\bigr)^{\perp} = 0 = \dim\bigl(\sum_n^{\oplus} \mathcal{K}_n(x)\bigr)^{\perp}$ a.e. So (for each n) we obtain a measurable field of unitaries

$$x \longmapsto V_n(x)\colon \mathcal{H}_n(x) \longrightarrow \mathcal{K}_n(x).$$

From here we can now construct the required field by setting

$$V(x) = \sum_n{}^{\oplus} V_n(x).$$

The lemma now follows.

LEMMA 1.21. *Let $x \longmapsto T(x)$ be a measurable field of bounded operators such that $\mathcal{R}(x) = \operatorname{rg} T(x)$ is non-closed a.e. Then there exists a measurable field of bounded operators $x \longmapsto S(x)$ for which $\operatorname{rg} S(x)$ is of type J_S and $\operatorname{rg} S(x) \supseteq \operatorname{rg} T(x)$ a.e.*

PROOF. Let $\mathcal{R}(x)$ be determined by the sequence $\{\mathcal{H}_n(x) : n \geq 0\}$. Because $\mathcal{R}(x)$ is non-closed, $\mathcal{H}_n(x) \neq 0$ for infinitely many n, a.e. Now let $\mathcal{K}_0(x) = \mathcal{H}_0(x)$ for all $x \in X_0 = \{x : \mathcal{H}_0(x) \neq 0\}$, $\mathcal{K}_0(x) = \mathcal{H}_1(x)$ for all $x \in X_1 = \{x : \mathcal{H}_0(x) = 0, \mathcal{H}_1(x) \neq 0\}$ *etc.*

By induction we obtain a sequence of measurable fields of Hilbert spaces $\mathcal{K}_n(x)$, $n \geq 0$ satisfying

(i) $\mathcal{K}_n(x) \neq 0$, $n \geq 0$ a.e.
(ii) $\mathcal{K}_n(x) \perp \mathcal{K}_m(x)$, $n \neq m$ a.e.
(iii) $\sum_{n\geq 0}^{\oplus} \mathcal{K}_n(x) = \sum_{n\geq 0}^{\oplus} \mathcal{H}_n(x)$ a.e.

Let $\{n_{rs}\}$ be a double sequence of non-negative integers for which

(i) each row $n_{r0}, n_{r1}, n_{r2}, \ldots$ is strictly increasing
(ii) $n_{r0} \geq r$ for all r
(iii) the n_{rs} are all distinct and
(iv) $\bigcup_{r,s}\{n_{rs}\} = \{0, 1, 2, 3, \ldots\}$.

To finish the proof, we define

$$\mathcal{H}_0'(x) = \Bigl(\sum_{s=0}^{\infty}{}^{\oplus} \mathcal{K}_{0s}(x)\Bigr) \oplus \Bigl(\sum_{n=0}^{\infty}{}^{\oplus} \mathcal{H}_n(x)\Bigr)^{\perp}$$

$$\mathcal{H}_1'(x) = \Bigl(\sum_{s=0}^{\infty}{}^{\oplus} \mathcal{K}_{1s}(x)\Bigr)$$

$$\mathcal{H}_2'(x) = \Bigl(\sum_{s=0}^{\infty}{}^{\oplus} \mathcal{K}_{2s}(x)\Bigr) \textit{ etc.}$$

Then the ranges $\mathcal{R}'(x)$ determined by the measurable field $\mathcal{H}_n'(x)$ are of type J_S and contain $\mathcal{R}(x)$ a.e.

LEMMA 1.22. *Suppose* $x \longmapsto S(x)$ *is a measurable field of bounded operators for which* $\mathcal{S}(x) = \operatorname{rg} S(x)$ *is dense and contains an infinite-dimensional closed subspace a.e. Then there exists a measurable field of bounded operators* $x \longmapsto S'(x)$ *such that* $\mathcal{S}'(x) = \operatorname{rg} S'(x) \subseteq \operatorname{rg} S(x)$ *and* $\mathcal{S}'(x)$ *is of type* J_S *a.e.*

PROOF. Let $\mathcal{S}(x)$ be determined by the measurable fields of Hilbert spaces $\mathcal{K}_n(x)$, as in Theorem 1.1 [6], and note that $\mathcal{K}_n(x)$ is infinite dimensional for some $n(x)$, a.e. (for otherwise, as pointed out in [6], the corresponding diagonal operator $D(x) = \sum_n 2^{-n} Q_n(x)$, where $Q_n(x)$ is the orthogonal projection onto $\mathcal{K}_n(x)$, would be compact and $\mathcal{S}(x)$ would contain no infinite dimensional subspaces, by Theorem 2.5 [6]). For each x, let $p(x)$ be the smallest positive integer for which this is so, that is, for which $\dim \mathcal{K}_{p(x)}(x) = \infty$. Write $\mathcal{K}_{p(x)} = \sum_{i=0}^{\infty}{}^{\oplus} \mathcal{L}_i(x)$ where $x \longmapsto \mathcal{L}_i(x)$ is a measurable field of infinite dimensional closed subspaces, for each $i = 0, 1, 2, \ldots$. Put $\mathcal{K}_i'(x) = \mathcal{K}_i(x) \oplus \mathcal{L}_i(x)$, $i \neq p$ and $\mathcal{K}_p'(x) = \mathcal{L}_p(x)$. Then the range $\mathcal{S}'(x)$ determined by $\{\mathcal{K}_n'(x)\}_{n=0}^{\infty}$ is of type J_S and is contained in $\mathcal{S}(x)$ a.e. We finish the proof by letting $S'(x)$ be the diagonal operator $D'(x)$ (See Theorem 1.1 (5). [6]).

We are now ready to end this section of the paper by proving Theorem A, stated in the Introduction.

PROOF OF THEOREM A. Similar to Theorem 3.6 [6], the proof rests on two facts:

(a) There exists a measurable field of bounded operators $x \longmapsto S(x)$ and a measurable field of unitaries $x \longmapsto V(x)$ such that $\mathcal{S}(x) = \operatorname{rg} S(x)$ is dense, contains a closed infinite dimensional subspace and $\mathcal{S}(x) \cap V(x)\mathcal{S}(x) = \{0\}$ a.e.

(b) If $\mathcal{S}(x)$ is as in (a), and $x \longmapsto T(x)$ is any measurable field of bounded operators with $\mathcal{R}(x) = \operatorname{rg} T(x)$ then $W(x)\mathcal{R}(x) \subseteq \mathcal{S}(x)$ for some measurable field of unitaries $x \longmapsto W(x)$.

The measurable field of unitaries needed in the statement of the theorem will then be $x \longmapsto U(x) = W^*(x)V(x)W(x)$.

To prove (a) we need only let $A(x) = A$, $\mathcal{S}(x) = \operatorname{rg} A(x)$ and $V(x) = V$, where A and V are the operators given in Theorem 3.6 (a) [6].

For the proof of (b), suppose $\mathcal{S}(x)$ is as in (a), and $\mathcal{R}(x)$ is as in the statement of the theorem, so non-closed a.e. Then by Lemma 1.28 there exists a measurable field of bounded operators $x \longmapsto R'(x)$ for which $\operatorname{rg} R'(x)$ is of type J_S and $\operatorname{rg} R'(x) \supseteq \mathcal{R}(x)$ a.e. By Lemmas 1.27 and 1.29 there exists a measurable field of bounded operators $R''(x)$ for which $\operatorname{rg} R''(x)$ is of type J_S, is contained in $\mathcal{S}(x)$ a.e. and such that the measurable field $x \longmapsto \operatorname{rg} R''(x)$ is unitarily equivalent to $x \longmapsto \operatorname{rg} R'(x)$.

But now (b) follows and Theorem A is established.

2. **Products of positive operators.** For this section $\mathcal{H}^\circ$ is a fixed separable infinite-dimensional Hilbert space and $\mathcal{H}(x) = \mathcal{H}^\circ$ a.e.

We begin with some elmentary results on the range function γ.

DEFINITION 2.1. For each bounded operator A

$$\gamma(A) = \inf\{\|Ah\| : \|h\| = 1 \text{ and } h \perp \ker A\}.$$

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LEMMA 2.2. *Suppose* $x \longmapsto T(x)$ *is a measurable field of bounded operators. Then* $x \longmapsto \gamma(x) = \gamma\big(T(x)\big)$ *defines a measurable function.*

LEMMA 2.3. *If* $T = \int_X^{\oplus} T(x)\,d\mu$ *is a bounded operator, then* T *has closed range if and only if there exists* $\varepsilon > 0$ *such that* $\gamma\big(T(x)\big) \geq \varepsilon > 0$ *a.e.*

PROOF. The result follows from the facts that $x \longmapsto \gamma\big(T(x)\big)$ is a measurable function and that $\gamma(T) = \operatorname{ess\,inf} \gamma\big(T(x)\big)$.

LEMMA 2.4. *A decomposable operator* $T = \int_X^{\oplus} T(x)\,d\mu$ *is invertible if and only if* $T(x)$ *is invertible a.e. and* $\gamma\big(T(x)\big)$ *is essentially bounded away from zero.*

The following proposition is a special case of Fillmore's theorem (Corollary to Theorem 3 [5]) which itself is a generalization of the Halmos-Kakutani Theorem [9] to the case of properly infinite von Neumann algebras.

PROPOSITION 2.5. *Let* U *be a decomposable operator. Then* U *is the product of four decomposable symmetries* J_1, J_2, J_3, J_4 *satisfying* $U(x) = J_1(x)\,J_2(x)\,J_3(x)\,J_4(x)$ *a.e.*

PROOF. The algebra $\int_X^{\oplus} \mathcal{B}(\mathcal{H}^{\circ})\,d\mu$ of decomposable operators is properly infinite.

REMARK. Necessary for our purposes is that for $i = 1, \ldots, 4$, +1 and -1 are eigenvalues of infinite multiplicity a.e. This can be easily seen from Fillmore's proof in [5].

REMARK. Our original proof was longer but drew explicitly on measurable selection theorems and the measurable structure of decomposable operators. Because measurable selection theorems are at the core of direct integral theory, we at least give a sketch of our elementary proof, which "point-wise" generalizes that of Halmos and Kakutani [9] (See also [7] Problem 142).

The key fact is that for each decomposable normal operator $N = \int_X^{\oplus} N(x)\,d\mu$ on $\mathcal{H} = \int_X^{\oplus} \mathcal{H}(x)\,d\mu$, there exists a measurable field $x \longmapsto \mathcal{H}_1(x)$ of subspaces which reduce $N(x)$ a.e. and such that both $\mathcal{H}_1(x)$ and $\mathcal{H}_1(x)^{\perp}$ are infinite dimensional. To prove this, note that the map $x \longmapsto \sigma\big(N(x)\big)$ is measurable. By analysing the original Halmos-Kakutani proof, and invoking the results of [11] and [1], the result follows. Now proceed as in [9] and observe that the symmetries obtained are decomposable.

LEMMA 2.6. *If* T *is a decomposable operator and* $\dim\ker T(x) = \dim\ker T(x)$ *a.e. then* T *is the product of two decomposable normal operators and the product of five decomposable Hermitian operators. If, in addition,* $\operatorname{rg} T$ *is closed a.e. then the normal and Hermitian operators may also be chosen to have closed ranges a.e.*

PROOF. Since $\dim\ker T(x) = \dim\ker T^*$ a.e. $T = UP$ where U is decomposable and unitary and $P = (T^*T)^{\frac{1}{2}}$. Now invoke Proposition 2.5.

The next result is Lemma 2.3 of [21], which we record as

LEMMA 2.7. *Suppose* $A \in \mathcal{B}(\mathcal{H}^\circ)$ *and* $T = N_1 \cdots N_n$ *where* $\dim\ker N_i = \dim\ker N_i^*$, $i = 1, \ldots, n$. *If T is one-sided invertible (one-sided Fredholm) then T is invertible (Fredholm).*

PROPOSITION 2.8. *Let T be a decomposable operator. Then if* $\operatorname{rg} T(x)$ *is closed a.e., the following are equivalent:*

(1) T is the product of finitely many decomposable normal operators;

(2) T is the product of finitely many decomposable Hermitian operators;

(3) $T = SP$, *for decomposable operators S, P for which* $S(x)$ *is invertible and* $P(x)$ *is an orthogonal projection a.e.*

(4) $\dim\ker T(x) = \dim\ker T^*(x)$ *a.e.*

PROOF. We have that (3) implies (2), by Lemma 2.6; and that (2) implies (1) is obvious.

Next we show that (1) implies (4). Since $x \mapsto \dim\ker T(x)$ is a measurable function, the measure space X is partitioned into the disjoint union of measurable sets X_k, $X_k = \{x : \dim\ker T(x) = k\}$, $k = 0, 1, 2, \ldots, \infty$. If $\dim\ker T(x) = \dim\ker T^*(x) = \infty$ a.e., then we are done. If $\dim\ker T(x) < \infty$ then, by Proposition 2.4 [21], $\dim\ker T^*(x) = \dim\ker T(x)$.

We conclude by showing that (4) implies (3). Let P be the orthogonal projection onto $(\ker T)^\perp$. The operator P is decomposable with $P = \int_X^\oplus P(x)\, d\mu$, $P(x)$ the orthogonal projection onto $\big(\ker T(x)\big)^\perp$ a.e. Let $x \mapsto R(x)$ be a measurable field of partial isometries with initial space $\ker T(x)$ and final space $\ker T^*(x)$. Define $S(x)\big(f(x)+k(x)\big) = T(x)f(x) + R(x)k(x)$, for $f(x) \in \big(\ker T(x)\big)^\perp$ and $k(x) \in \ker T(x)$. Then it is an easy exercise to show that $x \mapsto S(x)$ defines an essentially bounded measurable field of invertible operators. Letting $S = \int_X^\oplus S(x)\, d\mu$ we obtain $T = SP$, as needed for (3).

Following Wu [21], the next situation we consider is where $\operatorname{rg} T(x)$ is not closed a.e. Since the condition of having closed range is measurable ($x \mapsto \gamma\big(T(x)\big)$ is a measurable function), X will partition into disjoint measurable sets, according to whether or not $\gamma\big(T(x)\big) > 0$. We will then put these together to obtain our main factorization theorem, Theorem B.

PROPOSITION 2.9. *If* $\operatorname{rg} T(x)$ *is not closed a.e. then T is the product of three decomposable normal operators and six decomposable Hermitian operators.*

PROOF. Because of Lemma 2.6 we need only consider the case where $\dim\ker T(x) \neq \dim\ker T^*(x)$ a.e. And since $\operatorname{rg} T(x)$ is closed if and only if $\operatorname{rg} T^*(x)$ is closed, we further assume that $\dim\ker T(x) > \dim\ker T^*(x)$ a.e. Let P be the orthogonal projection onto $(\ker T)^\perp$, so that $P = \int_X^\oplus P(x)\, d\mu$ with $P(x)$ the orthogonal projection onto $\big(\ker T(x)\big)^\perp$ a.e.

As in Proposition 2.6 of [21] we will construct an operator S with $\dim\ker S = \dim\ker S^*$, such that $T = SP$. However, the operator S in our paper will have the additional properties of being decomposable with $\dim\ker S(x) = \dim\ker S^*$ a.e. To accomplish this, let

$$\ker T(x) = \mathcal{H}_1(x) \oplus \mathcal{H}_2(x)$$

where

$$x \longmapsto \mathcal{H}_i(x), \quad i = 1, 2$$

are measurable fields, and

$$\ker T = \int_X^{\oplus} \ker T(x)\, d\mu = \int_X^{\oplus} \mathcal{H}_1(x)\, d\mu \oplus \int_X^{\oplus} \mathcal{H}_2(x)\, d\mu.$$

Invoking Theorem A, let

$$x \longmapsto \mathcal{K}(x)$$

be a measurable field of closed infinite dimensional subspaces of

$$x \longmapsto [\operatorname{rg} T(x)]$$

such that

$$\mathcal{K}(x) \cap \operatorname{rg} T(x) = \{0\} \text{ a.e.}$$

We let $S(x)$ be a measurable field of operators which maps $\mathcal{H}_1(x)$ to $\{0\}$, $\mathcal{H}_2(x)$ isometrically into $\mathcal{K}(x)$ and equals $T(x)$ on $\big(\ker T(x)\big)^{\perp}$. As in [21], it follows that

$$\dim\ker S(x) = \dim \mathcal{H}_1(x) = \dim\ker T^*(x) = \dim\ker S^*(x) \text{ a.e.}$$

Define

$$S = \int_X^{\oplus} S(x)\, d\mu.$$

By Lemma 2.6, applied to S, the result follows.

The next lemma allows us to treat products of positive operators. Recall from [21] that every symmetry (in $\mathcal{B}(\mathcal{H}^0)$) is the product of six positive invertible operators. Because of the work of Azoff and Clancey on direct integrals of normal operators [1], this factorization also holds in the algebra of decomposable operators. Thus, we have

LEMMA 2.10. *Every decomposable symmetry is the product of six decomposable positive operators.*

PROOF. Combine [1] with [21]. We leave the details to the reader.

This now gives us

LEMMA 2.11. *Every decomposable unitary operator is the product of sixteen decomposable positive invertible operators.*

PROOF. By Proposition 2.5, every decomposable unitary operator is the product of four decomposable symmetries, for which 1 and -1 are eigenvalues of infinite multiplicty a.e. From the proof of Lemma 2.10 the assertion follows.

PROPOSITION 2.12. *If* $\dim\ker T(x) = \dim\ker T^*(x)$ *a.e. then* T *is the product of seventeen decomposable non-negative operators; if* $\operatorname{rg} T(x)$ *is not closed a.e., then* T *is the product of eighteen decomposable non-negative operators.*

PROOF. The first part of the assertion follows from Lemma 2.11 and the proof of Lemma 2.6. If $\operatorname{rg} T(x)$ is not closed a.e., then using the factorization in Proposition 2.9, $T = SP$, where $\dim\ker S^*(x) = \dim\ker S^*(x)$ and $P(x)$ is non-negative a.e. So by the first part, S is the product of seventeen decomposable non-negative operators, which together with P gives eighteen, as claimed.

COROLLARY 2.13. *A decomposable operator* $T = \int_X^{\oplus} T(x)\,d\mu$ *is the product of finitely many decomposable positive operators if and only if* $T(x)$ *is one-one with dense range a.e. In this case seventeen such factors will suffice.*

PROOF. Left to the reader.

We can now state and prove our main theorem, which extends Wu's theorem [21] to the algebra of decomposable operators.

THEOREM B. *Let* T *be a decomposable operator,* $T = \int_X^{\oplus} T(x)\,d\mu$ *acting on* $\mathcal{H} = \int_X^{\oplus} \mathcal{H}(x)\,d\mu$ *where* $\mathcal{H}(x)$ *is separable and infinite dimensional a.e. Then the following are equivalent:*

(1) T *is the product of finitely many decomposable normal operators;*

(2) T *is the product of finitely many decomposable Hermitian operators;*

(3) T *is the product of finitely many decomposable non-negative operators;*

(4) $T(x) = S(x)P(x)$ *or* $P(x)S(x)$ *(depending on whether* $\dim\ker T(x) \geq \dim\ker T^*(x)$ *or* $\dim\ker T(x) < \dim\ker T^*(x)$*) for some decomposable operator* $S = \int_X^{\oplus} S(x)\,d\mu$ *such that* $S(x)$ *is one-one with dense range a.e. and* $P = \int_X^{\oplus} P(x)\,d\mu$ *is an orthogonal projection;*

(5) The space X *partitions measurably into* $X_d \,\dot\cup\, X_r$ *so that for almost all* x *in* X_d $\dim\ker T(x) = \dim\ker T^*(x)$ *and for almost all* x *in* X_r, $\operatorname{rg} T(x)$ *is not closed.*

(6) T *is the norm limit of a sequence of decomposable invertible operators.*

Moreover, the number of factors required is bounded above; we need at most three normal operators, six Hermitian operators and eighteen non-negative operators.

PROOF. We first establish the equivalence of (5) and (6). If (5) is true, then by Propositions 2.9 and 2.12, T is a finite product of decomposable non-negative operators. Using [1], we obtain that each factor is a norm limit of decomposable invertible operators; hence so is T. If (6) is true, that is, there is a sequence $T_n = \int_X^{\oplus} T_n(x)\,d\mu$ of decomposable invertible operators such that $T = \lim_n T_n$, then $T(x) = \lim T_n(x)$ a.e., by elementary direct integral theory. We now call on [4] to conclude that $\dim\ker T(x) = \dim\ker T^*(x)$ or $\operatorname{rg} T(x)$ is not closed a.e.

From what we have done so far, we need only show now that (5) implies (4). But for this we refer the reader to our Proposition 2.9 and Corollary 2.10 of [21]. This completes the proof of Theorem B.

One immediate corollary is on approximation, and is modeled on Theorem 3 of [13].

To state and prove the result we introduce some notation: Let $\mathcal{A}$ be a C^*-algebra. Then for $k = 1, 2, 3, \ldots$

$$\mathcal{P}_k(\mathcal{A}) = \{T \in \mathcal{A} : T = P_1 \cdots P_k,\ P_i \text{ positive invertible, and in } \mathcal{A},\ i = 1, 2, \ldots k\}$$

and

$$\mathcal{P}_\infty(\mathcal{A}) = \bigcup_{k=1}^{\infty} \mathcal{P}_k(\mathcal{A})$$

Similarly, we have the sets $\mathcal{Q}_k(\mathcal{A})$, defined like $\mathcal{P}_k(\mathcal{A})$ except that the factors are allowed to be positive and not necessarily invertible. As usual, for a set $\mathcal{S} \subseteq \mathcal{A}$, $\bar{\mathcal{S}}$ denotes its norm closure.

COROLLARY 2.14. *Let $\mathcal{A}$ be the algebra of decomposable operators. Then the closures of the following sets are equal to $\bar{\mathcal{P}}_{17}(\mathcal{A})$.*

(1) The set of $T \in \mathcal{A}$ such that $T(x)$ is Fredholm with index $T(x) = 0$ a.e. denoted $\mathcal{F}_0(\mathcal{A})$.

(2) The set of $T \in \mathcal{A}$ such that $\dim\ker T(x) = \dim\ker T^*(x)$ *a.e., denoted $\mathcal{V}(\mathcal{A})$.*

(3) The set of invertible operators in $\mathcal{A}$, denoted $\mathcal{G}(\mathcal{A})$.

(4) $\mathcal{P}_n(\mathcal{A})$ for $n \geq 17$.

(5) $\mathcal{P}_\infty(\mathcal{A})$.

PROOF. If $n \geq 17$, we have the following inclusions:

$$\mathcal{P}_{17}(\mathcal{A}) \subseteq \mathcal{P}_n(\mathcal{A}) \subseteq \mathcal{P}_\infty(\mathcal{A}) \subseteq \mathcal{G}(\mathcal{A}) \subseteq \mathcal{F}_0(\mathcal{A}) \subseteq \mathcal{V}(\mathcal{A}).$$

Therefore it is enough to show that

$$\mathcal{V}(\mathcal{A}) \subseteq \bar{\mathcal{P}}_{17}(\mathcal{A}).$$

But, by Proposition 2.12, $\mathcal{V}(\mathcal{A}) \subseteq \mathcal{Q}_{17}(\mathcal{A})$; and $\mathcal{Q}_{17}(\mathcal{A}) \subseteq \bar{\mathcal{Q}}_{17}(\mathcal{A}) \subseteq \bar{\mathcal{P}}_{17}(\mathcal{A})$, by [1].

REMARK. In Theorem 3 of [13] it is shown that, for the algebra $\mathcal{B}(\mathcal{H}^0)$, five factors are enough in approximation by products of positive operators. Their proof depends on the result that a biquasitriangular operator is in $\bar{\mathcal{P}}_4\big(\mathcal{B}(\mathcal{H}^0)\big)$, which in turn rests on a deep structure theorem of Herrero (Theorem 6.15 of [10]). So the following question arises: Can Herrero's structure theorem be extended to the algebra of decomposable operators, thereby reducing the number of factors required in Corollary 2.14 from 17 to 5? On the basis of preliminary evidence we conjecture that the answer here is "Yes".

REFERENCES

1. E. A. Azoff and K. F. Clancey, *Spectral multiplicity for direct integrals of normal operators*, J. Operator Theory **3**(1980), 213–235.
2. C. S. Ballantine, *Products of positive definite matrices IV*, Linear Algebra Appl. **3**(1970), 79–114.
3. J. B. Conway, *A course in functional analysis*, Springer-Verlag, New York, 1985.
4. J. Feldman and R. V. Kadison, *The closure of the regular operators in a ring of operators*, Proc. Amer. Math. Soc. **5**(1954), 909–916.
5. P. Fillmore, *On products of symmetries*, Canad. J. Math. **18**(1966), 897–900.
6. P. Fillmore and J. Williams, *On operator ranges*, Adv. in Math. **7**(1971), 254–281.
7. P. Halmos, *A Hilbert-space problem book*, Second Ed., Springer-Verlag, New York, 1982.
8. ———, *Bad products of good matrices*, Linear and Multilinear Algebra **29**(1991), 1–20.
9. P. Halmos and S. Kakutani, *Products of symmetries*, Bull. Amer. Math. Soc. **64**(1958), 77–78.
10. D. A. Herrero, *Approximation of Hilbert-space operators I*, Pitman Res. Notes in Math. **72**, London 1982.
11. C. J. Himmellerg, *Measurable relations*, Fund. Math. **LXXXVII**(1975), 53–72.
12. R. V. Kadison and J. G. Ringrose, *Fundamentals in the theory of operator algebras I, II*, Academic Press, New York, 1983, 1986.
13. M. Khalkhali, C. Laurie, B. Mathes and H. Radjavi, *Approximation by products of positive operators*, J. Operator Theory, to appear.
14. A. E. Nussbaum, *Reduction theory for unbounded closed operators in Hilbert-space*, Duke Math. J. **31** (1964), 33–44.
15. T. Quinn, *Factorization in C^*-algebras: products of positive operators*, Ph.D. thesis, Dalhousie Univ., Halifax, 1992.
16. H. Radjavi, *Products of Hermitian matrices and symmetries*, Proc. Amer. Math. Soc. **26**(1970), 701.
17. ———, *On self-adjoint factorization of operators*, Canad. J. Math. **21**(1969), 1421–1426.
18. M. H. Stone, *On unbounded operators in Hilbert space*, J. Indian Math. Soc. **15**(1951), 155–192.
19. M. Takesaki, *Theory of operator algebras I*, Springer-Verlag, New York, 1979.
20. J. von Neumann, *Functional Operators II*, Ann. of Math. Stud. **22**, Princeton Univ. Press, Princeton, 1950.
21. P. Y. Wu, *Products of normal operators*, Canad. J. Math. (6) **XL**(1988), 1322–1330.
22. ———, *The operator factorization problems*, Linear Algebra Appl. **117**(1989), 35–63.

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