9

The Lorentz–Dirac equation

We return to the Lorentz model and add slowly varying external potentials. On a formal level one can carry out the expansion in ε just as for the Abraham model. The net result is that the rotational degrees of freedom decouple from the translational degrees of freedom, and the latter are governed by

$$m_0\dot{\mathbf{u}} = (e/c)\mathbf{F}\cdot\mathbf{u} + (e^2/6\pi c^3)(\ddot{\mathbf{u}} - c^{-2}(\dot{\mathbf{u}}\cdot\dot{\mathbf{u}})\mathbf{u}), \tag{9.1}$$

which includes radiation reaction. Equation (9.1) is the Lorentz–Dirac equation, written in microscopic units. m_0 is the experimental rest mass of the particle. We reintroduced the speed of light, c. $\mathbf{F}$ is the electromagnetic field tensor of the *external* fields, where for better readability we omit the subscript "ex" in this section. The scaling parameter ε has been reabsorbed into the definition of $\mathbf{F}$, which amounts to setting $\varepsilon = 1$. It should be kept in mind that the radiation reaction is a small correction to the Hamiltonian part.

If one fixes an inertial frame of reference and goes over to three-vectors, then the time component of the Lorentz–Dirac equation reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(m_0 c^2\,\gamma(\boldsymbol{v}) + e\phi(\boldsymbol{q}) - (e^2/6\pi c^3)\,\gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\big) = -(e^2/6\pi c^3)\,\gamma^4(\dot{\boldsymbol{v}}\cdot\kappa(\boldsymbol{v})\dot{\boldsymbol{v}}), \tag{9.2}$$

and the space part becomes

$$\begin{aligned} m_0\gamma\,\kappa(\boldsymbol{v})\dot{\boldsymbol{v}} = {} & e(\boldsymbol{E}(\boldsymbol{q}) + c^{-1}\boldsymbol{v}\times\boldsymbol{B}(\boldsymbol{q})) \\ & + (e^2/6\pi c^3)\,\gamma^2\kappa(\boldsymbol{v})\,[\ddot{\boldsymbol{v}} + 3\gamma^2\,c^{-2}\,(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\,\dot{\boldsymbol{v}}], \end{aligned} \tag{9.3}$$

where as before $\kappa(\boldsymbol{v}) = \mathbb{1} + c^{-2}\,\gamma^2\,\boldsymbol{v}\otimes\boldsymbol{v}$. Equation (9.3) differs from its semirelativistic sister (8.1) only through the proper relativistic kinetic energy. Equation (9.2) is identical to the energy balance (8.6), again with proper adjustment of the kinetic energy. Thus we can follow the blueprint of section 8.2 to establish the existence of the critical manifold and to derive an effective second-order equation for the motion on the critical manifold.

The Lorentz–Dirac equation makes definite predictions about the orbit of a charged particle, including the effects of radiation losses, and one would expect that these predictions can be verified experimentally. Of course, if radiation damping is neglected, there is a multitude of laboratory set-ups. The real challenge is to observe quantitatively the minute changes in the Hamiltonian orbit due to radiation losses. We will discuss two proposals in section 9.3. The first one is the motion of an electron in a Penning trap. In the quadratic approximation for the quadrupole field, this problem can still be handled analytically, which is done in section 9.2 along with a few other examples of independent interest. The second proposal is the motion of an electron when hit by an ultrastrong laser pulse. In this case the external potentials are time dependent and one has to rely on a numerical integration of the effective second-order equation.

9.1 Critical manifold, the Landau–Lifshitz equation

We write (9.3) in the standard form of singular perturbation theory; compare with section 8.2. Then

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}), \quad \varepsilon\dot{\boldsymbol{y}} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \varepsilon) \tag{9.4}$$

with

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}_2, \boldsymbol{y}), \tag{9.5}$$

$$\begin{aligned}\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \varepsilon) = {} & (6\pi c^3/e^2)\big(m_0\gamma^{-1}\boldsymbol{y} - e\gamma^{-2}\kappa(\boldsymbol{x}_2)^{-1}(\boldsymbol{E}(\boldsymbol{x}_1) + c^{-1}\boldsymbol{x}_2 \times \boldsymbol{B}(\boldsymbol{x}_1))\big) \\ & - 3\varepsilon\gamma^2 c^{-2}(\boldsymbol{x}_2 \cdot \boldsymbol{y})\boldsymbol{y}.\end{aligned} \tag{9.6}$$

To conform with (8.1) we reintroduced the small parameter ε. At zeroth order the critical manifold is $\{(\boldsymbol{x}, \boldsymbol{h}(\boldsymbol{x}))|\boldsymbol{x} \in \mathbb{R}^3 \times \mathbb{V}\}$ with $\boldsymbol{h}(\boldsymbol{q}, \boldsymbol{v}) = (e/m_0)\gamma^{-1}\kappa(\boldsymbol{v})^{-1}\big(\boldsymbol{E}(\boldsymbol{q}) + c^{-1}\boldsymbol{v} \times \boldsymbol{B}(\boldsymbol{q})\big)$. Linearizing (9.5), (9.6) at $\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x})$ the repelling eigenvalue is $(6\pi c^3/e^2)\,m_0\gamma^{-1} + \mathcal{O}(\varepsilon)$, which vanishes as $|\boldsymbol{v}|/c \to 1$. Thus we have to rely on the construction of section 8.2, which ensures that for given maximal velocity $\bar{v}$ one can choose ε small enough such that the orbit remains on the critical manifold for all times.

To order ε the effective second-order equation is given by (8.31), except that now $m(\boldsymbol{v}) = m_0\gamma\kappa(\boldsymbol{v})$. We work out the various terms and switch back to microscopic units. Then the motion on the critical manifold of the Lorentz–Dirac equation is governed by

$$\begin{aligned}
&\dot{\boldsymbol{q}} = \boldsymbol{v},\\
&m_0\gamma\,\kappa(\boldsymbol{v})\dot{\boldsymbol{v}} = e(\boldsymbol{E} + c^{-1}\boldsymbol{v}\times\boldsymbol{B}) + \frac{e^2}{6\pi c^3}\Big[\frac{e}{m_0}\gamma\,(\boldsymbol{v}\cdot\nabla)(\boldsymbol{E} + c^{-1}\boldsymbol{v}\times\boldsymbol{B})\\
&\qquad + \Big(\frac{e}{m_0}\Big)^2 c^{-1}\Big((\boldsymbol{E}\times\boldsymbol{B}) + c^{-1}(\boldsymbol{v}\cdot\boldsymbol{E})\boldsymbol{E} + c^{-1}(\boldsymbol{v}\cdot\boldsymbol{B})\boldsymbol{B}\\
&\qquad + \big(-\boldsymbol{E}^2 - \boldsymbol{B}^2 + c^{-2}(\boldsymbol{v}\cdot\boldsymbol{E})^2 + c^{-2}(\boldsymbol{v}\cdot\boldsymbol{B})^2\\
&\qquad + 2c^{-1}\boldsymbol{v}\cdot(\boldsymbol{E}\times\boldsymbol{B})\big)\gamma^2 c^{-1}\boldsymbol{v}\Big)\Big].
\end{aligned} \tag{9.7}$$

While singular perturbation theory provides a systematic method, Eq. (9.7) can also be derived formally. In (9.3) we regard $m_0\gamma\,\kappa(\boldsymbol{v})\dot{\boldsymbol{v}} = e\,(\boldsymbol{E} + c^{-1}\boldsymbol{v}\times\boldsymbol{B})$ as the unperturbed equation, differentiate it once, and substitute $\ddot{\boldsymbol{v}}$ inside the square brackets of (9.3). Resubstituting $\dot{\boldsymbol{v}}$ from the unperturbed equation results in Eq. (9.7). This argument is carried out more easily and in greater generality, because it allows for time-dependent potentials, in the covariant form of the Lorentz–Dirac equation. The unperturbed equation is

$$m_0\dot{\mathbf{u}} = (e/c)\mathbf{F}(\mathbf{q})\cdot\mathbf{u}. \tag{9.8}$$

One differentiates with respect to the eigentime,

$$(m_0 c/e)\ddot{\mathbf{u}} = (\mathbf{u}\cdot\nabla_g)\mathbf{F}(\mathbf{q})\cdot\mathbf{u} + \mathbf{F}(\mathbf{q})\cdot\dot{\mathbf{u}}. \tag{9.9}$$

Substituting (9.9) in (9.1) and resubstituting (9.8) yields

$$\begin{aligned}
m_0\dot{\mathbf{u}} = (e/c)\mathbf{F}\cdot\mathbf{u} + \frac{e^2}{6\pi c^3}\big[&(e/m_0 c)(\mathbf{u}\cdot\nabla_g)\mathbf{F}\cdot\mathbf{u} + (e/m_0 c)^2\mathbf{F}\cdot\mathbf{F}\cdot\mathbf{u}\\
&-(e/m_0 c^2)^2(\mathbf{F}\cdot\mathbf{u})\cdot(\mathbf{F}\cdot\mathbf{u})\mathbf{u}\big].
\end{aligned} \tag{9.10}$$

In three-vector notation the space part of Eq. (9.10) coincides with (9.7), except for the additional term $(e/m_0)\gamma(\partial_t\boldsymbol{E} + c^{-1}\boldsymbol{v}\times\partial_t\boldsymbol{B})$ because of the time dependence of the fields. As usual, the time component of (9.10) provides the energy balance.

Equation (9.10) and its formal derivation appeared for the first time in the second volume of the *Course in Theoretical Physics* by Landau and Lifshitz. Hence it seems to be appropriate to call Eq. (9.10) the Landau–Lifshitz equation. The error in going from (9.1) to (9.10) is of the same order as that in the derivation of the Lorentz–Dirac equation itself. *Thus we regard the Landau–Lifshitz equation as* *the effective equation governing the motion of a charged particle in the adiabatic limit.*

9.2 Some applications

(i) *Zero magnetic field, one-dimensional motion.* We assume $\boldsymbol{B} = 0$ and ϕ_{ex} to vary only along the 1-axis. Setting $\boldsymbol{v} = (v, 0, 0)$, $\boldsymbol{q} = (x, 0, 0)$, and $\boldsymbol{E} = (-\phi', 0, 0)$, the Landau–Lifshitz equation becomes

$$m_0 \gamma^3 \dot{v} = -e\phi'(x) - \frac{e^2}{6\pi c^3} \frac{e}{m_0} \gamma \phi''(x) v. \tag{9.11}$$

The radiation reaction is proportional to $-\phi''(x)v$, which can be regarded as a spatially varying friction coefficient. For a convex potential, $\phi'' > 0$, such as an oscillator potential, this friction coefficient is strictly positive and the resulting motion is damped until the minimum of ϕ is reached. In general, however, ϕ'' will not have a definite sign, like in the case of the double well potential $\phi(x) \simeq (x^2 - 1)^2$ or the washboard potential $\phi(x) \simeq -\cos x$. At locations where $\phi''(x) < 0$ one has antifriction and the mechanical energy increases. This gain is always dominated by losses as can be seen from the energy balance

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big[m_0 \gamma + e\phi + \frac{e^2}{6\pi c^3} \frac{e}{m_0} \gamma \phi' v\Big] \\ &\quad = -\frac{e^2}{6\pi c^3}\Big(\frac{e}{m_0}\Big)^2 \phi'^2 - \frac{1}{m_0}\Big(\frac{e^2}{6\pi c^3} \frac{e}{m_0}\Big)^2 \gamma \phi' \phi'' v \,. \end{aligned} \tag{9.12}$$

The last term in (9.12) does not have a definite sign. But its prefactor is down by the factor $e^2/m_0 c^3$ and therefore it is outweighed by $-\phi'^2$.

Equation (9.11) has one peculiar feature. If $\phi(x) = -Ex$, $E > 0$, over some interval $[a_-, a_+]$, then $\phi'' = 0$ over that interval and the friction term vanishes. The particle entering at a_- is uniformly accelerated to the right until it reaches a_+. From Larmor's formula we know that the energy radiated per unit time equals $(e^2/6\pi c^3)(e/m_0)^2 E^2$. Since the mechanical energy is conserved, the radiated energy must come entirely from the Schott energy stored in the near field. The same behavior is found for the Lorentz–Dirac equation. If, locally, $\boldsymbol{E} =$ const. and $\boldsymbol{B} = 0$, then the Hamiltonian part is solved by hyperbolic motion, i.e. a constantly accelerated relativistic particle. For this solution the radiation reaction vanishes which means that locally the critical manifold happens to be independent of ε. The radiated energy originates exclusively from the Schott energy.

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(ii) *Zero magnetic field, central potential.* For zero magnetic field the Landau–Lifshitz equation simplifies to

$$m_0\,\gamma\,\kappa(\boldsymbol{v})\dot{\boldsymbol{v}} = e\,\boldsymbol{E} + \frac{e^2}{6\pi c^3}\Big[\frac{e}{m_0}\,\gamma\,(\boldsymbol{v}\cdot\nabla)\boldsymbol{E} + \Big(\frac{e}{m_0 c}\Big)^2\Big((\boldsymbol{v}\cdot\boldsymbol{E})\boldsymbol{E} - \gamma^2\,\boldsymbol{E}^2\boldsymbol{v} + \gamma^2 c^{-2}(\boldsymbol{v}\cdot\boldsymbol{E})^2\boldsymbol{v}\Big)\Big]. \tag{9.13}$$

We take $\boldsymbol{E} = -\nabla\phi_{\mathrm{ex}}$ and assume that ϕ_{ex} is central. Let us set $\boldsymbol{q} = \boldsymbol{r}$, $|\boldsymbol{r}| = r$, $\hat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}|$, $\phi_{\mathrm{ex}}(\boldsymbol{q}) = \phi(r)$ which implies $\boldsymbol{E} = -\phi'\hat{\boldsymbol{r}}$. Then (9.13) becomes

$$m_0\,\gamma\,\kappa(\boldsymbol{v})\dot{\boldsymbol{v}} = -e\,\phi'\hat{\boldsymbol{r}} + \frac{e^2}{6\pi c^3}\Big[\frac{e}{m_0}\,\gamma\,\Big(-(\boldsymbol{v}\cdot\hat{\boldsymbol{r}})\phi''\hat{\boldsymbol{r}} - \frac{1}{r}(\boldsymbol{v} - (\boldsymbol{v}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}})\phi'\Big) + \Big(\frac{e}{m_0 c}\Big)^2\phi'^2\Big((\boldsymbol{v}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \gamma^2\,\boldsymbol{v} + \gamma^2\,c^{-2}(\boldsymbol{v}\cdot\hat{\boldsymbol{r}})^2\,\boldsymbol{v}\Big)\Big]. \tag{9.14}$$

The angular momentum $\boldsymbol{L} = \boldsymbol{r} \times m_0\gamma\boldsymbol{v}$ satisfies

$$\dot{\boldsymbol{L}} = \frac{e^2}{6\pi c^3}\Big[-\frac{e}{m_0}\,\frac{1}{r}\,\phi' - \Big(\frac{e}{m_0 c}\Big)^2\gamma^2\left(1 - c^{-2}(\boldsymbol{v}\cdot\hat{\boldsymbol{r}})^2\right)\phi'^2\Big]\boldsymbol{L}. \tag{9.15}$$

Thus the orientation of $\boldsymbol{L}$ is conserved and the motion lies in the plane perpendicular to $\boldsymbol{L}$. No further reduction seems to be possible and one would have to rely on numerical integration. Only for the harmonic oscillator, $\phi(r) = \frac{1}{2}\,m_0\omega_0^2 r^2$, can a closed form solution be achieved.

(iii) *Zero electrostatic field and constant magnetic field.* We set $\boldsymbol{B} = (0, 0, B)$ with constant B. Then (9.7) simplifies to

$$m_0\,\gamma\,\kappa(\boldsymbol{v})\dot{\boldsymbol{v}} = \frac{e}{c}(\boldsymbol{v}\times\boldsymbol{B}) + \frac{e^2}{6\pi c^3}\Big(\frac{e}{m_0 c}\Big)^2\big[(\boldsymbol{v}\cdot\boldsymbol{B})\boldsymbol{B} - \gamma^2\boldsymbol{B}^2\boldsymbol{v} + \gamma^2\,c^{-2}(\boldsymbol{v}\cdot\boldsymbol{B})^2\boldsymbol{v}\big]. \tag{9.16}$$

We multiply by $\kappa(\boldsymbol{v})^{-1}$ to obtain

$$m_0\,\gamma\,\dot{\boldsymbol{v}} = \frac{e}{c}(\boldsymbol{v}\times\boldsymbol{B}) + \frac{e^2}{6\pi c^3}\Big(\frac{e}{m_0 c}\Big)^2\big[(\boldsymbol{v}\cdot\boldsymbol{B})\boldsymbol{B} - \boldsymbol{B}^2\boldsymbol{v}\big]. \tag{9.17}$$

The motion parallel to $\boldsymbol{B}$ decouples with $\dot{v}_3 = 0$. We set $v_3 = 0$ and $\boldsymbol{v} = (\boldsymbol{u}, 0)$, $\boldsymbol{u}^{\perp} = (-u_2, u_1)$. Then the motion in the plane orthogonal to $\boldsymbol{B}$ is governed by

$$\gamma\,\dot{\boldsymbol{u}} = \omega_{\mathrm{c}}(\boldsymbol{u}^{\perp} - \beta\omega_{\mathrm{c}}\boldsymbol{u}), \tag{9.18}$$

with the cyclotron frequency $\omega_{\mathrm{c}} = eB/m_0 c$ and $\beta = e^2/6\pi c^3 m_0$. Equation (9.18) holds over the entire velocity range. For an electron $\beta\omega_{\mathrm{c}} = 8.8\times 10^{-18}B$ [gauss]. Thus even for very strong fields the friction is small compared to the inertial terms.

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Equation (9.18) can be integrated as

$$\frac{d}{dt}\gamma = -\beta\omega_c^2(\gamma^2 - 1) \tag{9.19}$$

with solution

$$\gamma(t) = \left[\gamma_0 + 1 + (\gamma_0 - 1)e^{-2\beta\omega_c^2 t}\right]\left[\gamma_0 + 1 - (\gamma_0 - 1)e^{-2\beta\omega_c^2 t}\right]^{-1}, \tag{9.20}$$

which tells us how $\boldsymbol{u}(t)^2$ shrinks to zero. To determine the angular dependence we introduce polar coordinates as $\boldsymbol{u} = u(\cos\varphi, \sin\varphi)$. Then

$$\frac{du}{d\varphi} = -\beta\omega_c u, \quad \frac{d\varphi}{dt} = \gamma^{-1}\omega. \tag{9.21}$$

Thus $u(\varphi)$ shrinks exponentially,

$$u(\varphi) = u(0)\,e^{-\beta\omega_c\varphi}. \tag{9.22}$$

Since $\beta\omega_c = 8.8 \times 10^{-18} B$ [gauss] for an electron, even for strong fields the change of u over one revolution is tiny.

To obtain the evolution of the position $\boldsymbol{q} = (\boldsymbol{r}, 0)$, $|\boldsymbol{r}| = r$, we use the fact that for zero radiation reaction, $\beta = 0$,

$$r = \frac{u}{\omega_c}\gamma\,. \tag{9.23}$$

By (9.22) this relation remains approximately valid for non-zero β. Inserting $u(t)$ from (9.20) one obtains

$$r(t) = r_0\,e^{-\beta\omega_c^2 t}\left[1 + ((\gamma_0 - 1)/2)(1 - e^{-2\beta\omega_c^2 t})\right]^{-1} \tag{9.24}$$

with r_0 the initial radius and $u(0)/c = (\gamma_0 - 1)^{1/2}/\gamma_0$ the initial speed which are related through (9.23). In the ultrarelativistic regime, $\gamma_0 \gg 1$, and for times such that $\beta\omega_c^2 t \ll 1$, (9.24) simplifies to

$$r(t) = r_0\frac{1}{1 + \gamma_0\beta\omega_c^2 t} \tag{9.25}$$

and the initial decay is according to the power law t^{-1} rather than exponential.

For an electron $\beta\omega_c^2 = 1.6 \times 10^{-6}(B\ [\text{gauss}])^2\ \text{s}^{-1}$. Therefore if one chooses a field strength $B = 10^3$ gauss and an initial radius of $r_0 = 10$ cm, which corresponds to the ultrarelativistic case of $\gamma = 6 \times 10^4$, then the radius shrinks within 0.9 s to $r(t) = 1\ \mu$m by which time the electron has made 2×10^{14} revolutions.

(iv) *The Penning trap*. An electron can be trapped for a very long time in the combination of a homogeneous magnetic field and an electrostatic quadrupole potential, which has come to be known as a Penning trap. Its design has been optimized towards high-precision measurements of the gyromagnetic g-factor of the

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electron. Our interest here is that the motion in the plane orthogonal to the magnetic field consists of two coupled modes, which means that the damping cannot be guessed by pure energy considerations using Larmor's formula. One really needs the full power of the Landau–Lifshitz equation.

An ideal Penning trap has the electrostatic quadrupole potential

$$e\phi(\boldsymbol{x}) = \frac{1}{2} m\omega_{\mathrm{z}}^2\Big(-\frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 + x_3^2\Big), \tag{9.26}$$

which satisfies $\Delta\phi = 0$, superimposed with the uniform magnetic field

$$\boldsymbol{B} = (0, 0, B). \tag{9.27}$$

The quadrupole field provides an axial restoring force whereas the magnetic field is responsible for the radial restoring force, which however could be outweighed by the inverted part of the harmonic electrostatic potential.

We insert $\boldsymbol{E} = -\nabla\phi$ and $\boldsymbol{B}$ in the Landau–Lifshitz equation. The terms proportional to $(\boldsymbol{v}\cdot\nabla)\boldsymbol{E},\ \boldsymbol{E}\times\boldsymbol{B},\ (\boldsymbol{v}\cdot\boldsymbol{B})\boldsymbol{B}$, and $\boldsymbol{B}^2\boldsymbol{v}$ are linear in $\boldsymbol{v}$, respectively $\boldsymbol{q}$. The remaining terms are either cubic or quintic and will be neglected. This is justified provided

$$\frac{|\boldsymbol{v}|}{c} \ll 1 \tag{9.28}$$

and

$$(m_0\omega_{\mathrm{z}}^2/e)\, r_{\max} \ll B, \text{ i.e. } \quad r_{\max} \ll c(\omega_{\mathrm{c}}/\omega_{\mathrm{z}}^2), \tag{9.29}$$

if $r_{\max}$ denotes the maximal distance from the trap center. With these assumptions the Landau–Lifshitz equation decouples into an in-plane motion and an axial motion governed by

$$\dot{\boldsymbol{u}} = \frac{1}{2}\omega_{\mathrm{z}}^2\boldsymbol{r} + \omega_{\mathrm{c}}\boldsymbol{u}^{\perp} - \beta\Big[\Big(\omega_{\mathrm{c}}^2 - \frac{1}{2}\omega_{\mathrm{z}}^2\Big)\boldsymbol{u} + \frac{1}{2}\omega_{\mathrm{c}}\omega_{\mathrm{z}}^2\boldsymbol{r}^{\perp}\Big], \tag{9.30}$$

$$\ddot{z} = -\omega_{\mathrm{z}}^2 z - \beta\omega_{\mathrm{z}}^2\dot{z}. \tag{9.31}$$

Here $\boldsymbol{q} = (\boldsymbol{r}, z),\ \boldsymbol{v} = (\boldsymbol{u}, \dot{z}),\ (x_1, x_2)^{\perp} = (-x_2, x_1)$.

The axial motion is just a damped harmonic oscillation with frequency ω_z and friction coefficient

$$\gamma_{\mathrm{z}} = \beta\omega_{\mathrm{z}}^2. \tag{9.32}$$

The in-plane motion can be written in matrix form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi = (A + \beta V)\psi \tag{9.33}$$

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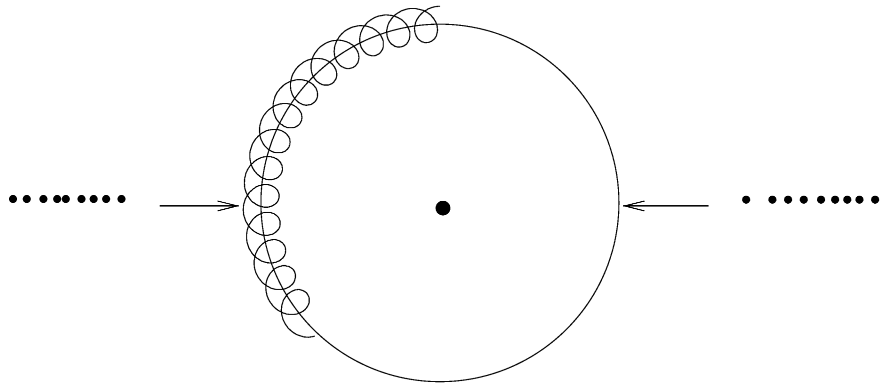

Figure 9.1: Orbit of an electron in a Penning trap seen from above.

with $\psi = (\boldsymbol{r}, \boldsymbol{u})$ and $A_{11} = 0$, $A_{12} = \mathbb{1}$, $A_{21} = \omega_{\mathrm{z}}^2 \mathbb{1}$, $A_{22} = \mathrm{i}\omega_{\mathrm{z}}\sigma_y$, $V_{11} = 0$, $V_{12} = 0$, $V_{21} = \mathrm{i}\omega_{\mathrm{c}}\omega_{\mathrm{z}}^2\sigma_y$, $V_{22} = (\omega_{\mathrm{z}}^2 - \omega_{\mathrm{c}}^2)\mathbb{1}$, where σ_y is the Pauli spin matrix with eigenvectors $\chi_\pm$, $\sigma_y\chi_\pm = \pm\chi_\pm$. The unperturbed motion is governed by the 4×4 matrix A. It has the eigenvectors $\psi_{+,\pm} = (\pm\mathrm{i}(1/\omega_+)\chi_\mp, \chi_\mp)$ with eigenvalues $\pm\mathrm{i}\omega_+$ and $\psi_{-,\pm} = (\pm\mathrm{i}(1/\omega_-)\chi_\mp, \chi_\mp)$ with eigenvalues $\pm\mathrm{i}\omega_-$, where

$$\omega_\pm = \frac{1}{2}\left(\omega_{\mathrm{c}} \pm \sqrt{\omega_{\mathrm{c}}^2 - 2\omega_{\mathrm{z}}^2}\right). \tag{9.34}$$

The mode with frequency ω_+ is called the cyclotron mode and that with ω_- is called the magnetron mode. Experimentally $\omega_{\mathrm{c}} \gg \omega_{\mathrm{z}}$ and therefore $\omega_+ \ll \omega_-$. The orbit is then an epicycle with rapid cyclotron and slow magnetron motion, as shown in figure 9.1. The adjoint matrix A^* has eigenvectors orthogonal to the ψ's. They are given by $\varphi_{+,\pm} = (\mp\mathrm{i}(\omega_{\mathrm{z}}^2/\omega_+)\chi_\mp, \chi_\mp)$ with eigenvalues $\pm\mathrm{i}\omega_+$ and $\varphi_{-,\pm} = (-(\omega_{\mathrm{z}}^2/\omega_-)\chi_\mp, \chi_\mp)$ with eigenvalues $\mp\mathrm{i}\omega_-$.

Since β is small, the eigenfrequencies of $A + \beta V$ can be computed in first-order perturbation. The cyclotron mode attains a negative real part corresponding to the friction coefficient

$$\gamma_+ = \frac{e^2}{6\pi c^3 m_0} \frac{\omega_+^3}{\omega_+ - \omega_-} \tag{9.35}$$

and the magnetron mode attains a positive real part corresponding to the antifriction coefficient

$$\gamma_- = \frac{e^2}{6\pi c^3 m_0} \frac{\omega_-^3}{\omega_- - \omega_+}. \tag{9.36}$$

As the electron radiates, it lowers its potential energy by increasing the magnetron radius.

Experimentally $B = 6 \times 10^4$ gauss and the voltage drop across the trap is 10 V. This corresponds to $\omega_z = 4 \times 10^8$ Hz, $\omega_+ = 1.1 \times 10^{12}$ Hz, $\omega_- = 7.4 \times 10^4$ Hz. The conditions (9.28), (9.29) are easily satisfied. For the lifetimes $(1/\gamma_z) = 5 \times 10^8$ s, $(1/\gamma_+) = 8 \times 10^{-2}$ s, and $(1/\gamma_-) = -2 \times 10^{23}$ s are obtained. Thus the magnetron motion is stable, as observed through keeping a single electron trapped over weeks. The cyclotron motion decays within fractions of a second. The axial motion is in fact damped by coupling to the external circuit and decays also within a second.

The variation with the magnetic field can be more clearly discussed in terms of the dimensionless ratio $(\omega_c/\omega_z) = \lambda$. Then

$$\omega_\pm = \omega_z \frac{1}{2}\left(\lambda \pm \sqrt{\lambda^2 - 2}\right), \quad \gamma_\pm = \pm\beta\omega_z^2\left(\lambda \pm \sqrt{\lambda^2 - 2}\right)^3 / 8\sqrt{\lambda^2 - 2}. \tag{9.37}$$

For large λ, $\omega_+ \cong \lambda$, $\omega_- \cong \lambda^{-1}$, while $\gamma_+ \cong \lambda^2$, $\gamma_- \cong \lambda^{-4}$. As $\lambda \to \sqrt{2}$, we have $\omega_+ = \omega_- = \omega_z/\sqrt{2}$. However, the friction coefficients diverge as $(\lambda - \sqrt{2})^{-1/2}$. Let us call B_c the critical field at which the mechanical motion becomes unstable. For $B > B_c$, one still has periodic motion with frequency $\omega_z/\sqrt{2}$, but the onsetting instability is revealed through the vanishing lifetime. In the mentioned experiment $\lambda = 2.7 \times 10^3$ and for fixed ω_z the critical field strength is $B_c = 30$ gauss.

9.3 Experimental status of the Lorentz–Dirac equation

Energy loss through radiation is a well-established phenomenon. Indeed, in synchrotron sources electrons slow down because of radiation losses, and energy has to be supplied to maintain a steady electron current. The supplied power is computed on the basis of Larmor's formula, and synchrotron sources are one prominent example to confirm its validity. On the other hand, the Lorentz–Dirac equation goes way beyond mere energy balancing and claims to predict the orbit of an electron. Here synchrotron sources provide no test, since the modification of the orbit due to radiation damping is lost in the noise of experimental uncertainties. As a fair summary, thus we can say that while qualitative aspects of radiation damping are well tested, there is no single experiment which probes quantitatively the predictions of the electron motion by the Lorentz–Dirac equation. We propose and discuss here two experiments which are within the reach of present-day techniques.

To cope with the smallness of the radiation reaction, in essence, only two approaches seem feasible. The first one is to wait long enough until the effects accumulate to something which may be detected, a route followed in the Penning-trap experiment. The other option is to use ultrastrong fields. In either case, there is no way to monitor directly the electron orbit and one has to rely on indirect evidence, like lifetimes or emission spectra.

(i) *Penning trap.* In the previous section we discussed the electron orbits for the Penning trap with the quadrupole potential in the quadratic approximation. The Lorentz–Dirac equation predicts, in particular, the lifetime of the cyclotron mode. For the field strengths used in the high-precision measurement of the g-factor, this lifetime is measured to 0.8 s in good agreement with the theoretical result. To have a more stringent test what would be needed is a systematic determination of how the lifetime depends on the magnetic field strength. Another option of interest is to turn the $\boldsymbol{B}$-field out off the symmetry axis. For this case we have not computed the cyclotron lifetime, but could have done so by the scheme explained, with the welcome complication that all three modes couple. The dependence of the cyclotron lifetime on the orientation of the $\boldsymbol{B}$-field would be a valuable test of the validity of the Lorentz–Dirac equation.

(ii) *Ultrastrong laser pulse.* A strong laser pulse hits a bound electron. Since the atom ionizes instantaneously, the electron is subject only to the time-dependent laser field. Thus we set $\boldsymbol{q}^0 = 0$, $\boldsymbol{v}^0 = 0$, and for the external fields

$$\begin{aligned}
&\boldsymbol{E}(\boldsymbol{x}, t) = h(\omega t - \boldsymbol{k} \cdot \boldsymbol{x})\boldsymbol{E}_0 \cos(\omega t - \boldsymbol{k} \cdot \boldsymbol{x}), \\
&\boldsymbol{B}(\boldsymbol{x}, t) = h(\omega t - \boldsymbol{k} \cdot \boldsymbol{x})\boldsymbol{B}_0 \cos(\omega t - \boldsymbol{k} \cdot \boldsymbol{x}), \\
&|\boldsymbol{E}_0| = |\boldsymbol{B}_0|, \quad \boldsymbol{E}_0 \cdot \boldsymbol{k} = 0 = \boldsymbol{B}_0 \cdot \boldsymbol{k}, \quad \boldsymbol{E}_0 \cdot \boldsymbol{B}_0 = 0.
\end{aligned} \tag{9.38}$$

h is a shape function. The motion of the electron is governed by the Landau–Lifshitz equation (9.7) augmented by the term

$$\frac{e^2}{6\pi c^3} \frac{e}{m_0} \gamma \frac{\partial}{\partial t} (\boldsymbol{E} + c^{-1} \boldsymbol{v} \times \boldsymbol{B}) \tag{9.39}$$

because of the time dependence of the external fields. Our dynamical problem is in fact two dimensional with the motion of the electron lying in the plane spanned by $\boldsymbol{E}_0$ and $\boldsymbol{k}$. Nevertheless one has to rely on numerical integration, and we discuss the example from Keitel *et al.* (1998).

The ultra-intense laser field has an intensity of 10^{22} W cm^{-2}. The frequency is chosen to be $\omega = 3.54 \times 10^{15}$ s^{-1}, which is in the near-infrared regime. We follow the motion of the electron up to 3000 laser cycles, i.e. up to the final time $t = 3000(2\pi/3.54 \times 10^{15})$ s $= 0.53 \times 10^{-11}$ s. Over that time span the shape function is assumed to interpolate linearly between zero and the full field strength.

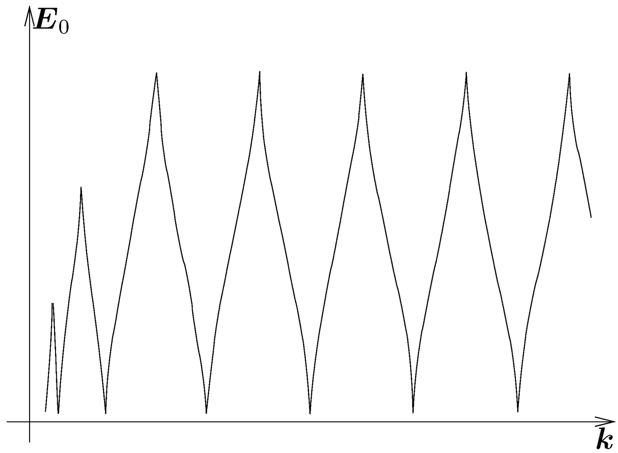

Figure 9.2: Orbit of an electron when hit by an ultrastrong laser pulse.

The electron motion is highly relativistic, as can be seen from the strong redshift corresponding to only the seven electron cycles displayed in figure 9.2. The electron is displaced by 0.1586 cm in the propagation direction and has a maximal amplitude of 0.795×10^{-3} cm in the electric field direction.

The effects of radiation damping are minute. In the propagation direction the distance is *in*creased by the fraction 7×10^{-7} and in the electric field direction it is *de*creased by the fraction 10^{-2}. Thus a direct verification of the radiation reaction is out of reach. However, in the emission spectrum the radiation damping results in a roughly 1% change as compared to the frictionless solution with the Lorentz force from the external fields of (9.38). In an experiment the radiation spectrum has to be measured with such precision that, after the theoretical spectrum, computed without radiation reaction, has been subtracted, there is still a significant background which allows for a quantitative comparison with the emission spectrum predicted by the Lorentz–Dirac equation.

Notes and references

Section 9

The name Lorentz–Dirac is standard but historically inaccurate. Some authors, e.g. Rohrlich (1997), therefore propose Abraham–Lorentz–Dirac instead. The radiation reaction term was originally derived in Abraham (1905); compare with chapters 7 and 8. Von Laue (1909) realized its covariant form. In the Pauli (1921) encyclopaedia article on relativity the equation is stated as in (9.1). Dirac's contribution is explained in section 3.3. Plass (1961) is a summary of exact solutions of the Lorentz–Dirac equation.

Section 9.1

Detailed case studies of the Lorentz–Dirac equation, including its center manifold, are listed in the Notes to section 8.2. Baylis and Huschilt (2002) critically explore the relation to the Landau–Lifshitz equation. The substitution trick seems to have been common knowledge. For example, without further comment it is used by Pauli (1929) and Heitler (1936) in the particular case of a harmonic oscillator. In its full generality the Landau–Lifshitz equation (9.10) appears already in the first edition of *Volume II: The Classical Theory of Fields* of the Landau–Lifshitz *Course in Theoretical Physics*. At no point is the reader given a hint on the geometrical picture of the solution flow and on the errors involved in the approximation. To me it is rather surprising that the contribution of Landau and Lifshitz is ignored in essentially all discussions of radiation reaction, one notable exception being Teitelbom *et al.* (1980). Spohn (1998, 2000a) uses singular perturbation theory to rederive the Landau–Lifshitz equation. The appearance of singular perturbation theory is difficult to track. For a particular application it is clearly stated by Burke (1970). There have been attempts to replace the Lorentz–Dirac equation by a second-order equation (Mo and Papas 1971; Shen 1972b; Bonnor 1974; Parrot 1987; Valentini 1988; Ford and O'Connell 1991, 1993). Based on Ford and O'Connell (1991), Jackson (1999) uses the substitution trick for a radiation damped harmonic oscillator and discusses several applications. In the case of arbitrary time-dependent potentials, only Landau and Lifshitz provide the correct center manifold equation. The structure discussed here reappears whenever a low-dimensional system is coupled to a wave equation; for an application in acoustics see Templin (1999).

Section 9.2

Uniform acceleration is discussed in Fulton and Rohrlich (1960) and Rohrlich (1990). A constant magnetic field is of importance for synchrotron sources. Since the electron is maintained on a circular orbit, Larmor's formula is precise enough. Landau and Lifshitz (1959) give a brief discussion. The power law for the ultrarelativistic case is noted in Spohn (1998). Shen (1972a, 1978) discusses at which field strengths quantum corrections will become important. His results are only partially reliable, since his starting point is not the Landau–Lifshitz equation. The Penning trap is reviewed by Brown and Gabrielse (1986), which includes a discussion of the classical orbits and their lifetimes. They state the results (9.35), (9.36) as based on a quantum resonance computation. Since the final answer does not contain $\hbar$, it must follow from the Landau–Lifshitz equation (Spohn 2000a). In the classical framework, general trap potentials can be handled through numerical integration routines for ordinary differential equations. The self-force in the case of synchroton radiation is studied by Burko (2000).

Section 9.3

The Penning-trap experiment is proposed in Spohn (2000a). The numerical results on ultrastrong laser pulses are taken from Keitel *et al.* (1998). Another proposal, which apparently never received the proper funding, is to measure the mega-gauss magnetic bremsstrahlung for ultrarelativistic electrons (Erber 1971; Shen 1970).