Bull. Aust. Math. Soc. **83** (2011), 262–266
doi:10.1017/S0004972710002005

A NOTE ON BADLY APPROXIMABLE LINEAR FORMS

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(Received 8 June 2010)

Abstract

In this paper we investigate the analogue of the classical badly approximable setup in which the distance to the nearest integer $\|\cdot\|$ is replaced by the sup norm $|\cdot|$. In the case of one linear form we prove that the hybrid badly approximable set is of full Hausdorff dimension.

2010 *Mathematics subject classification*: primary 11J25; secondary 11J13.

Keywords and phrases: Diophantine approximation, systems of linear forms, Hausdorff dimension.

1. Introduction

Let $X = (x_{ij}) \in \mathbb{I}^{mn} := (0, 1]^{mn}$ be an $m \times n$ matrix. Let

$$q_1 x_{1i} + q_2 x_{2i} + \cdots + q_m x_{mi} \quad (1 \leq i \leq n)$$

be a system of n linear forms in m variables. The system will be written more concisely as $\mathbf{q}X$. The classical result of Dirichlet [3] states that for any point $X \in \mathbb{I}^{mn}$, there exist infinitely many integer points $\mathbf{q} \in \mathbb{Z}^m$ such that

$$\|\mathbf{q}X\| := \max_{1 \leq i \leq n} \|q_1 x_{1i} + q_2 x_{2i} + \cdots + q_m x_{mi}\| < |\mathbf{q}|^{-m/n}, \tag{1.1}$$

where $|\mathbf{q}|$ denotes the supremum norm; that is, $|\mathbf{q}| := \max\{|q_1|, |q_2|, \ldots, |q_m|\}$, and $\|\cdot\|$ denotes the distance to the nearest integer in $\mathbb{Z}^n$. The right-hand side of (1.1) may be sharpened by a constant $c(m, n)$ but the best permissible values for $c(m, n)$ are unknown except for $m = n = 1$. A point $X \in \mathbb{I}^{mn}$ is said to be *badly approximable* if the right-hand side of (1.1) cannot be improved by an arbitrary positive constant. Denote the set of all such points as $\mathbf{Bad}(m, n)$; that is, $X \in \mathbf{Bad}(m, n)$ if there exists a constant $C(X) > 0$ such that

$$\|\mathbf{q}X\| > C(X)|\mathbf{q}|^{-m/n} \quad \forall \mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}.$$

The set $\mathbf{Bad}(1, 1)$ is the standard set of badly approximable numbers and corresponds to those irrationals with bounded continued fraction expansion. A consequence of a fundamental theorem of Khintchine in the theory of Diophantine

Research supported by Higher Education Commission of Pakistan: 3-6-25/HEC/Overseas/2006/822.

approximation is that $\mathbf{Bad}(1, 1)$ is of zero Lebesgue measure. Nevertheless, a classical result of Jarník [7] states that $\mathbf{Bad}(1, 1)$ is a large set in the sense that it has maximal dimension. More precisely, $\dim \mathbf{Bad}(1, 1) = 1$ where $\dim A$ denotes the Hausdorff dimension of the set A—see [5] for the definition of Hausdorff dimension. In higher dimensions, the Khintchine–Groshev theorem [8] implies that $\mathbf{Bad}(m, n)$ is of mn-dimensional Lebesgue measure zero and a result of Schmidt [9] states that $\dim \mathbf{Bad}(m, n) = mn$.

In this note we investigate the hybrid of $\mathbf{Bad}(m, n)$ in which the distance to the nearest integer $\|\cdot\|$ is replaced by the sup norm $|\cdot|$. The corresponding well-approximable theory has been well developed over the years and the hybrid well-approximable sets naturally appear in operator theory and KAM theory—see [2, 4].

A consequence of the Dirichlet type theorem established by Dickinson in [1] is the following statement.

LEMMA (Dickinson). *For each $X \in \mathbb{I}^{mn}$ there exist infinitely many nonzero integer vectors $\mathbf{q} \in \mathbb{Z}^m$ such that*

$$|\mathbf{q}X| < m|\mathbf{q}|^{-m/n+1}.$$

In view of this lemma, it is natural to consider the following badly approximable set. Let $\mathbf{Bad}^*(m, n)$ denote the set of $X \in \mathbb{I}^{mn}$ for which there exists a constant $C(X) > 0$ such that

$$|\mathbf{q}X| > C(X)|\mathbf{q}|^{-m/n+1} \quad \forall \mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}. \tag{1.2}$$

REMARK 1.1. In the case where $m = n$, it is easily seen that

$$\mathbb{I}^{m^2} \setminus \{X \in \mathbb{I}^{m^2} : \det X = 0\} = \mathbf{Bad}^*(m, m).$$

Now

$$|\{X \in \mathbb{I}^{m^2} : \det X = 0\}|_{m^2} = 0$$

where $|\cdot|_k$ denotes k-dimensional Lebesgue measure. Hence, it follows that

$$|\mathbf{Bad}^*(m, m)|_{m^2} = 1$$

and so $\dim \mathbf{Bad}^*(m, m) = m^2$. The upshot of this is that in the case $m = n$ the corresponding badly approximable set is of full dimension.

In the case where $m > n$, the Khintchine–Groshev type theorem recently established in [6] implies that the mn-dimensional Lebesgue measure of $\mathbf{Bad}^*(m, n)$ is zero; that is,

$$|\mathbf{Bad}^*(m, n)|_{mn} = 0.$$

Naturally, one would expect that the following analogue of Schmidt's theorem is true.

CONJECTURE. For $m > n$, $\dim \mathbf{Bad}^*(m, n) = mn$.

In this note we establish the conjecture for one linear form ($n = 1$) in m variables.

THEOREM 1.2. *For $m \geq 1$,*

$$\dim \mathbf{Bad}^*(m, 1) = m.$$

https://doi.org/10.1017/S0004972710002005 Published online by Cambridge University Press

2. Proof of Theorem 1.2

In view of the above remark, we can assume without loss of generality that $m \geq 2$. Next, since $\mathbf{Bad}^*(m, 1) \subseteq \mathbb{I}^m$, we immediately have that

$$\dim \mathbf{Bad}^*(m, 1) \leq m.$$

Thus the proof of Theorem 1.2 follows on obtaining the complementary lower bound estimate. For this we shall make use of the following result.

LEMMA 2.1. *Let S be a subset of $\mathbb{I}^k$ and let*

$$\Lambda := \{(x, xS) : x \in \mathbb{I}\}.$$

If $\dim S = k$, *then* $\dim \Lambda = k + 1$.

PROOF OF LEMMA 2.1 Trivially, $\Lambda \subseteq \mathbb{I}^{k+1}$ and so it follows that $\dim \Lambda \leq k + 1$. For the reverse inequality, suppose that $\dim \Lambda = h < k + 1$. Then given any $\epsilon > 0$ and $\delta > 0$ such that $h < k + 1 - \delta < k + 1$, there exists a covering $\mathcal{C}$ of Λ by $(k + 1)$-dimensional hypercubes such that

$$\sum_{C \in \mathcal{C}} |C|^{k+1-\delta} < \epsilon.$$

For any fixed $x \in \mathbb{I}$ the set $A := \{x\} \times xS$ can be covered by the collection

$$\mathcal{C}(x) := \{(\{x\} \times \mathbb{I}^k) \cap C : C \in \mathcal{C}\}$$

of k-dimensional hypercubes. Let

$$\lambda_C(x) = \begin{cases} 1 & \text{if } (\{x\} \times \mathbb{I}^k) \cap C \neq \emptyset \\ 0 & \text{if } (\{x\} \times \mathbb{I}^k) \cap C = \emptyset, \end{cases}$$

so that

$$\int_0^1 \lambda_C(x)\, dx = |C|$$

and

$$\sum_{C \in \mathcal{C}(x)} |C|^{k-\delta} = \sum_{C \in \mathcal{C}} \lambda_C(x) |C|^{k-\delta}.$$

Then

$$\begin{aligned} \int_0^1 \sum_{C \in \mathcal{C}(x)} |C|^{k-\delta}\, dx &= \sum_{C \in \mathcal{C}} \bigg(\int_0^1 \lambda_C(x)\, dx \bigg) |C|^{k-\delta} \\ &= \sum_{C \in \mathcal{C}} |C|^{k+1-\delta} \\ &< \epsilon. \end{aligned}$$

Hence $\mathcal{C}(x)$ is a cover for A such that

$$\sum_{C \in \mathcal{C}(x)} |C|^{k-\delta} < \epsilon.$$

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In particular, $\mathcal{C}(1)$ is a cover for S and thus $\dim S \leq k - \delta < k$. This contradicts our hypothesis that $\dim S = k$. Hence, $h \geq k + 1$ as required. □

With reference to Lemma 2.1, with $k = m - 1$ and $S := \mathbf{Bad}(m - 1, 1)$, we show that Λ is contained in the set $\mathbf{Bad}^*(m, 1)$.

For each $\mathbf{x} \in \mathbf{Bad}(m - 1, 1)$ there exists some constant $c(\mathbf{x}) > 0$ such that

$$|q_1 + q_2x_2 + \cdots + q_mx_m| > c(\mathbf{x})|\mathbf{q}^*|^{-(m-1)} \quad \forall (q_1, \mathbf{q}*) \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$$

where $\mathbf{q}* := (q_2, \ldots, q_m)$. Multiplying by $x_1 \in \mathbb{I}$, we get that

$$|q_1x_1 + q_2x_1x_2 + \cdots + q_mx_1x_m| > x_1c(\mathbf{x})|\mathbf{q}^*|^{-(m-1)} \quad \forall (q_1, \mathbf{q}*) \in \mathbb{Z}^m \setminus \{\mathbf{0}\}. \quad (2.1)$$

Now let $\mathbf{q} := (q_1, \ldots, q_m)$. Then $|\mathbf{q}| = |\mathbf{q}^*|$ if $|q_1| \leq |\mathbf{q}^*|$ and $|\mathbf{q}^*|^{-(m-1)} > |\mathbf{q}|^{-(m-1)}$ if $|q_1| > |\mathbf{q}^*|$. This, together with (2.1), implies that

$$|q_1x_1 + q_2x_1x_2 + \cdots + q_mx_1x_m| > c(\mathbf{x}, x_1)|\mathbf{q}|^{-(m-1)} \quad \forall \mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\},$$

where $c(\mathbf{x}, x_1) := x_1c(\mathbf{x}) > 0$. The upshot of this is that

$$\Lambda \subseteq \mathbf{Bad}^*(m, 1).$$

Lemma 2.1 implies that $\dim \mathbf{Bad}^*(m, 1) \geq m$ and thereby completes the proof of Theorem 1.2.

3. A final comment

In the case where $m < n$, although the right-hand side of the inequality appearing in (1.2) is an increasing function as $|\mathbf{q}| \to \infty$, we cannot rule out the possibility that $|\mathbf{q}X|$ grows faster than $O(|\mathbf{q}|^{-m/n+1})$ for some X. However, we suspect that this is rare and it is reasonable to expect that

$$\dim \mathbf{Bad}^*(m, n) = 0.$$

We hope to pursue this and the conjecture in the near future.

Acknowledgements

I am grateful to my supervisor Sanju Velani for suggesting this problem, and for enthusiastic help and encouragement. I would like to thank the referee for a careful reading of the paper.

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