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TOTALLY REAL RIGID ELEMENTS AND GALOIS THEORY

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ABSTRACT. Abelian closed subgroups of the Galois group of the pythagorean closure of a formally real field are described by means of the inertia group of suitable valuation rings.

1. **Introduction.** Let F be a formally real field and denote by F_π its pythagorean closure. The purpose of this note is to investigate subextensions $F \subset E \subset F_\pi$ such that the Galois group $G_\pi(E) = G(F_\pi; E)$ is abelian. Griffin (1976) stated for such a field E that either $E(\sqrt{-1})$ contains all 2-power roots of unity or $G_\pi(E)$ is cyclic ([Gri], Proposition 11). Later, Ware (1983) established that if $G_\pi(E)$ is abelian, for every totally positive element $t \in \Sigma\dot{E}^2 \setminus \dot{E}^2$, the set of all elements of $\dot{E}$ which are represented by the binary quadratic form $X^2 + tY^2$ is $\dot{E}^2 \cup t\dot{E}^2$ ([W2], Corollary 3.11). He also gave an example showing that the converse is not true (Remark 3.13(ii)).

Let us call (as usual) an element t with the above property *rigid*. We shall establish clearly the link between totally positive rigid elements of F and intermediate fields $F \subset E \subset F_\pi$ with $G(F_\pi; E)$ abelian. As a consequence, we describe completely these subextensions. We also state that among the fields E such that $G_\pi(E)$ is abelian and $E|F$ is normal, there exists a unique minimal one, with such properties. Proofs will be based on valuation theoretic methods. In an earlier paper [En] we showed that the existence of "enough" totally real rigid elements in a field F implies that F admits a valuation ring A which extends uniquely to F_π. This result will be the main tool in this paper.

Let us call a valuation ring with the above property *π-henselian*. In the next section we examine the properties of π-henselian valuation rings. In Section 3 we describe $G_\pi(F)$ for a field F which admits a π-henselian valuation ring and in the last three sections we state the results concerning abelian subgroups of $G_\pi(F)$.

CONVENTIONS. Although the paper is concerned formally real fields we have to consider general cases because of the residue fields of valuation rings.

In what follows all fields will have characteristic different from 2 and for any field F, $\dot{F}$, $\dot{F}^2$, and $\Sigma\dot{F}^2$ will denote the multiplicative groups of nonzero elements, squares, and sums of squares, respectively. Let F_π and $F(2)$ be the pythagorean closure and the quadratic closure of F, respectively. If F is not formally real then $F_\pi = F(2)$. Therefore, we shall denote the *quadratic closure of non-formally real fields* by F_π in order to

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simplify the statements. In the same way, we shall refer to a quadratic closed field as *"non-formally real pythagorean field."*

For every valuation ring A we denote by A^*, m_A, $k_A = A/m_A$, φ_A, Γ_A, and v_A the group of units of A, the maximal ideal, the residue field, the canonical homomorphism, the value group and a valuation corresponding to A, respectively. In the whole article, except for Sections 2 and 3, all valuation rings considered have non-formally real residue field of characteristic not 2. To be precise, in the next section $\operatorname{char} k_A = 2$ is allowed and in Section 3 we also consider the case k_A formally real.

2. **π-Henselian valuation rings.** We are going to continue with the study of π-henselian valuation rings which we started in [En], however, with no restriction on k_A now.

For a normal extension of fields $L|F$, we say that a valuation ring A of F is L-henselian if A extends uniquely to L. In ([Br], Lemma 1.2) Bröcker showed that L-henselianity is equivalent to Krasner's Lemma and the lifting property for simple roots applied to polynomials splitting into linear factors over L. We are mainly interested in the F_π-henselian valuation rings ($L = F_\pi$), which we call π-henselian for short. According to our convention, if F is not formally real, π-henselianity coincides with the well-known 2-henselianity.

We shall next see that the characterization of 2-henselian henselian valuation rings due to Dress ([D], Satz 2) already holds for general π-henselian valuation rings. Results of this nature are common when dealing with relative henselianity and are very useful to work out calculations.

LEMMA 2.1. *For each valuation ring A of a field F of characteristic $\neq 2$ the following conditions are equivalent:*

(i) A is π-henselian.

(ii) $(1+4m_A)\cap\Sigma\dot{F}^2 \subset \dot{F}^2$.

(iii) For $s, t \in \Sigma\dot{F}^2$ such that $v_A(t-s) > v_A(4)+v_A(s)$ it follows that $s \in \dot{F}^2$ if and only if $t \in \dot{F}^2$.

PROOF. (i) $\Rightarrow$ (ii) Let $x \in (1+4m_A)\cap\Sigma\dot{F}^2$. Then $\sqrt{x} \in F_\pi$ and $x = 1+4a$, for some $a \in m_A$. For $f(X) = X^2 + X - a$, it follows that $f(X)$ splits over F_π and $f(X) \equiv (X+1)X \pmod{m_A[X]}$. Therefore, by ([Br], Lemma 1.2), $f(X)$ has its roots in F. Hence $\sqrt{x} \in F$, as desired.

(ii) $\Rightarrow$ (iii) For $s, t \in \Sigma\dot{F}^2$ such that $v_A(t-s) > v_A(4s)$ it follows that $(1/4)(ts^{-1}-1) \in m_A$. Hence $ts^{-1} \in (1+4m_A)\cap\Sigma\dot{F}^2$ and so $ts^{-1} \in \dot{F}^2$, as required.

(iii) $\Rightarrow$ (i) Take an extension C of A to the algebraic closure of F and let F^Z be the corresponding henselization ([E], Section 17). According to ([E], 15.6-c), we have to prove that $F^Z \cap F_\pi = F$. To this end, since $G(F_\pi; F)$ is a pro-2-group, it is enough to show that $F^Z \cap F_\pi$ contains no quadratic extension $F(\sqrt{s})$, $s \in \Sigma\dot{F}^2 \setminus \dot{F}^2$. Finally, by ([E], Theorem 17.17), the last statement will be true if we prove that A has exactly one prolongation to each extension $F(\sqrt{s})$, of the above type.

Let $s \in \Sigma\dot{F}^2 \setminus \dot{F}^2$ and, for every $z \in F(\sqrt{s})$, denote by $\overline{z}$ the image of z through the non-trivial F-automorphism of $F(\sqrt{s})$. We want to show that $v(z) = v(\overline{z})$, where v is a valuation corresponding to an extension of A to $F(\sqrt{z})$. Take $z = a + b\sqrt{s}$ and $f(X) = X^2 - 2aX + (a^2 - sb^2)$. Observe first that if $a = 0$ or $b = 0$, then clearly $v(z) = v(\overline{z})$. So, we assume $a, b \neq 0$. Observe now that if α is a root of $f(X)$, then one of the following cases holds:

(1) $v(\alpha^2) = v(a^2 - sb^2) \leq v(2a\alpha)$
(2) $v(\alpha^2) = v(2a\alpha) \leq v(a^2 - sb^2)$
(3) $v(2a\alpha) = v(a^2 - sb^2) \leq v(\alpha^2)$

In the first case, $v(z^2) = v(z\overline{z})$. So $v(z) = v(\overline{z})$ and we are done.

We now claim that in the cases (2) and (3), $v(\alpha) = v(2a)$. Therefore, again $v(z) = v(\overline{z})$ and the proof is completed.

Proof of the claim. Since case (2) is clear let us consider case (3). As $v(2a\alpha) \leq v(\alpha^2)$, then $v(2a) \leq v(\alpha)$. On the other side, condition (iii) implies that $v(a^2 - sb^2) \leq v(4a^2)$. Hence $v(2a\alpha) \leq 2v(2a)$ and so $v(\alpha) \leq v(2a)$, proving the claim. ■

If $\operatorname{char} k_A \neq 2$, we can cut 4 from the conditions (ii) and (iii) above. Observe also that if we replace $\Sigma\dot{F}^2$ by $\dot{F}$ in these conditions we get the characterization of 2-henselian valuation rings ([D], Satz 2). Therefore, it is clear that if F is not formally real π-henselianity coincides with 2-henselianity.

Following with this analogy, we shall see in the next results that the set of π-henselian valuation rings of a field has a description similar to the 2-henselian case.

Let us first state a technical lemma.

LEMMA 2.2. *Let A be a π-henselian valuation ring of a field F with* $\operatorname{char} F \neq 2$. *If $a \in F$ verifies* $1 + a^2 \notin \dot{F}^2$, *then* $-v_A(2) \leq v_A(a) \leq v_A(2)$.

PROOF. If $2^{-1}a \in m_A$, then $1 + a^2 = 1 + 4(2^{-1}a)^2 \in (1 + 4m_A) \cap \Sigma\dot{F}^2$. So, Proposition 2.1(iii) implies $1 + a^2 \in \dot{F}^2$. Similarly, $(2a)^{-1} \in m_A$ yields $1 + a^{-2} = 1 + 4(2a)^{-1} \in (1 + 4m_A) \cap \Sigma\dot{F}^2$. Hence $1 + a^2 = a^2(1 + a^{-2}) \in \dot{F}^2$. ■

We now describe k_A for pythagorean fields.

PROPOSITION 2.3. *Let C be a valuation ring of field K such that* $\operatorname{char} K \neq 2$.

(a) If C is π-henselian and k_C is a pythagorean formally real field, then K is also pythagorean and formally real.

(b) If K is pythagorean and $\operatorname{char} k_C \neq 2$, *then k_C is also pythagorean. Moreover, if K is not formally real, then $\dot{k}_C = \dot{k}_C^2$ and $\Gamma_C = 2\Gamma_C$.*

(c) Assume now that $K = F_\pi$ for some subfield F and $A = C \cap F$ is π-henselian. If $\operatorname{char} k_C \neq 2$, *then k_C is the pythagorean closure of k_A. Furthermore, if K is a non-formally real field, then Γ_C is also the* 2*-divisible hull of Γ_A.*

PROOF. (a) It is well-known that if k_C is formally real so is K ([L2], Lemma 3.7, p. 23). Going for a contradiction let us assume that K is not pythagorean. Then $1 + a^2 \notin \dot{K}^2$ for some $a \in K$. Since char $k_C = 0$, $a \in C^*$, by Lemma 2.2. As k_C is formally real, $1+a^2 \notin m_C$. Moreover, k_C is pythagorean and so there exists $b \in C^*$ such that $\varphi_C(1 + a^2) = \varphi_C(b^2)$. Therefore, $(1 + a^2)b^{-2} \in (1 + m_C)$. Since $(1 + a^2)b^{-2} \in \Sigma\dot{K}^2$, Proposition 2.1 implies that $(1 + a^2)b^{-2} \in \dot{K}^2$, a contradiction.

(b) For $x, y \in C^*$, let $z \in K$ such that $x^2 + y^2 = z^2$. Then $\varphi_C(x)^2 + \varphi_C(y)^2 = \varphi_C(z)^2$, and so k_C is pythagorean.

By a result due to Krull (see [E], Theorem 27.1, p. 206), if k_C admits any extension of degree 2, or if there exists a subgroup Δ of the divisible hull of Γ_C such that $(\Delta : \Gamma_C) = 2$, then there exists a quadratic extension of L. But this is not possible since K is a non-formally real pythagorean field.

(c) From general valuation theory we know that k_C is a normal extension of k_A such that $[k_C : k_A]$ is a 2-power (as supernatural number) and Γ_C/Γ_A is a torsion group where each element has 2-power order. Hence, if k_A is not formally real, it follows from (b) that k_C is the quadratic closure of k_A. Otherwise, let ℓ be a pythagorean closure of k_A contained in k_C. By ([E], Theorem 19.13, p. 152), there exists an intermediate extension $F \subset E \subset F_\pi$ such that $C \cap E$ has ℓ as its residue field. Since A is π-henselian, so is $C \cap E$. Therefore, by (a), E is pythagorean, which implies $E = F_\pi$ ($E \subset F_\pi$). Thus $k_C = \ell$, as desired.

The last statement follows from (b). ∎

Recall that two valuation rings A and B of a field F are said to be *independent* if there is no non-trivial valuation ring C of F containing both A and B.

PROPOSITION 2.4 (F. K. SCHMIDT [SCHM]). *Let F be a field such that* char $F \neq 2$ *and assume that there exist π-henselian valuation rings A and B which are independent. Then $F = F_\pi$.*

PROOF. Going for a contradiction let us assume that $F \neq F_\pi$. Then there exists $a \in F$ such that $t = 1 + a^2 \notin \dot{F}^2$. Take now $b \in F$ such that $v_A(b) > v_A(2)$. By Lemma 2.2 $-v_B(2) \leq v_B(a) \leq v_B(2)$ and $s = 1 + b^2 \in \dot{F}^2$. Since A and B are independent, by the Approximation Theorem ([E], 11.16, p. 80), for $\gamma \in \Gamma_A$ and $\delta \in \Gamma_B$ such that $\gamma > \text{maximum}\{v_A(4), v_A(b)\}$ and $\delta > \text{maximum}\{v_B(8t), v_B(a)\}$ there exists $c \in F$ satisfying $v_A(c - b) > \gamma$ and $v_B(c - a) > \delta$. Therefore $v_A(c - b) > v_A(b)$ which implies that $v_A(c) = v_A(b)$. In the same way $v_B(c) = v_B(a)$.

Take now $r = 1+c^2$. Then $v_A(r-s) = v_A(c^2-b^2) = v_A(c-b)+v_A(c+b) > \gamma+v_A(2) \geq \gamma$. In the same way $v_B(r - t) > \delta - v_B(2)$. Consequently, $v_A(r - s) > v_A(4) = v_A(4) + v_A(s)$. Thus, by Proposition 2.1(iii), $r \in \dot{F}^2$. On the other side, as $v_B(r - t) > v_B(4) + v_B(t)$, it follows that $r \notin \dot{F}^2$, a contradiction. ∎

Maybe it is worth mentioning that a relative version of the F. K. Schmidt's result was stated by Bröcker for prime closed Galois extensions ([Br], Proposition 1.4). Our next result is a relative version of ([EE], Proposition).

COROLLARY 2.5. *Let A and B be valuation rings of a field F with* $\operatorname{char} F \neq 2$*. Assume that A is π-henselian and* $\operatorname{char} k_B \neq 2$*. If B is not comparable to A, then k_B is pythagorean.*

PROOF. Let $C = AB$. Since $\operatorname{char} k_B \neq 2$, also $\operatorname{char} k_C \neq 2$. Denote by D the unique prolongation of C to F_π. Set $\bar{A} = \varphi_C(A)$ and $\bar{B} = \varphi_C(B)$, respectively. Using Lemma 2.1(ii), a simple computation shows that the π-henselianity of A and C implies that $\bar{A}$ is π-henselian (or see Lemma 1.3 of [Br]). Let $\tilde{B}$ be an extension of $\bar{B}$ to k_D and denote by k^Z the decomposition field of $\tilde{B}$ over k_C, (see [E], Section 15, p. 109). By Proposition 2.3(b), the residue field of D is pythagorean. On the other side, the unique extension of $\bar{A}$ to k^Z is also π-henselian. By construction $\bar{A}$ and $\bar{B}$ are independent. Therefore, their prolongations to k^Z are also independent. Thus, by Proposition 2.4, k^Z is pythagorean and so $k^Z = k_D$. Since $\bar{B}$ and $\tilde{B} \cap k^Z$ have the same residue field ([E], Theorem 15.8, p. 112), we can conclude that the residue field of $\bar{B}$ is pythagorean. Finally, as B and $\bar{B}$ have the same residue field, the result is proved. ∎

In the next corollary we shall see that the set $\mathcal{H}$ of all proper π-henselian valuation rings A of F such that $\operatorname{char} k_A \neq 2$ has the same aspect as the set of all henselian valuation rings of a field (see [EE], Corollary 1), or the set of all Ω-henselian valuation rings of a p-closed normal extension $\Omega|F$ ([EK], Lemma 4.1).

COROLLARY 2.6. *For a non-pythagorean field F with* $\operatorname{char} F \neq 2$ *let $\mathcal{H}$ be the set introduced above and put $\mathcal{H}_1 = \{A \in \mathcal{H} \mid k_A$ is not pythagorean$\}$, $\mathcal{H}_2 = \mathcal{H} \setminus \mathcal{H}_1$. Then:*

(a) $\mathcal{H}_1$ is totally ordered by inclusion, provided it is not empty. Moreover, there exists $A_{(1)} \in \mathcal{H}$ such that $A_{(1)} \subset A$ for every $A \in \mathcal{H}_1$.

(b) If $\mathcal{H}_2 \neq \emptyset$, there exists $A_{(2)} \in \mathcal{H}_2$ such that each $B \in \mathcal{H}_2$ satisfies $B \subset A_{(2)}$.

(c) If both, $\mathcal{H}_1$ and $\mathcal{H}_2$, are non-empty, then $A_{(2)} \subseteq A_{(1)}$ and there is no valuation ring B of F such that $A_{(2)} \subsetneq B \subsetneq A_{(1)}$. Furthermore, if $A_{(2)} \neq A_{(1)}$, then $A_{(1)} \in \mathcal{H}_1$.

PROOF. The first part of (a) is clear by of the previous proposition. Take $A_{(1)} = \bigcap A$, $A \in \mathcal{H}_1$. Since $\mathcal{H}_1$ is totally ordered, $A_{(1)}$ is a valuation ring and has maximal ideal $m = \bigcup m_A$, $A \in \mathcal{H}_1$. As $2 \notin m_A$, for every $A \in \mathcal{H}_1$, also $2 \notin m$. Thus $A_{(1)}$ has residue field of characteristic different from 2. Finally, Lemma 2.1(ii) implies that $A_{(1)}$ is π-henselian.

(b) Since F is non-pythagorean, every pair of elements of $\mathcal{H}_2$ are dependent, by Proposition 2.4. Hence, $\mathcal{H}_2$ is a directed set, ordered by inclusion. Then $A_{(2)} = \bigcup B$, $B \in \mathcal{H}_2$, is a subring of F. Let k be the residue field of $A_{(2)}$. We claim that k is pythagorean. By the claim $A_{(2)} \in \mathcal{H}_2$, as desired. To prove the claim, let $\bar{B}$ be the image of B in k, for every $B \in \mathcal{H}_2$. By construction $k = \bigcup \bar{B}$, $B \in \mathcal{H}_2$. Thus, for every $a, b \in \dot{k}^2$, there is $\bar{B}$ such that $a, b, a+b \in \bar{B}^*$ (recall that $\mathcal{H}_2$ is a directed set). By assumption, there exists $c \in (\bar{B}^*)^2$ satisfying $\varphi_{\bar{B}}(c) = \varphi_{\bar{B}}(a+b)$. Therefore, $(a+b)c^{-1} \in (1 + m_{\bar{B}}) \cap \Sigma \dot{k}^2$. Observe that Lemma 2.1(ii) and the π-henselianity of B and $A_{(2)}$ yield $\bar{B}$ π-henselian. Also $2 \in \bar{B}^*$. Hence $a + b \in \dot{k}^2$, by Lemma 2.1(ii), and the claim is proved.

(c) By the previous proposition, every $A \in \mathcal{H}_1$ is comparable to $A_{(2)}$. Due to the properties of their residue fields, $A_{(2)} \subset A$. Hence $A_{(2)} \subset A_{(1)}$. The other assertions are clear. ∎

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REMARK 2.7. Recall from [En] that a π-henselian valuation ring A of a field F which is comparable to each of the others is called *distinguished* (Definition 2.17). Observe that each valuation ring in $\mathcal{H}_1$, and also $A_{(2)}$ if $\mathcal{H}_2 \neq \emptyset$, is distinguished. Therefore, if $\mathcal{H} \neq \emptyset$, F admits a distinguished valuation ring.

On the other side, if $\mathcal{H}_2 \neq \emptyset$ and $B \in \mathcal{H}_2$, then every valuation ring $\overline{C}$ of k_B is π-henselian. Hence, by Lemma 2.1(ii), the lift $C = \varphi_B^{-1}(\overline{C})$ is also π-henselian. Hence B has to contain non-comparable π-henselian valuation rings, unless k_B is an algebraic extension of a finite field. Consequently, if A is a distinguished valuation ring of F such that $\operatorname{char} k_A \neq 2$, the above corollary implies either $A = A_{(2)}$, or $A \in \mathcal{H}_1$.

We may now rewrite Proposition 2.18 of [En] in a more complete form.

COROLLARY 2.8. *For a normal subextension $F \subset L \subsetneq F_\pi$, if L admits a π-henselian valuation ring A with* $\operatorname{char} k_A \neq 2$*, then there exists a π-henselian valuation ring B containing A such that $B \cap F$ is also π-henselian.*

Moreover: If A is distinguished, $B = A$ verifies the statement above. If $A \cap F$ is not π-henselian, B can be chosen such that k_B is pythagorean.

PROOF. If A is distinguished, with the same proof of Proposition 2.18 in [En] it follows that that $A \cap F$ is also π-henselian. Therefore, if $A \cap F$ is not π-henselian, by the remark above, $A \subsetneq A_{(2)}$. Hence $B = A_{(2)}$ has the desired properties. ∎

We end this section reviewing the link between π-henselian valuation rings and rigid elements. For every $a \in \dot{F}$ we denote $D\langle 1, a\rangle = \{x^2 + ay^2 \neq 0 \mid x, y \in F\}$.

An element $t \in \dot{F}$ is called *rigid* if $t \notin \dot{F}^2$ and $D\langle 1, t\rangle = \dot{F}^2 \cup t\dot{F}^2$. In this paper we are mainly interested in rigid elements $t \in \Sigma\dot{F}^2$. We also denote by $B_\pi(F) = \{t \in \Sigma\dot{F}^2 \mid t$ is not rigid$\}$. Recall that for a formally real field F a rigid element $t \in \Sigma\dot{F}^2$ is not birigid (when t and $-t$ are rigid) ([BCW], Proposition 1). On the other side, if F ($\operatorname{char} F \neq 2$) is not formally real, a rigid element t is birigid ([CR], Corollary). An element t which is not birigid is called *basic* and it is well-known that the set B of basic elements is a subgroup of $\dot{F}$ ([W], Proposition 2.4). As observed above $\Sigma\dot{F}^2 \subset B$ if F is formally real and $B_\pi(F) = B$ otherwise.

REMARK 2.9 ([EN]). Let F be a formally real field with a π-henselian valuation ring A such that k_A is a non-formally real field of characteristic different from 2.

(A) It was stated in [En] that $B_\pi(F) \subset A^*\dot{F}^2$ is a subgroup of $\dot{F}$ (Propositions 2.5 and 2.7). Actually, $B_\pi(F) \subset (A^* \cap \Sigma\dot{F}^2)\dot{F}^2$, since $B_\pi(F) \subset \Sigma\dot{F}^2$.

(B) The inclusion $\varphi_A\big(A^* \cap B_\pi(F)\big) \subset B_\pi(k_A)$ is always true. Moreover, if $k_A \neq \pm\dot{k}_A^2$, then $\varphi_A\big(A^* \cap B_\pi(F)\big) = B_\pi(k_A)$ (Proposition 2.5).

(C) There exists a π-henselian valuation ring B of F such that $B \subseteq A$, $\operatorname{char} k_B \neq 2$ and $\big(\dot{k}_B : B_\pi(k_B)\big) \leq 2$ (Proposition 2.6(1)).

For a field F where $B_\pi(F)$ is a subgroup of $\dot{F}$ such that $\big(\Sigma\dot{F}^2 : B_\pi(F)\big) > 2$ we have the converse of (A):

(D) F admits a distinguished π-henselian valuation ring A which verifies the following conditions: $\Sigma\dot{F}^2 \neq (A^* \cap \Sigma\dot{F}^2)\dot{F}^2$, $\operatorname{char} k_A \neq 2$, $\big(\dot{k}_A : B_\pi(k_A)\big) \leq 2$ and $\varphi_A\big(A^* \cap B_\pi(F)\big) = B_\pi(k_A)$ (Theorem 2.8 and Corollary 2.15).

(E) If $B_\pi(F) = \dot{F}^2$, the residue field of the above valuation ring also verifies: $(\dot{k}_A : \dot{k}_A^2) \leq 2$, $B_\pi(k_A) = \dot{k}_A^2$, $-1 \in \dot{k}_A^2$ and denoting by ξ_n a primitive 2^n root of unity, $\xi_n \in k_A$ if and only if $\xi_n \in F(i)$ ([En], Proposition 2.14).

3. **The Galois group of a π-henselian valued field.** From now on, F stands for a formally real field and for all valuation rings A we assume that $\operatorname{char} k_A \neq 2$.

Following [E], if L is a normal extension of F and C is a valuation ring of L, we denote the decomposition group, the inertia group and the ramification group of C over F by $G^Z(C;F)$, $G^T(C;F)$ and $G^V(C;F)$, respectively. Let $K^Z(C;F)$, $K^T(C;F)$ and $K^V(C;F)$ be the corresponding fixed fields. By ([E], 15.1-b, p. 109), $A = C \cap F$ is L-henselian if and only if $G^Z(C;F) = G(L;F)$.

According to ([E], Theorem 20.12, p. 163), there exists a continuous surjective homomorphism $\Psi\colon G^T(C;F) \longrightarrow \operatorname{Hom}(\Gamma_C/\Gamma_A, \dot{k}_C)$ whose kernel is $G^V(C;F)$. Recall that $G^V(C;F)$ is the unique $\bar{p}$-Sylow subgroup of $G^T(C;F)$, where $\bar{p} = 1$ if $\operatorname{char} k_A = 0$ and $\bar{p} = \operatorname{char} k_A$ otherwise ([E], 20.18, p. 167). Therefore, under the present condition ($G^T(C;F)$ is a pro-2-group), since $\operatorname{char} k_A \neq 2$, $G^V(C;F)$ is trivial. Hence, $G^T(C;F) \simeq \operatorname{Hom}(\Gamma_C/\Gamma_A, \dot{k}_C)$ is an abelian group.

Consider now a formally real field F and let $F(2)$ be a quadratic closure of F. For a valuation ring A ($\operatorname{char} k_A \neq 2$) of F let D be an extension of A to $F(2)$. Recall from ([E], Theorem 19.1, p. 145) that the canonical projection φ_D gives rise to a split exact sequence

$$(*) \qquad 1 \longrightarrow G^T(D;F) \longrightarrow G^Z(D;F) \longrightarrow G(k_D;k_A) \longrightarrow 1.$$

By Proposition 2.3(b), k_D is the quadratic closure of k_A ($F(2)$ is not formally real). Hence $G(k_D;k_A) = G_\pi(k_A)$ if k_A is not formally real.

Summing up the comments above:

REMARK 3.1. If k_A is not formally real, $G^Z(D;F) \simeq G^T(D;F) \rtimes G_\pi(k_A)$ and $G^T(D;F)$ is an abelian group.

We now consider the following framework:

Let A be a π-henselian valuation ring of a formally real field F ($\operatorname{char} k_A \neq 2$*), C the unique extension of A to F_π and let D be any extension of C to $F(2)$.*

Let us also denote by $G_2(K)$ the Galois group $G\big(F(2);K\big)$, for every intermediate field $F \subset K \subset F(2)$.

Recall from ([E] 15.6, p. 111, 19.10, p. 151 and 20.15, p. 166) the relations:

$$(\dagger) \qquad \begin{array}{ll} G_2\big(K^Z(C;F)\big) = G^Z(D;F)G_2(F_\pi) & G_2\big(K^T(C;F)\big) = G^T(D;F)G_2(F_\pi) \\ G^Z(D;F_\pi) = G^Z(D;F) \cap G_2(F_\pi) & G^T(D;F_\pi) = G^T(D;F) \cap G_2(F_\pi) \end{array}$$

We are now able to prove that $G_\pi(F)$ has a decomposition like the one described in the above remark.

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PROPOSITION 3.2. *Under the conditions introduced above, the following statements are true.*

(a) If k_A is formally real, $G_\pi(F) \simeq G_\pi(k_A)$. Otherwise,

(b) $G_\pi(F) \simeq G^T(C;F) \rtimes G\big(K^T(C;F);F\big) \simeq G^T(C;F) \rtimes G_\pi(k_A)$.

PROOF. Observe first that $G_\pi(F) = G^Z(C;F)$, since A is π-henselian.

(a) Recall that $C \cap K^T(C;F)$ has residue field k_C ([E], Theorem 19.12, p. 152). Therefore, by Proposition 2.3(c), $K^T(C;F)$ is pythagorean. Hence $K^T(C;F) = F_\pi$. So $G^T(C;F)$ is trivial and $G_\pi(F) \simeq G_\pi(k_A)$, as desired.

In order to prove (b) we claim that $G^Z(D;F_\pi) = G^T(D;F_\pi)$ and $G^T(C;F) \simeq G^T(D;F)/G^T(D;F_\pi)$.

PROOF OF THE CLAIM. By Proposition 2.3(c), k_C is pythagorean. But, since k_A is not formally real, this already means that k_C is quadratically closed. Hence, $k_C = k_D$. As we know that $D \cap K^Z(D;F_\pi)$ and C have the same residue field ([E], Theorem 15.8, p. 112), it follows from ([E], 19.11, p. 151) that $K^Z(D;F_\pi) = K^T(D;F_\pi)$ and so $G^Z(D;F_\pi) = G^T(D;F_\pi)$, as required.

By the relations (†), presented before the proposition, it follows that

$$\begin{aligned} G^T(C;F) &\simeq G_2\big(K^T(C;F)\big)/G_2(F_\pi) \simeq G^T(D;F)G_2(F_\pi)/G_2(F_\pi) \\ &\simeq G^T(D;F)/\big(G^T(D;F) \cap G_2(F_\pi)\big) \simeq G^T(D;F)/G^T(D;F_\pi), \end{aligned}$$

and the claim is proved.

Continuing with the proof of (b), observe first that the relations (†) imply

$$\begin{aligned} G_\pi(F) &= G_2(F)/G_2(F_\pi) \simeq G^Z(D;F)G_2(F_\pi)/G_2(F_\pi) \\ &\simeq G^Z(D;F)/\big(G^Z(D;F) \cap G_2(F_\pi)\big) \simeq G^Z(D;F)/G^Z(D;F_\pi). \end{aligned}$$

The first statement of the claim yields then, $G_\pi(F) = G^Z(D;F)/G^T(D;F_\pi)$. Finally, Remark 3.1 and the second statement of the claim imply the result. ∎

Case (a) can be deduced from [Be], Theorem 7' (p. 85) and Theorem 21 (p. 55). Actually, (a) shows that the case k_A formally real is not interesting for the study of $G_\pi(F)$. Therefore, we will be assuming that k_A is not formally real, for every A.

In order to have a more precise description of $G_\pi(F)$ let us recall a few facts about 2-power roots of the unity. Let $\mu_\infty \subset F(2)$ be the group of all 2-power roots of the unity. Since char $k_A \neq 2$, we may assume that the restriction of φ_D to μ_∞ is the identity.

Fix the following convention: Take inside μ_∞ a system of 2^n roots of unity: $\xi_1 = -1$, $\xi_2 = i = \sqrt{-1}$, $\xi_3, \ldots$, chosen so that $\xi_{n+1}^2 = \xi_n$ for all $n \geq 2$. For every $n \geq 3$ let $h_n = \xi_n + \xi_n^{-1}$.

Following [Gri] let us denote by FH the field which arises from F by adjoining $\{h_n \mid n \geq 3\}$. By ([Gri], Proposition 6 and Corollary 7), $FH \subset F_\pi$ is a normal subextension such that either $G(FH;F) \simeq \mathbb{Z}_2$ or $FH = F$. The last case occurs if and only if $\mu_\infty \subset F(i)$. It is also clear that $FH(i) = F(\mu_\infty)$ and so $\mu_\infty \subset F_\pi(i)$.

Keep the conditions and notations we have introduced so far.

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LEMMA 3.3. *There exists exactly one subextension $F \subset F_0 \subset K^T(C;F)$ such that the residue field of $C \cap F_0$ is $k_A(\mu_\infty)$. Furthermore,*

(a) if $-1 \in k_A$, then $F_0 = FH$;

(b) if $-1 \notin k_A$, a unit $u_0 \in A^* \cap \Sigma \dot{F}^2$ can be chosen such that $F_0 = FH(\sqrt{u_0})$ and $C \cap F(\sqrt{u_0})$ has residue field $k_A(i)$.

PROOF. We first recall that k_C is quadratically closed, by Proposition 2.3(b), since k_A is not formally real. So $\mu_\infty \subset k_C$. We know by Theorem 19.13b of [E] (p. 152) that there exists a bijective and inclusion-preserving correspondence between the set of all subextensions $F \subset L \subset K^T(C;F)$ and the set of all subextensions $k_A \subset k \subset k_C$ =the quadratic closure of k_A, where every L is associated with the residue field of $C \cap L$. Thus, there exists just one such subextension F_0 for which $k_A(\mu_\infty)$ is the residue field of $C \cap F_0$.

As we have assumed that the restriction of φ_D to μ_∞ is the identity it follows that the restriction of φ_D to $\{h_n \mid n \geq 3\}$ is also the identity. Therefore, $k_A H = k_A\left(\{h_n \mid n \geq 3\}\right) \subset k_A(\mu_\infty)$ and also $k_A(\mu_\infty) = k_A H(i)$. Hence, FH is the subextension of $K^T(C;F)$ which corresponds to $k_A H$ and also $FH \subset F_0$ verifies $[F_0 : FH] \leq 2$.

Now it is clear that if $i \in k_A$, then $k_A(\mu_\infty) = k_A H$ and so $F_0 = FH$, which proves (a).

(b) As k_A is not formally real -1 is a sum of squares in k_A. Thus there exists $u_0 \in A^* \cap \Sigma \dot{F}^2$ such that $\varphi_A(u_0) = -1$. Therefore, $F(\sqrt{u_0}) \subset F_\pi$ and the extension B of A to $F(\sqrt{u_0})$ verifies $k_B = k_A(i)$. Since $C \cap K^T(C;F)$ is π-henselian, Lemma 1.2 of [Br] implies that $\sqrt{u_0} \in K^T(C;F)$ and so $F(\sqrt{u_0}) \subset K^T(C;F)$ is the subextension which corresponds to $k_A(i)$. As $k_A(i) \subset k_A(\mu_\infty)$, by the inclusion-preserving property, $F(\sqrt{u_0}) \subset F_0$. Thus $FH(\sqrt{u_0}) \subset F_0$. Since the residue field of $C \cap FH(\sqrt{u_0})$ clearly contains μ_∞, we can conclude that $FH(\sqrt{u_0}) = F_0$. ∎

We are now able to refine the description of $G_\pi(F)$ given by Lemma 3.2(b).

PROPOSITION 3.4. *Let F be a formally real field and A a π-henselian valuation ring of F such that k_A is a non-formally real field of characteristic not 2. Let C be its unique extension to F_π. Then:*

(a) If $\xi_n \in k_A$ for every $n \geq 1$, then $G_\pi(F) \simeq G^T(C;F) \times G_\pi(k_A)$.

(b) If $\xi_n \in k_A$ and $\xi_{n+1} \notin k_A$ for some $n \geq 2$ then

$$G_\pi(F) \simeq \left(G^T(C;F) \times G_\pi\big(k_A(\mu_\infty)\big)\right) \rtimes \mathbb{Z}_2,$$

where the factor $\mathbb{Z}_2$ corresponds to $G(k_A(\mu_\infty); k_A)$ and has a generator σ such that $\tau^\sigma = \tau^{2^n+1}$ for every $\tau \in G^T(C;F)$.

(c) If $i \notin k_A$, let $u_0 \in A^* \cap \Sigma \dot{F}^2$ and F_0 be the unit and the field introduced in Lemma 3.3(b). Then $G^T\big(C; F(\sqrt{u_0})\big) = G^T(C;F)$ and $G_\pi\big(F(\sqrt{u_0})\big)$ can be described either as in (a) or (b). Moreover, for every $\phi \in G_\pi(F) \setminus G_\pi\big(F(\sqrt{u_0})\big)$ such that $\phi^2 \in G_\pi(F_0)$, $\tau^\phi = \tau^{-1}$, for each $\tau \in G^T(C;F)$. Furthermore, if $G\big(k_A(\mu_\infty); k_A\big) \simeq \mathbb{Z}_2$, then $\xi_n \in k_A(i)$ and $\xi_{n+1} \notin k_A(i)$ for some $n \geq 3$, $G_\pi(F)$ has a description as in (b) and the factor $\mathbb{Z}_2$, which corresponds to $G\big(k_A(\mu_\infty); k_A\big)$, has a generator σ such that $\tau^\sigma = \tau^{2^{n-1}-1}$ for every $\tau \in G^T(C;F)$.

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PROOF. As we saw in the claim during the proof of Proposition 3.2 $G^T(C;F) \simeq G^T(D;F)/G^T(D;F_\pi)$. Hence (a), (b) and the last two assertions of (c) follow from Proposition 3.2(b) and ([EK], Proposition 1.1). The first statement of (c) follows from Lemma 3.3(b). ∎

Observe that since $G_\pi(F)$ is torsion free the exact sequence

$$1 \longrightarrow G_\pi\big(F(\sqrt{u_0})\big) \longrightarrow G_\pi(F) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

does not split. Hence we can give a description of $G_\pi(F)$ only in the case $G\big(k_A(\mu_\infty);k_A\big) \simeq \mathbb{Z}_2$.

The last proposition has the following interesting corollary.

COROLLARY 3.5. *For a field F as in the previous proposition every subgroup of $G^T(C;F)$ is a normal subgroup of $G_\pi(F)$.*

PROOF. The statement is clear if F verifies either (a) or (b) of the above proposition.

For F as in (c) and U a subgroup of $G^T(C;F)$, we have that U is a normal subgroup of $G_\pi\big(F(\sqrt{u_0})\big)$ by (a) and (b).

Next we consider the case where $F(\sqrt{u_0})$ verifies the condition (a). In this case $F(\sqrt{u_0}) = F_0$. So $\phi^2 \in G_\pi(F_0)$ for each $\phi \in G_\pi(F) \setminus G_\pi\big(F(\sqrt{u_0})\big)$. Thus, by (c) of the previous proposition, $\phi U \phi^{-1} = U$ and the statement is proved in this case.

For the other case, by Lemma 3.3(b), $FH \cap F(\sqrt{u_0}) = F$ and $F_0 = FH(\sqrt{u_0})$. Applying the previous case to FH we see that U is a normal subgroup of $G_\pi(FH)$. As $G_\pi(F) = G_\pi(FH)G_\pi\big(F(\sqrt{u_0})\big)$, U is also a normal subgroup of $G_\pi(F)$. ∎

We shall now state a link between $G^T(C;F)$ and $\Sigma\dot{F}^2/(A^*\cap\Sigma\dot{F}^2)\dot{F}^2$. For the extension C of A to F_π let us write $K = K^T(C;F)$ for short.

PROPOSITION 3.6. *For a field F with a π-henselian valuation ring A the inclusion $F \subset K$ induces an isomorphism from $\Sigma\dot{F}^2/(A^*\cap\Sigma\dot{F}^2)\dot{F}^2$ onto $\Sigma\dot{K}^2/\dot{K}^2$.*

Consequently, $G^T(C;F)$ and $\Sigma\dot{F}^2/(A^*\cap\Sigma\dot{F}^2)\dot{F}^2$ have the same rank.

PROOF. Let $x \in \Sigma\dot{F}^2$ be an element such that $x = z^2$ for some $z \in K$. Since $(K, C\cap K)$ is a non-ramified extension of (F, A), there exists $y \in \dot{F}$ such that $v_C(y) = v_C(z)$. Consequently, $v_C(x) = v_C(y^2)$ and so $x \in A^*\dot{F}^2$. Thus, the map from $\Sigma\dot{F}^2/(A^*\cap\Sigma\dot{F}^2)\dot{F}^2$ to $\Sigma\dot{K}^2/\dot{K}^2$ is injective. In order to show the surjectivity of the map, for each $z \in \Sigma\dot{K}^2 \setminus \dot{K}^2$ let $U = G_\pi\big(K(\sqrt{z})\big)$. By the previous corollary U is a normal subgroup of $G_\pi(F)$. By Proposition 3.2(b), $G_\pi(F)/U \simeq \big(G^T(C;F)/U\big) \rtimes G(K;F) \simeq \mathbb{Z}/2\mathbb{Z} \rtimes G(K;F)$. Therefore, if L is the fixed field of $G(K;F)$ inside $K(\sqrt{z})$, there exists $x \in F$ such that $L = F(\sqrt{x})$. Since $L \subset F_\pi$ we may take $x \in \Sigma\dot{F}^2$. By Galois theory $K(\sqrt{z}) = K(\sqrt{x})$. Hence $z \in x\dot{K}^2$, which proves the surjectivity.

Finally, as $G^T(C;F)$ and $\Sigma\dot{K}^2/\dot{K}^2$ have the same rank, the result is proved. ∎

We would like to remark that for a field F which admits a π-henselian valuation ring A such that k_A is a non-formally real field of characteristic not 2, Proposition 3.2 implies that $G_\pi(F)$ has normal abelian subgroups. In the next section we shall prove a partial converse.

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4. **Normal abelian subgroups of rank** ≥ 2**.** Let us start the study of the link between normal abelian subgroups of $G_\pi(F)$ and rigid elements of F. The first result completes the analysis made by Ware in [W2], Theorem 3.9, Corollary 3.11 and Remark 3.12, for fields satisfying $B_\pi(F) = \dot{F}^2$.

In order to avoid repeating conditions, we assume for the rest of the paper that k_A is *a non-formally real field of characteristic not* 2.

PROPOSITION 4.1. *Let F be a formally real field such that* $(\Sigma\dot{F}^2 : \dot{F}^2) > 2$. *Then the following conditions are equivalent.*

(i) $G_\pi(F)$ *is abelian.*

(ii) $B_\pi(F) = \dot{F}^2$ *and* $\xi_n \in F(i)$ *for every* $n \geq 1$.

(iii) F admits a distinguished π*-henselian valuation ring A such that* $B_\pi(k_A) = \dot{k}_A^2$, $(\dot{k}_A : \dot{k}_A^2) \leq 2$ *and* $\xi_n \in k_A$ *for every* $n \geq 1$.

PROOF. (i) $\Rightarrow$ (ii) The assumption $(\Sigma\dot{F}^2 : \dot{F}^2) > 2$ implies that $G_\pi(F) \not\simeq \mathbb{Z}_2$. Hence, by ([Gri], Proposition 11 and Corollary 7), $\xi_n \in F(i)$, for every $n \geq 2$. Finally, by ([W2], Corollary 3.11), $B_\pi(F) = \dot{F}^2$.

(ii) $\Rightarrow$ (iii) By Remark 2.9, (D) and (E), F admits a distinguished π-henselian valuation ring A such that k_A has the desired properties.

(iii) $\Rightarrow$ (i) Observe that $(\dot{k}_A : \dot{k}_A^2) \leq 2$ implies that $G_\pi(k_A)$ is either trivial or isomorphic to $\mathbb{Z}_2$. Therefore (i) follows from Proposition 3.4(a). ∎

We shall next improve the above result. To this end let us state a technical lemma.

LEMMA 4.2. *Let F be a field with a* π*-henselian valuation ring A. Then:*

(a) $(\Sigma\dot{F}^2 : \dot{F}^2)$ *is finite if and only if* $\Sigma\dot{F}^2/(A^* \cap \Sigma\dot{F}^2)\dot{F}^2$ *and* $(\dot{k}_A : \dot{k}_A^2)$ *are finite. In which case* $(\Sigma\dot{F}^2 : \dot{F}^2) = \big(\Sigma\dot{F}^2 : (A^* \cap \Sigma\dot{F}^2)\dot{F}^2\big)(\dot{k}_A : \dot{k}_A^2)$.

(b) If $(\Sigma\dot{F}^2 : \dot{F}^2) > 4$, $\big(\Sigma\dot{F}^2 : (A^* \cap \Sigma\dot{F}^2)\dot{F}^2\big) \leq 2$ *and* $\big(\dot{k}_A : B_\pi(k_A)\big) \leq 2$, *then there exists a* π*-henselian valuation ring* $B \subseteq A$ *of F such that* $B_\pi(F) = (B^* \cap \Sigma\dot{F}^2)\dot{F}^2$ *and* $B_\pi(k_B) = \dot{k}_B$.

PROOF. (a) As k_A is not formally real by assumption, the restriction of φ_A to $A^* \cap \Sigma\dot{F}^2$ is a surjective map onto $\dot{k}_A$ and has $(1 + m_A) \cap \Sigma\dot{F}^2$ as its kernel. Since A is π-henselian, Lemma 2.1(ii) implies that $(1 + m_A) \cap \Sigma\dot{F}^2 \subset (A^*)^2$. Therefore, φ_A induces an isomorphism from $(A^* \cap \Sigma\dot{F}^2)/(A^*)^2$ to $\dot{k}_A/\dot{k}_A^2$. On the other side, as $(A^* \cap \Sigma\dot{F}^2)\dot{F}^2/\dot{F}^2 \simeq (A^* \cap \Sigma\dot{F}^2)/(A^*)^2$, we get the following exact sequence

$$1 \longrightarrow \dot{k}_A/\dot{k}_A^2 \longrightarrow \Sigma\dot{F}^2/\dot{F}^2 \longrightarrow \Sigma\dot{F}^2/(A^* \cap \Sigma\dot{F}^2)\dot{F}^2 \longrightarrow 1,$$

from which (a) follows.

(b) We first observe that since $\big(\Sigma\dot{F}^2 : (A^* \cap \Sigma\dot{F}^2)\dot{F}^2\big) \leq 2$, (a) implies that $(\dot{k}_A : \dot{k}_A^2) > 2$. Consequently, by Remark 2.9(B), $\varphi_A\big(A^* \cap B_\pi(F)\big) = B_\pi(k_A)$.

We now consider two cases. First case: $B_\pi(k_A) = \dot{k}_A$. Hence, as we saw in the prove of (a), $\varphi_A(A^* \cap \Sigma\dot{F}^2) = \dot{k}_A = B_\pi(k_A) = \varphi_A\big(A^* \cap B_\pi(F)\big)$. Which implies that $A^* \cap B_\pi(F) = A^* \cap \Sigma\dot{F}^2$. As $\dot{F}^2 \subset B_\pi(F)$ by the very definition of $B_\pi(F)$, we have that

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$(A^* \cap \Sigma\dot{F}^2)\dot{F}^2 \subset B_\pi(F)$. Since the other inclusion is always true (Remark 2.9(A)), the statement is proved in this case by taking $B = A$.

We now look for B in the case $\big(\dot{k}_A : B_\pi(k_A)\big) = 2$. Since we have seen that $(\dot{k}_A : \dot{k}_A^2) > 2$, we can conclude that $B_\pi(k_A) \neq \dot{k}_A^2$. We claim that $\dot{k}_A^2$ is not *exceptional* ([AEJ], Definition 2.15). In fact, if $B_\pi(k_A) = \pm\dot{k}_A^2$, the above conditions show that $-1 \notin \dot{k}_A^2$, and $\dot{k}_A^2$ is not addively closed, since $\dot{k}_A \neq \dot{k}_A^2$.

Consequently, Proposition 2.11 of [EN] implies that k_A has a π-henselian non-dyadic valuation ring $\bar{B}$ such that $\dot{k}_A \neq \bar{B}^*\dot{k}_A^2$. On the other side, by ([AEJ], Proposition 1.9(1)), $B_\pi(k_A)$ is contained in $\bar{B}^*\dot{k}_A^2$, and so these two groups must be equal (recall that $\big(\dot{k}_A : B_\pi(k_A)\big) = 2$).

We now take $B \subset A$ the valuation ring of F such that $\varphi_A(B) = \bar{B}$. Observe that the residue field of B, being equal to the residue field of $\bar{B}$, is a non-formally real field such that $\operatorname{char} k_B \neq 2$. Next, as A and $\bar{B}$ are π-henselian it follows from Lemma 2.1(ii) that B is also π-henselian (or use Lemma 1.3 from [Br]).

We claim that B is the desired valuation ring. By Remark 2.9(A), it is enough to prove that $(B^* \cap \Sigma\dot{F}^2)\dot{F}^2 \subset B_\pi(F)$. As in the prove of (a), we have that $\varphi_A(B^* \cap \Sigma\dot{F}^2) = \bar{B}^*$. Thus $\varphi_A\big((B^* \cap \Sigma\dot{F}^2)(A^*)^2\big) = \bar{B}^*\dot{k}_A^2 = B_\pi(k_A)$. But, we saw in the beginning of the prove that $\varphi_A\big(A^* \cap B_\pi(F)\big) = B_\pi(k_A)$. And so, putting the things together, $(B^* \cap \Sigma\dot{F}^2)(A^*)^2 = A^* \cap B_\pi(F)$. Therefore, $(B^* \cap \Sigma\dot{F}^2)\dot{F}^2 \subset B_\pi(F)$, as desired. Finally, since $B_\pi(k_A) = \bar{B}^*\dot{k}_A^2$, an easy verification shows that $B_\pi(k_B) = \dot{k}_B$. ■

PROPOSITION 4.3. *Let F be a formally real field such that $(\Sigma\dot{F}^2 : \dot{F}^2) > 4$. Then the following conditions are equivalent.*

(i) There exists a normal abelian subgroup U of $G_\pi(F)$ of rank ≥ 2.

(ii) F admits a π-henselian valuation ring A such that $\operatorname{rank} G^T(C;F) \geq 2$ *for the extension C of A to F_π.*

(iii) $B_\pi(F)$ is a group such that $\big(\Sigma\dot{F}^2 : B_\pi(F)\big) > 2$.

PROOF. (i) ⇒ (ii) Let L be the fixed field of U. Since $\operatorname{rank} U \geq 2$, then $(\Sigma\dot{L}^2 : \dot{L}^2) > 2$ and so Proposition 4.1 applies to L. Let A' be a distinguished π-henselian valuation ring of L verifying the condition (iii) of 4.1. Therefore, Corollary 2.8 implies that $A = A' \cap F$ is a π-henselian valuation ring of F. By Remark 2.9(C), we can assume without loss of generality that $\big(\dot{k}_A : B_\pi(k_A)\big) \leq 2$.

If $\big(\Sigma\dot{F}^2 : (A^* \cap \Sigma\dot{F}^2)\dot{F}^2\big) > 2$ the proof is done by Proposition 3.6. If $\big(\Sigma\dot{F}^2 : (A^* \cap \Sigma\dot{F}^2)\dot{F}^2\big) \leq 2$, let B be the valuation ring given by Lemma 4.2(b). We claim that $\big(\Sigma\dot{F}^2 : (B^* \cap \Sigma\dot{F}^2)\dot{F}^2\big) > 2$. Going for a contradiction we assume that $\big(\Sigma\dot{F}^2 : (B^* \cap \Sigma\dot{F}^2)\dot{F}^2\big) \leq 2$. Then, it follows from Proposition 3.6 that $G^T(C;F)$ is either trivial or isomorphic to $\mathbb{Z}_2$, where C is the extension of B to F_π. Therefore the same is true for $G^T(C;L)$, since it is a subgroup of $G^T(C;F)$. Let B' be the extension of B to L. Since $\operatorname{rank} U \geq 2$, Proposition 3.2(b) implies that $G_\pi(k_{B'})$ is a non-trivial abelian subgroup of $G_\pi(k_B)$. It is also a normal subgroup, since $L|F$ is a normal extension. Observe now that since $(\Sigma\dot{F}^2 : \dot{F}^2) > 4$, then $(k_B : \dot{k}_B^2) > 2$ by Lemma 4.2(a). Therefore,

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Lemma 4.3 of [EN] leads to the contradiction $B_\pi(k_B) \neq \dot{k}_B$ (recall (b) of 4.2). Therefore the claim is proved and so, replacing A by B, (ii) follows from Proposition 3.6.

(ii) $\Rightarrow$ (iii) Since F admits a π-henselian valuation ring A, we know by Remark 2.9(A) that $B_\pi(F)$ is a group. Since $B_\pi(F) \subset (A^* \cap \Sigma\dot{F}^2)\dot{F}^2$ and $\big(\Sigma\dot{F}^2 : (A^* \cap \Sigma\dot{F}^2)\dot{F}^2\big) > 2$ it follows that $\big(\Sigma\dot{F}^2 : B_\pi(F)\big) > 2$.

(iii) $\Rightarrow$ (ii) It follows from Remark 2.9(D) that F admits a π-henselian valuation ring A such that $\big(\dot{k}_A : B_\pi(k_A)\big) \leq 2$. If $\big(\Sigma\dot{F}^2 : (A^* \cap \Sigma\dot{F}^2)\dot{F}^2\big) > 2$ the statement follows from Proposition 3.6. In the other case, by Lemma 4.2(b), there exists a π-henselian valuation ring $B \subset A$ of F such that $B_\pi(F) = (B^* \cap \Sigma\dot{F}^2)\dot{F}^2$. Thus B fulfills the conditions of (ii) (Proposition 3.6).

(ii) $\Rightarrow$ (i) is trivial, since $G^T(C;F)$ is abelian. ∎

In the next section we partially remove the two restrictions, $(\Sigma\dot{F}^2 : \dot{F}^2) > 4$ and rank $U \geq 2$, in the study of normal abelian subgroups of $G_\pi(F)$.

5. **The case** $(\Sigma\dot{F}^2 : \dot{F}^2) = 4$**.** In this section we shall study $G_\pi(F)$ for fields satisfying $(\Sigma\dot{F}^2 : \dot{F}^2) = 4$. This case requires a particular care since Remark 2.9(D) does not apply to F, if $B_\pi(F) \neq \dot{F}^2$. We first prove three preparatory results. Let us adapt Theorem 3.4 of ([L], p. 202) to sum of squares.

LEMMA 5.1. *Let F be a formally real field, $d \in \Sigma\dot{F}^2 \setminus \dot{F}^2$ and $K = F(\sqrt{d})$. The following sequence is exact*

$$1 \longrightarrow \{\dot{F}^2, d\dot{F}^2\} \longrightarrow \Sigma\dot{F}^2/\dot{F}^2 \longrightarrow \Sigma\dot{K}^2/\dot{K}^2 \xrightarrow{\bar{N}} (D\langle 1, -d\rangle \cap \Sigma\dot{F}^2)/\dot{F}^2 \longrightarrow 1,$$

where $\bar{N}$ is induced by the norm $N: K \longrightarrow F$ and the others maps are natural.

PROOF. Recall that $D\langle 1, -d\rangle$ is the image of N. We next show that $N(\Sigma\dot{K}^2) \subset \Sigma\dot{F}^2$. Thus, the image of $\bar{N}$ is contained in $D\langle 1, -d\rangle \cap \Sigma\dot{F}^2/\dot{F}^2$. Denote by $\bar{z}$ the conjugate of each $z \in K$. If $z \in \Sigma\dot{K}^2$ then $\bar{z} \in \Sigma\dot{K}^2$, too. Therefore, $N(z) \in \Sigma\dot{K}^2 \cap \dot{F}$. Since $K|F$ is normal, each order of F extends to K. On the other side, Artin-Schreier Theory states that the set of sum of squares is the intersection of the positive cones of all orders of the field. Hence $\Sigma\dot{K}^2 \cap F = \Sigma\dot{F}^2$. So $N(z) \in \Sigma\dot{F}^2$, as required. We now prove the surjectivity. Take $x \in D\langle 1, -d\rangle \cap \Sigma\dot{F}^2$ and $z \in \dot{K}$ such that $N(z) = x$. By ([W2], Lemma 3.10) there is $a \in \dot{F}$ such that $az \in \Sigma\dot{K}^2$. Thus $a^2x = N(az)$ shows that $\bar{N}$ is surjective.

The sequence is exact at $\Sigma\dot{K}^2/\dot{K}^2$. For $z \in \Sigma\dot{K}^2$ such that $N(z) = x^2 \in \dot{F}^2$, $N(zx^{-1}) = 1$. Thus, by Hilbert's Theorem 90, $zx^{-1} = y/\bar{y} = y^2/N(y)$, for some $y \in \dot{K}$. Hence, $x/N(y) \in \Sigma\dot{K}^2 \cap \dot{F} = \Sigma\dot{F}^2$ and $z\dot{K}^2 = \big(x/N(y)\big)\dot{K}^2$. Therefore the kernel of $\bar{N}$ is contained in image of $\Sigma\dot{F}^2/\dot{F}^2$. Since the other inclusion is clear the statement is proved.

The sequence is exact at $\Sigma\dot{F}^2/\dot{F}^2$. Let $x \in \Sigma\dot{F}^2$ such that $x \in \dot{K}^2$ and write $x = (a + b\sqrt{d})^2$. Then $2ab = 0$. If $b = 0$, then $x \in \dot{F}^2$. Otherwise $x \in d\dot{F}^2$. ∎

LEMMA 5.2. *For every $a, b, \in \dot{F}$ the following is true:*

(a) $a \in D\langle 1, ab\rangle$ if and only if $a \in D\langle 1, -b\rangle$.

(b) $ab \in D\langle 1, a\rangle$ if and only if $ab \in D\langle 1, -b\rangle$.

(c) $b \in D\langle 1, -b\rangle$ if and only if $b \in D\langle 1, 1\rangle$.

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PROOF. Let $x, y \in \dot{F}^2$.

(a) If $a = x + aby$, then $ax = a^2 - ba^2y \in D\langle 1, -b\rangle$. Conversely, $a = x - by$ implies $ax = a^2 + aby \in D\langle 1, ab\rangle$.

(b) If $ab = x + ay$, then $abx = (ab)^2 - ba^2y \in D\langle 1, -b\rangle$. Conversely, $ab = x - by$ implies that $abx = (ab)^2 + ab^2y \in D\langle 1, a\rangle$.

(c) Just take $a = 1$ in (b). ∎

From now on we assume $(\Sigma\dot{F}^2 : \dot{F}^2) = 4$. Therefore, (as F is non-pythagorean) either $D\langle 1, 1\rangle = \Sigma\dot{F}^2$ or $(\Sigma\dot{F}^2 : D\langle 1, 1\rangle) = 2$.

PROPOSITION 5.3. *Let F be a formally real field such that $(\Sigma\dot{F}^2 : \dot{F}^2) = 4$. Then*

(a) $D\langle 1, 1\rangle = \Sigma\dot{F}^2$, if and only if either $B_\pi(F) = \dot{F}^2$ or $B_\pi(F) = \Sigma\dot{F}^2$.

(b) If $D\langle 1, 1\rangle \neq \Sigma\dot{F}^2$, then $B_\pi(F) = D\langle 1, 1\rangle$.

PROOF. (a) Observe first that since $D\langle 1, 1\rangle = \Sigma\dot{F}^2$, $B_\pi(F)$ is a group, by ([En], Corollary 2.13(2)). Take $a, b \in \Sigma\dot{F}^2$ for which $\Sigma\dot{F}^2 = \dot{F}^2 \dot\cup a\dot{F}^2 \dot\cup b\dot{F}^2 \dot\cup ab\dot{F}^2$ and assume $B_\pi(F) \neq \dot{F}^2$. Without loss of generality we may assume $b \in B_\pi(F)$. Therefore, $\dot{F}^2 \cup b\dot{F}^2 \subsetneq D\langle 1, b\rangle$. Thus $D\langle 1, b\rangle = \Sigma\dot{F}^2$. Hence, $ab \in D\langle 1, b\rangle$ which implies, by (b) of the last lemma, that $ab \in D\langle 1, -a\rangle$. By assumption $a \in D\langle 1, 1\rangle$. Thus, (c) of the above lemma implies that $a \in D\langle 1, -a\rangle$. Therefore, $a^2b \in D\langle 1, -a\rangle$ and $b \in D\langle 1, -a\rangle$. Hence, (a) of the last lemma implies that $b \in D\langle 1, ab\rangle$ and so $ab \in B_\pi(F)$. Then $B_\pi(F) = \Sigma\dot{F}^2$.

Conversely, assume $B_\pi(F) = \Sigma\dot{F}^2$ and take $x \in \Sigma\dot{F}^2 \setminus \dot{F}^2$. Since $(D\langle 1, x\rangle : \dot{F}^2) > 2$, then $D\langle 1, x\rangle = \Sigma\dot{F}^2$. Therefore, if $y \in \Sigma\dot{F}^2$ verifies $y, xy \notin \dot{F}^2$, as $y \in D\langle 1, xy\rangle$ and $xy \in D\langle 1, y\rangle$, by (b) and (c) of the above lemma, $y, xy \in D\langle 1, -x\rangle$. Thus $x \in D\langle 1, -x\rangle$ and (c) implies that $x \in D\langle 1, 1\rangle$.

If $B_\pi(F) = \dot{F}^2$, the statement was proved by Ware, ([W2], Corollary 3.11).

(b) According to (a), $B_\pi(F) \neq \Sigma\dot{F}^2$. Therefore, it is enough to prove that $D\langle 1, 1\rangle \subset B_\pi(F)$. By assumption, there are $x, y, z \in \dot{F}^2$ such that $a = x + y + z \notin D\langle 1, 1\rangle$. Therefore, $b = y + z \in D\langle 1, 1\rangle \setminus \dot{F}^2$ and $a \in D\langle 1, b\rangle$. Since $a \notin \dot{F}^2 \cup b\dot{F}^2$, then $b \in B_\pi(F)$. So $D\langle 1, 1\rangle = \dot{F}^2 \cup b\dot{F}^2 \subset B_\pi(F)$, as desired. ∎

PROPOSITION 5.4. *Let F be a field as in the previous proposition such that $D\langle 1, 1\rangle \neq \Sigma\dot{F}^2$. If $E = F(\sqrt{b})$ for $b \in D\langle 1, 1\rangle \setminus \dot{F}^2$, then $(\Sigma\dot{E}^2 : \dot{E}^2) = 4$ and $B_\pi(E) = \dot{E}^2$.*

PROOF. For $a \in \Sigma\dot{F}^2 \setminus D\langle 1, 1\rangle$ we have that $\Sigma\dot{F}^2 = \dot{F}^2 \dot\cup a\dot{F}^2 \dot\cup b\dot{F}^2 \dot\cup ab\dot{F}^2$. Observe that the choice of b implies that $D\langle 1, 1\rangle = \dot{F}^2 \dot\cup b\dot{F}^2$. By (b) of the previous proposition $B_\pi(F) = D\langle 1, 1\rangle$. Hence, $a, ab \notin B_\pi(F)$. Thus $a \notin D\langle 1, ab\rangle$. Hence, by (a) of Lemma 5.2, $a \notin D\langle 1, -b\rangle$. On the other side, as $b \in D\langle 1, 1\rangle$, Lemma 5.2(c) implies that $b \in D\langle 1, -b\rangle$. Hence $ab \notin D\langle 1, -b\rangle$ and consequently $D\langle 1, -b\rangle \cap \Sigma\dot{F}^2 = \dot{F}^2 \cup b\dot{F}^2$. Therefore, the exact sequence of Lemma 5.1 implies that $(\Sigma\dot{E}^2 : \dot{E}^2) = 4$, as required.

We now claim that $a \in D_E\langle 1, 1\rangle \setminus \dot{E}^2$ and $a \notin B_\pi(E)$. It follows from the claim that $B_\pi(E) \neq \Sigma\dot{E}^2, D_E\langle 1, 1\rangle$. Therefore, the last proposition implies that $B_\pi(E) = \dot{E}^2$ and the proof is complete.

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PROOF OF THE CLAIM. Since $b \in B_\pi(F)$ $(= D\langle 1,1\rangle)$, $D\langle 1,b\rangle = \Sigma\dot{F}^2$ and so $a \in D\langle 1,b\rangle$. Since $b \in \dot{E}^2$, then $a \in D_E\langle 1,1\rangle$. Again, the exact sequence of Lemma 5.1 implies that $a \notin \dot{E}^2$. We now prove that $a \notin B_\pi(E)$. Take $x \in D_E\langle 1,a\rangle$. By the "Norm Principle" ([EL], 2.13), $N(x) \in D_F\langle 1,a\rangle$, where, as in Lemma 5.1, N is the norm map. Since $a \notin B_\pi(F)$, $D\langle 1,a\rangle = \dot{F}^2 \cup a\dot{F}^2$. As we saw in the first paragraph, $D\langle 1,-b\rangle \cap \Sigma\dot{F}^2 = \dot{F}^2 \cup b\dot{F}^2$. Therefore, since $N(x) \in D\langle 1,-b\rangle \cap \Sigma\dot{F}^2$, it follows that $N(x) \in \dot{F}^2$. Once again the exact sequence of Lemma 5.1 implies that $x = yz$ for some $y \in \Sigma\dot{F}^2$ and $z \in \dot{E}^2$. For $y \in \Sigma\dot{F}^2$ there are $\varepsilon,\eta \in \{0,1\}$ and $t \in \dot{F}^2$ such that $y = a^\varepsilon b^\eta t$. Consequently, $x \in a^\varepsilon\dot{E}^2$ and so $D_E\langle 1,a\rangle = \dot{E}^2 \cup a\dot{E}^2$. Hence $a \notin B_\pi(E)$, as desired. ∎

We now carry on with the study of $G_\pi(F)$. Let us consider the case where there exists a normal extension $L|F$ such that $L \subset F_\pi$ and $G_\pi(L) \simeq \mathbb{Z}_2$. Denote by $C(U) = \{g \in G_\pi(F) \mid gh = hg\ \forall h \in U\}$, the centralizer of U, for every subgroup U of $G_\pi(F)$. Since $G_\pi(F)$ is torsion free ([Be], Theorem 7, p. 81), we can make use of the Proposition 3.1 of [EN]. Then, there exists $F \subset L' \subset L$ such that either

(1) $G_\pi(L') \simeq \mathbb{Z}_2$ and $C\big(G_\pi(L')\big) = G_\pi(L')$, or

(2) $G_\pi(L') \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

In the case (1) we shall prove that $(\Sigma\dot{F}^2 : \dot{F}^2) = 4$ and describe $G_\pi(F)$. Case (2) will be study later.

PROPOSITION 5.5. *Let F be a formally real field for which there exists a normal extension $F \subsetneq L \subset F_\pi$ such that $G_\pi(L) \simeq \mathbb{Z}_2$ and $C\big(G_\pi(L)\big) = G_\pi(L)$. Then, $(\Sigma\dot{F}^2 : \dot{F}^2) = 4$, $\big(B_\pi(F) : \dot{F}^2\big) \leq 2$ and F admits a π-henselian valuation ring A such that $\operatorname{char} k_A \neq 2$ and $G_\pi(L) = G^T(C;F)$, where C is the extension of A to F_π.*

Moreover, $\xi_{n+1} \notin F(i)$, for some $n \geq 2$ and $G_\pi(F) \simeq \mathbb{Z}_2 \rtimes \mathbb{Z}_2$, where the components have generators τ,σ verifying one of the following conditions:

(a) If $B_\pi(F) = \dot{F}^2$, then τ, σ can be chosen such that $\sigma\tau\sigma^{-1} = \tau^{2^{n+1}+1}$.

(b) If $\big(B_\pi(F) : \dot{F}^2\big) = 2$, then $n \geq 3$ and we can find generators τ, σ such that $\sigma\tau\sigma^{-1} = \tau^{2^{n-1}-1}$.

PROOF. Let $\theta\colon G_\pi(F) \longrightarrow \operatorname{Aut}\big(G_\pi(L)\big) \simeq \mathbb{Z}_2 \times \mathbb{Z}/2\mathbb{Z}$ be the homomorphism given by $\theta(g)(h) = ghg^{-1}$, for every $g \in G_\pi(F)$ and $h \in G_\pi(L)$. By assumption, kernel $\theta = G_\pi(L)$.

We claim that image$\theta \simeq \mathbb{Z}_2$.

To prove the claim we only need to show that imageθ contains no element of order 2. Assume this is not so and let $g \in G_\pi(F)$ such that $\theta(g)$ has order 2. Hence $g^2 \in C\big(G_\pi(L)\big) = G_\pi(L)$ and $ghg^{-1} = h^{-1}$ (the unique order 2 automorphism of $\mathbb{Z}_2$). Therefore $(g^2)^{-1} = gg^2g^{-1} = g^2$. This leads to a contradiction, since $G_\pi(F)$ is torsion free.

By the claim $G_\pi(F) \simeq \mathbb{Z}_2 \rtimes \mathbb{Z}_2$ and $G(L;F) \simeq \mathbb{Z}_2$. Hence $(\Sigma\dot{F}^2 : \dot{F}^2) = 4$ and the commutator subgroup $[G:G]$ of $G_\pi(F)$ satisfies $[G:G] \subset G_\pi(L)$. As $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is a homomorphic image of $G_\pi(F)$, we conclude that $[G:G] \neq G_\pi(L)$. Thus, there exists $n \geq 1$ such that $[G:G] = G_\pi(L)^{2^n}$. We now consider 2 cases.

If $n = 1$, we see that $\mathbb{Z}/4\mathbb{Z}$ is a homomorphic image of $G_\pi(F)$ and $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ is not. It is well known that there exists a Galois extension $E|F$ such that $G(E;F) \simeq \mathbb{Z}/4\mathbb{Z}$

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if and only if there exists a non-square $c \in F$, which is a sum of 2 squares and $F(\sqrt{c}) \subset E$ ([L], Ex. 8, p. 217). On the other side, from Griffin ([Gri], Lemma 8), we can choose $E \subset F_\pi$. Therefore, in the case $n = 1$, we can conclude that $\dot{F}^2 \subsetneq D\langle 1,1\rangle \subsetneq \Sigma\dot{F}^2$. So, Proposition 5.3(b) yields $B_\pi(F) = D\langle 1,1\rangle$ and $\big(\Sigma\dot{F}^2 : B_\pi(F)\big) = 2$.

In the other case, $[G : G] = G_\pi(L)^{2^n}$, for some $n \geq 2$, we shall prove that the dihedral group of order 8, D_8, is not a homomorphic image of $G_\pi(F)$. Thus, by ([W2], Theorem 3.9), $B_\pi(F) = \dot{F}^2$.

Suppose that there exists a subgroup T of $G_\pi(F)$ such that $G_\pi(F)/T \simeq D_8$ and let λ be a generator for $G_\pi(L)$. Hence the image of λ in $G_\pi(F)/T$ has order 1, 2 or 4. But, if $\lambda^s \in T$, for some $1 \leq s \leq 4$, then $[G : G] \subset T$, since $[G : G] = G_\pi(L)^{2^n}$ $(n \geq 2)$ is generated by λ^{2^n}. Therefore $G_\pi(F)/T$ is an abelian group, contradicting the supposition.

We now prove the existence of A as in the statement. In the case $B_\pi(F) = \dot{F}^2$, by Remark 2.9(E), F admits a distinguished π-henselian valuation ring A such that $(\dot{k}_A : \dot{k}_A^2) \leq 2$ and $-1 \in \dot{k}_A^2$.

In the other case, $B_\pi(F) = D\langle 1,1\rangle$. If $b \in B_\pi(F) \setminus \dot{F}^2$ and $E = F(\sqrt{b})$, then $(\Sigma\dot{E}^2 : \dot{E}^2) = 4$ and $B_\pi(E) = \dot{E}^2$, by Proposition 5.4. Then the last considerations apply to E. Let B be a distinguished π-henselian valuation ring of E such that $(\dot{k}_B : \dot{k}_B^2) \leq 2$ and let C be its extension to F_π. By Lemma 4.2(a) and Proposition 3.6, $G^T(C;E)$ is non-trivial. Next, we take $A = B \cap F$. According to Corollary 2.8, A is a π-henselian valuation ring of F. It is clear that C is the extension of A to F_π. Thus $G^T(C;F)$ is also non-trivial, and now, Proposition 3.6 and Lemma 4.2(a) imply that $(\dot{k}_A : \dot{k}_A^2) \leq 2$. Moreover, Proposition 3.2(a) implies that k_A is not formally real.

Consequently, in both cases, $B_\pi(F) = \dot{F}^2$ or $B_\pi(F) = D\langle 1,1\rangle$, F admits a π-henselian valuation ring A such that k_A is not formally and $(\dot{k}_A : \dot{k}_A^2) \leq 2$. Thus $G_\pi(k_A)$ is either trivial or isomorphic to $\mathbb{Z}_2$. The trivial case cannot occur, otherwise $G_\pi(F) = G^T(C;F)$ is an abelian group, contradicting the assumption $C\big(G_\pi(L)\big) = G_\pi(L)$. Thus $G_\pi(k_A) \simeq \mathbb{Z}_2$ and also $G^T(C;F) \simeq \mathbb{Z}_2$, by the above considerations.

Once again the assumption $C\big(G_\pi(L)\big) = G_\pi(L)$ implies that $G_\pi(L) \cap G^T(C;F) \neq 1$. Otherwise, $G^T(C;F) \subset C\big(G_\pi(L)\big)$ because $G_\pi(L)$ and $G^T(C;F)$ are normal subgroups of $G_\pi(F)$. Hence $\big(G_\pi(L) : G_\pi(L) \cap G^T(C;F)\big)$ is finite. But, as $G_\pi(L) \cap G^T(C;F) = G^T(C;L)$ ([E], 19.10-b) and $G_\pi(k_A)$ contains no finite subgroups, it follows that $G_\pi(L) = G^T(C;F)$, as desired.

Our assumption on $G_\pi(L)$ implies $C\big(G^T(C;F)\big) = G^T(C;F)$. Hence, it follows from Proposition 3.4(a) that there exists $n \geq 2$ such that $\xi_n \notin k_A$. Furthermore, if $-1 \notin \dot{k}_A^2$, the above argument shows that $\xi_n \notin k_A(i)$, for some $n > 2$ (see 3.4(c)). Since the residue field of the extension of A to $F(i)$ is $k_A(i)$ we can conclude that $\xi_n \notin F(i)$, for some $n > 2$.

We now prove (a) and (b). Observe first that the last considerations imply that $k_A(\mu_\infty)$ is the quadratic closure of k_A. So $G(k_A(\mu_\infty); k_A) = G_\pi(k_A)$.

In the case $B_\pi(F) = \dot{F}^2$ we have observed that $-1 \in k_A$. Then, Proposition 3.4(b) implies that $G_\pi(F)$ has the required description.

In the other case, $B_\pi(F) = D\langle 1,1\rangle \neq \Sigma\dot{F}^2$, the lemma below implies that $-1 \notin \dot{k}_A^2$.

LEMMA 5.6. *Let A be a π-henselian valuation ring of a formally real field F. If $-1 \in \dot{k}_A^2$, then $\Sigma\dot{F}^2 = D\langle 1,1\rangle$ (char $k_A \neq 2$).*

PROOF. Let $t \in \Sigma\dot{F}^2$ and wright $t = x_1 + \cdots + x_n$ where $x_1,\ldots,x_n \in \dot{F}^2$. We may assume that $n \geq 2$. Take $1 \leq i \leq n$ such that $v_A(x_i) \leq v_A(x_j)$, for every $1 \leq j \leq n$. Without loss of generality we may assume $i = 1$. Then $tx_1^{-1} = 1 + x_1^{-1}(x_2 + \cdots + x_n) \in A$ and also $x_1^{-1}(x_2+\cdots+x_n) \in A$. If $x_1^{-1}(x_2+\cdots+x_n) \in m_A$, then $tx^{-1} \in 1+m_A$ and so $tx^{-1} \in \dot{F}^2$, by Lemma 2.1(ii). Since this contradicts $n \geq 2$, it follows that $x_1^{-1}(x_2+\cdots+x_n) \in A^*$. We now consider two cases. If $tx_1^{-1} \in m_A$ then $\varphi_A\big(x_1^{-1}(x_2+\cdots+x_n)\big) = -1$. Take $u \in A^* \cap \dot{F}^2$ such that $\varphi_A(u) = -1$. Then $x_1^{-1}(x_2+\cdots+x_n)u^{-1} \in (1+m_A) \cap \Sigma\dot{F}^2$. Thus $x_1^{-1}(x_2+\cdots+x_n) \in \dot{F}^2$ (2.1(ii)) and $t \in D\langle 1,1\rangle$ as desired. If $tx_1^{-1} \in A^*$ there exist $a,b \in A^* \cap \dot{F}^2$ such that $\varphi_A(tx_1^{-1}) = \varphi_A(a) + \varphi_A(b)$, since $-1 \in \dot{k}_A^2$ (every element of k_A is a sum of at most two squares). Once again $tx_1^{-1}(a+b)^{-1} \in (1+m_A) \cap \Sigma\dot{F}^2$ will imply $t \in D\langle 1,1\rangle$. ■

Therefore, back in the proof of the proposition, we see that (b) follows from Proposition 3.4(c). ■

Conversely,

PROPOSITION 5.7. *Let F be a formally real field for which $(\Sigma\dot{F}^2 : \dot{F}^2) = 4$ and there exists $n \geq 2$ such that $\xi_{n+1} \notin F(i)$. Then there exists a normal subextension $F \subset L \subset F_\pi$ such that $G_\pi(L) \simeq \mathbb{Z}_2$ and $C\big(G_\pi(L)\big) = G_\pi(L)$.*

PROOF. Let us denote by L the field FH introduced before Lemma 3.3. Then $L|F$ is a normal subextension such that $G(L;F) \simeq \mathbb{Z}_2$. Thus, $G_\pi(F) \simeq G_\pi(L) \rtimes \mathbb{Z}_2$. Since $G_\pi(F)$ has rank 2 and $G_\pi(L)$ is torsion free, $G_\pi(L) \simeq \mathbb{Z}_2$.

Going for a contradiction let us assume that $C\big(G_\pi(L)\big) \neq G_\pi(L)$. Denote by E the fixed field of $C\big(G_\pi(L)\big)$. Since $F \subset E \subset L$ and $G(L;F) \simeq \mathbb{Z}_2$, it follows that $E|F$ is a finite extension. Therefore, there exists $m \geq n+1$ such that $\xi_m \notin E(i)$. It follows from this and ([Gri], Proposition 11) that $C\big(G_\pi(L)\big)$ is not abelian.

On the other side, arguing as above, we have a decomposition $C\big(G_\pi(L)\big) \simeq G_\pi(L) \rtimes \mathbb{Z}_2$. Consequently, due to its nature, $C\big(G_\pi(L)\big)$ has to be an abelian group, a contradiction. ■

We now consider the case $C\big(G_\pi(L)\big) \neq G_\pi(L)$.

PROPOSITION 5.8. *Let F be a field for which there exists a normal subextension $F \subsetneq L \subset F_\pi$ such that $G_\pi(L) \simeq \mathbb{Z}_2$, $C\big(G_\pi(L)\big) \neq G_\pi(L)$ and for every normal subextension $F \subset E \subset L$, $G_\pi(E) \not\simeq \mathbb{Z}_2$. Then F admits a π-henselian valuation ring A such that char $k_A \neq 2$ and $G^T(C;F)$ is non-trivial, where C is the extension of A to F_π. Moreover, if B is the extension of A to L, then $\xi_n \in k_B$, for every $n \geq 1$. Furthermore, $(\Sigma\dot{F}^2 : \dot{F}^2) \geq 4$ and $G_\pi(F)$ is abelian if equality occurs.*

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PROOF. By ([EN], Proposition 3.1), the above assumptions imply that there exists a subgroup N of $C\big(G_\pi(L)\big)$ such that $N \simeq \mathbb{Z}_2$ and $NG_\pi(L) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. Let E be the fixed field of $NG_\pi(L)$. According to Proposition 4.1, E admits a π-henselian valuation ring B' such that its residue field k verifies: char $k \neq 2$, $(\dot{k} : \dot{k}^2) \leq 2$ and $\xi_n \in k$ for every $n \geq 1$. Observe now that the extension B of B' to L is π-henselian and k_B shares with k the above properties. By Corollary 2.8, we may assume that $A = B \cap F$ is π-henselian.

Let C be the extension of B' to F_π. Since rank $G_\pi(E) = 2$ and $(\dot{k} : \dot{k}^2) \leq 2$, it follows that $G^T(C; E)$ is non-trivial. As C is also the extension of A to F_π it follows that $G^T(C; F)$ is also non-trivial.

Finally, as $G_\pi(E)$ is not a cyclic group the same is true for $G_\pi(F)$. Hence $(\Sigma\dot{F}^2 : \dot{F}^2) \geq 4$. On the other side, if $(\Sigma\dot{F}^2 : \dot{F}^2) = 4$, by Lemma 4.2(a), $(\dot{k}_A : \dot{k}_A^2) \leq 2$. Therefore, either $G_\pi(k_A)$ is trivial or $G_\pi(k_A) \simeq \mathbb{Z}_2$. In the first case $G_\pi(F) = G^T(C; F)$ is abelian. In the other case, Lemma 4.2(a) and Proposition 3.6 imply that $G^T(C; F) \simeq \mathbb{Z}_2$. We now claim that $\xi_n \in k_A$, for every $n \geq 1$. If this is not so $k_A(\mu_\infty)$ is the quadratic closure of k_A. But, $\mu_\infty \subset k$ (the residue field of B'). Hence $k = k_A(\mu_\infty)$ which implies $K^T(C; F) \subset E$. Since this cannot occur, because rank $G_\pi(E) >$ rank $G^T(C; F)$, we get a contradiction. Consequently, Proposition 3.4(a) yields $G_\pi(F)$ abelian. ∎

6**. The general case.** Now, by combining the results of the previous sections we shall consider general abelian subgroups of $G_\pi(F)$.

THEOREM 6.1. *Let F be a formally real field such that* $(\Sigma\dot{F}^2 : \dot{F}^2) > 2$. *Then the following conditions are equivalent.*

(i) There exists a normal abelian subgroup U of $G_\pi(F)$.

(ii) F admits a π-henselian valuation ring A such that $\Sigma\dot{F}^2 \not\subset A^*\dot{F}^2$ and k_A is a non-formally real field with char $k_A \neq 2$.

Furthermore, assuming that $G_\pi(F)$ is not abelian, A can be chosen such that $U \subset G^T(C; F)$, for the unique extension C of A to F_π.

PROOF. The implication (i) $\Rightarrow$ (ii) is consequence of Proposition 4.3, Proposition 5.5 and Proposition 5.8. The other direction is clear by Proposition 3.4.

For the last statement we assume $G_\pi(F)$ non-abelian and let U be a non-trivial normal abelian subgroup of $G_\pi(F)$. Observe first that, by Proposition 3.4(a), $\dot{k}_A = \dot{k}_A^2$ cannot occur, since $G_\pi(F)$ is not abelian. Moreover, if there exists A such that $G_\pi(k_A)$ has no non-trivial normal abelian subgroup, then A has the required property. Indeed, for the extension C of A to F_π, the quotient $UG^T(C; F)\big/G^T(C; F)$ is trivial by the assumption. Hence $U \subset G^T(C; F)$, as desired. On the other side, by Remark 2.9(C), A can be chosen such that $\big(\dot{k}_A : B_\pi(k_A)\big) \leq 2$. But, if $\dot{k}_A = B_\pi(k_A)$ and $(\dot{k}_A : \dot{k}_A^2) > 2$, Lemma 4.3 of [EN] implies that there is no non-trivial normal abelian subgroup of $G_\pi(k_A)$ and the result is proved. Therefore, to finish the proof we have to consider two cases: $(\dot{k}_A : \dot{k}_A^2) = 2$ or $(\dot{k}_A : \dot{k}_A^2) > 2$ and $\big(\dot{k}_A : B_\pi(k_A)\big) = 2$.

In the first case we shall see that A has the required property. Let C be the extension of A to F_π. Since k_A is not formally real, Proposition 2.3(c) implies that k_C is the quadratic closure of k_A. Since $G_\pi(F)$ is not abelian, $\xi_n \notin k_A$ for some $n \geq 2$ (Proposition 3.4).

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Hence $k_C = k_A(\mu_\infty)$. Now, if rank $U \geq 2$ and L is the fixed field of U, Proposition 4.1(iii) and Remark 2.7 imply that $C \cap L$ has residue field k_C. From ([E], Theorem 19.11, p. 151) it follows that $K^T(C;F) \subset E$ and $U \subset G^T(C:F)$, as desired.

In the case rank $U = 1$, if $C(U) = U$, the result was proved in Proposition 5.5. In the other case ($C(U) \neq U$), observe first that we may replace U by a maximal normal abelian subgroup of $G_\pi(F)$ (Use Zorn's Lemma and recall that the topological closure of a normal abelian subgroup is still normal and abelian). Thus we can assume that U is maximal. Hence, by Proposition 5.8 and Remark 2.7, for the fixed field L of U, the residue field of $C \cap L$ contains ξ_n, for every $n \geq 1$. Once again $K^T(C;F) \subset E$, as in the above case and the result is proved.

We now consider the case $(\dot{k}_A : \dot{k}_A^2) > 2$ and $\big(\dot{k}_A : \underline{B}_\pi(k_A)\big) = 2$. According to ([AEJ], Corollary 2.17) k_A admits a π-henselian valuation ring $\overline{B}$ such that $B_\pi(k_A) = \overline{B}^* \dot{k}_A^2$ (observe that the above conditions imply that $\dot{k}_A^2$ is not exceptional ([AEJ], Definition 2.15)). Let $B = \varphi_A^{-1}(\overline{B})$. Since A and $\overline{B}$ are π-henselian, so is B (by Lemma 2.1(ii)). Furthermore, by Proposition 1.9 of [AEJ], $B_\pi(k_B) = \dot{k}_B$. Therefore, by what we have proved before, B has the desired property. ∎

The next theorem improves Griffin's result.

THEOREM 6.2. *With the same conditions of the previous theorem assume that U is an abelian subgroup of $G_\pi(F)$ such that* rank $U \geq 2$. *Then there exists a valuation ring C of F_π with* char $k_C \neq 2$ *for which either $U \subset G^T(C;F)$, or there exists a subextension, $F \subset E \subset F_\pi$ such that $G_\pi(E) \simeq G^T(C;F) \times \mathbb{Z}_2$ and $U \subset G_\pi(E)$.*

PROOF. Apply Proposition 4.1 to the fixed field of U. ∎

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