BULL. AUSTRAL. MATH. SOC. 83C15, 35Q20
VOL. 37 (1988) [471–472]

APPLICATIONS OF SYMMETRY GROUPS AND THE WAHLQUIST-ESTABROOK PROCEDURE IN GENERAL RELATIVITY

PHILLIP CHARLES HARMSWORTH

This thesis examines the application of symmetry groups and the Wahlquist-Estabrook procedure [**10, 6**] to the Einstein gravitational field equations for spacetimes with a two-parameter Abelian isometry group.

The first chapter reviews the geometric formulation of partial differential equations and their symmetries in the language of jet bundles. Some methods are discussed for the construction of pseudopotentials associated with certain systems of equations. A new method is described identifying the resulting (prolongation) structures with finite dimensional Lie algebras, which augments that of Dodd and Fordy [**4**]. The final section outlines the formulation of a system of partial differential equations with an associated set of nontrivial pseudopotentials as a zero-curvature condition on a principal bundle [**8**].

A discussion of the application of these methods to the Ernst [**5**] and Cosgrove [**1, 2**] formulations of the vacuum field equations occupies the second chapter, which reviews the results of Cosgrove [**3**], Harrison [**7**] and Neugebauer [**9**]. It is argued that the existing formulations of the pseudopotentials associated with the Ernst system are incomplete.

The third chapter introduces two approaches to this problem. Neugebauer's formulation of the field equations is used to produce the defining equations for the pseudopotentials, and particular solutions to these are considered. The first produces an apparently nontrivial prolongation structure, which can be mapped to solvable Lie algebras of up to twelve dimensions (by the method described in the first chapter), while the second reduces the problem of solving a set of nonlinear partial differential equations (with matrix-valued dependent variables) to the solution of an Euler-Darboux equation (also with matrix-valued dependent variables). It is shown that the symmetry group of this equation is isomorphic to the non-local **Q** symmetry of Cosgrove. Harrison's pseudopotential formulation is shown to result from the only similarity solution of this equation that leads to a non-trivial prologation structure.

Received 23 October 1987. Thesis submitted to University of Sydney, February 1987. Degree approved October 1987. Supervisor: Associate Professor E.D. Fackerell

The symmetry group of the stationary axisymmetric field equations with a rigidly rotating perfect fluid is calculated in the fourth chapter. The possible reductions of the field equations under one-parameter subgroups are given, and some first integrals for a few special cases of the resulting systems of ordinary differential equations are derived.

Most of the new material in this thesis occupies the third and fourth chapters. Part of Chapter 1, describing the use of the Cartan-Killing metric to assist in the identification of prologation algebras, and the comparison in Chapter 2 of Neugebauer's results with those for the Korteweg-de Vries and sine-Gordon equations are also original.

References

[1] C.M. Cosgrove, 'A new formulation of the field equations for the stationary axisymmetric vacuum gravitational field I. General theory', *J.Phys. A:Math. Gen* **11** (1978a), 2389-2404.

[2] C.M. Cosgrove, 'A new formulation of the field equations for the stationary axisymmetric gravitational field II. Separable solutions', *J. Phys. A:Math. Gen* **11** (1978b), 2405-2430.

[3] C.M. Cosgrove, 'Stationary axisymmetric gravitational fields: an asymptotic flatness preserving transformation', in *Proc. Einstein Centenary Summer School, Perth W.A.* **124**: *Lecture Notes in Physics*, Edited by C. Edwards, 22-31 Jan, 1979 (Springer, 1980).

[4] R.K. Dodd and A. Fordy, 'The prolongation structures of quasi-polynomial flows', *Proc. Roy. Soc. Lond. Ser A* **385** (1983), 389-429.

[5] F.J. Ernst, 'New formulation of the axially symmetric gravitational field problem', *Phys. Rev.* **167** (1968), 1175-1178.

[6] F.B. Estabrook and H.D. Wahlquist, 'Prolongation structures of nonlinear evolution equations II', *J. Math. Phys.* **17** (1976), 1293-1297.

[7] B.K. Harrison, 'Unification of Ernst-equation Bäcklund transformations using a modified Wahlquist-Estabrook technique', *J. Math. Phys.* **24** (1983), 2178-2187.

[8] R. Hermann, 'Pseudopotentials of Estabrook and Wahlquist, the geometry of solitons and the theory of connections', *Phys. Rev. Lett.* **37** (1976), 835-836.

[9] G. Neugebauer, 'Bäcklund transformations of axially symmetric gravitational fields', *J. Phys. A: Math. Gen.* **12** (1979), L67-L70.

[10] H.D. Wahlquist and F.B. Estabrook, 'Prolongation structures of nonlinear evolution equations', *J. Maths. Phys.* **16** (1975), 1–7.

Department of Applied Mathematics
University of Sydney
New South Wales
Australia. 2006