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INJECTIVITY IN THE TOPOS OF COMPLETE HEYTING ALGEBRA VALUED SETS

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1. Introduction. Let $\mathscr{A}$ be a complete Heyting algebra (CHA). An *$\mathscr{A}$-valued set* is a pair (X, δ) where X is a set and δ is a function from $X \times X$ to $\mathscr{A}$ such that

$$\delta(x, y) = \delta(y, x) \quad \text{and} \quad \delta(x, y) \wedge \delta(y, z) \leqq \delta(x, z)$$

for all x, y z in X. $\mathscr{A}$-valued sets form a category $\mathscr{S}(\mathscr{A})$ as follows: a morphism from (X, δ) to (Y, δ) is a function f from $X \times Y$ to $\mathscr{A}$ such that

(i) $f(x) \wedge \delta(x, x') \leqq f(x', y)$, $f(x, y) \wedge \delta(y, y') \leqq f(x, y')$,

(ii) $f(x, y) \wedge f(x, y') \leqq \delta(y, y')$, and

(iii) $\bigvee_y f(x, y) = \delta(x, x)$

for all x, x' in X and y, y' in Y; if $f:(X, \delta) \rightarrow (Y, \delta)$ and $g:(Y, \delta) \rightarrow (Z, \delta)$ are morphisms then $gf:(X, \delta) \rightarrow (Z, \delta)$ is given by

$$(gf)(x, z) = \bigvee_y f(x, y) \wedge g(y, z);$$

the identity morphism $1_{(X,\delta)}$ at (X, δ) is just δ.

The use of a CHA as truth-value algebra includes both the complete boolean algebras used in boolean-valued set theory and the lattices of open subsets of topological spaces used in sheaf theory. The δ function has two purposes: $\epsilon(x) = \delta(x, x)$ measures the extent to which x is granted membership in (X, δ) (in the case of a sheaf of functions on a topological space, it gives the domain of x), and $\delta(x, y)$ itself measures the extent to which x and y are equal in (X, δ) (in the topological case, it gives the largest open set on which x and y agree). Thus the present conception of an $\mathscr{A}$-valued set is somewhat broader than is customary in multivalued set theories (fuzzy set theory for example) where one or the other of the above two considerations is frequently absent. A morphism is to be thought of as described by the ($\mathscr{A}$-valued) characteristic function of its graph, so that $f(x, y)$ measures the extent to which y is associated by f to x. (It might have been technically more convenient to construct first the category of $\mathscr{A}$-valued sets and relations.) In the definition of morphism, (i) is

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extensionality; the replacement of equals by equals ([**13**]), (ii) states that f is single-valued, and (iii) that f is everywhere defined.

The aim of this paper is to describe injectives and injective hulls in $\mathcal{S}(\mathcal{A})$. Sections 2 and 3 are devoted to the concepts and results needed for this description, which is itself obtained in Section 4. Section 2 contains various preliminaries such as the construction of products, etc. in $\mathcal{S}(\mathcal{A})$, leading to a direct proof that $\mathcal{S}(\mathcal{A})$ is a topos; although this fact is a consequence of the equivalence, shown in Section 3, of $\mathcal{S}(\mathcal{A})$ to the topos of sheaves on $\mathcal{A}$, the explicit construction of power objects in $\mathcal{S}(\mathcal{A})$ is needed later on, as is that of the object $(\widetilde{X, \delta})$ of 'partial elements' of (X, δ), also in Section 2. Much of what is in Sections 2 and 3 has been obtained independently by Fourman and Scott [**2**].

Injectivity in this paper is to be understood in the usual external sense: an object X of a category $\mathcal{C}$ is injective if for all monos $m:A \rightarrowtail B$ and morphisms $f:A \to X$, there exists a morphism $g:B \to X$ such that $f = gm$. In a series of papers [**10**], [7], [**9**], Linton, Paré and Johnstone have studied this and related notions of injectivity in a topos. In his thesis [**11**], Meyer also investigated injectivity in a topos and in particular he proved that, in a Grothendieck topos, every object has an injective hull ([**11**, (4.10)]; a more general result has been obtained recently by Ebrahimi [**1**]). The same is true a fortiori for the situation considered here, namely that of $\mathcal{S}(\mathcal{A})$; however the present approach is quite different from Meyer's and it leads to a simple explicit description of injective hulls in $\mathcal{S}(\mathcal{A})$.

An earlier version ([**5**]) of this paper was accepted for publication in the Canadian Mathematical Bulletin but for various reasons I discontinued the process of seeing it through into print. In addition to the present contents, [**5**] contained: a brief account of logic in $\mathcal{S}(\mathcal{A})$; the Lawvere-Tierney version (for $\mathcal{S}(\mathcal{A})$) of the independence of the continuum hypothesis [**14**]; a sketch of a proof that $\mathcal{S}(\mathcal{A})$ is equivalent to the category of sets within the universe $V^{(\mathcal{A})}$ of $\mathcal{A}$-valued set theory; a mention of sheaves on an arbitrary site from the point of view of $\mathcal{A}$-valued sets; and some elementary remarks on boolean powers and ultrapowers in relation to $\mathcal{S}(\mathcal{A})$.

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2. Preliminary results. The representation of (some) morphisms of $\mathcal{S}(\mathcal{A})$ by actual functions, introduced by Scott [**13**], is very useful. Let (X, δ) and (Y, δ) be $\mathcal{A}$-valued sets, let $f_0:X \to Y$, and define $\bar{f}_0:X \times Y \to \mathcal{A}$ by

$$\overline{f_0}(x, y) = \epsilon(x) \wedge \delta(f_0(x), y).$$

If $\overline{f_0}$ is a morphism from (X, δ) to (Y, δ) we say that f_0 *represents* $\overline{f_0}$. For example, 1_X represents $1_{(X,\delta)}$.

(2.1) (a). *Let f be a morphism from (X, δ) to (Y, δ) and $f_0:X \rightarrow Y$ a function. Then f_0 represents f if and only if*

$$f(x, y) \leqq \delta(f_0(x), y) \quad \text{for all } x \text{ in } X \text{ and } y \text{ in } Y.$$

(b) *A function $f_0:X \rightarrow Y$ represents some morphism from (X, δ) to (Y, δ) if and only if*

$$\delta(x, x') \leqq \delta(f_0(x), f_0(x')) \quad \text{for all } x, x' \text{ in } X.$$

(c) *Let f_0 and f_1 represent morphisms from (X, δ) to (Y, δ). Then they represent the same morphism if and only if*

$$\epsilon(x) \leqq \delta(f_0(x), f_1(x)) \quad \text{for all } x \text{ in } X.$$

(d) *Let f_0 represent a morphism from (X, δ) to (Y, δ). If g is a morphism from (Y, δ) to (Z, δ) then*

$$(g\overline{f_0})(x, z) = \epsilon(x) \wedge g(f_0(x), z) \quad \text{for all } x \text{ in } X \text{ and } z \text{ in } Z;$$

if g_0 represents a morphism from (Y, δ) to (Z, δ) then

$$\overline{g_0}\,\overline{f_0} = \overline{g_0 f_0}.$$

Proof. In (a) it is clear that if f_0 represents f then the stated inequality holds, and the converse follows from:

$$\begin{aligned} f(x, y) \leqq \epsilon(x) \wedge \delta(f_0(x), y) &= \bigvee_{y'} f(x, y') \wedge \delta(f_0(x), y) \\ &= \bigvee_{y'} f(x, y') \wedge \delta(f_0(x), y') \wedge \delta(f_0(x), y) \\ &\leqq \bigvee_{y'} f(x, y') \wedge \delta(y', y) = f(x, y). \end{aligned}$$

The remaining results are quite straightforward.

An $\mathscr{A}$-valued set is said to be *ample* if every morphism to it is represented by a function; such $\mathscr{A}$-valued sets are characterized in Section 3.

(2.2) *$\mathscr{S}(\mathscr{A})$ has finite products and a terminal object.*

Proof. We show that the product of (X, δ) and (Y, δ) is given by $(X \times Y, \delta)$ where

$$\delta((x, y), (x', y')) = \delta(x, x') \wedge \delta(y, y')$$

and the projections p and q to (X, δ) and (Y, δ) are represented by the projections from $X \times Y$ to X and Y respectively. (Note. $(X \times Y, \delta)$ and

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$(X, \delta) \times (Y, \delta)$ always denote the $\mathscr{A}$-valued set just constructed, and similarly for more factors.) Let $f:(Z, \delta) \to (X, \delta)$ and $g:(Z, \delta) \to (Y, \delta)$ be morphisms; then the unique morphism

$$h = (f, g):(Z, \delta) \to (X \times Y, \delta)$$

such that $ph = f$ and $qh = g$ is given by

$$h(z, (x, y)) = f(z, x) \wedge g(z, y).$$

To see this, take any such h. Then $ph = f$ gives

$$(1) \quad \bigvee_{x',y'} h(z, (x', y')) \wedge \epsilon(y') \wedge \delta(x', x) = f(z, x)$$

for all z in Z and x' in X. Now

$$h(z, (x', y')) \wedge \epsilon(y') \wedge \delta(x', x)$$
$$= h(z, (x', y')) \wedge \delta((x', y'), (x, y')) \leqq h(z, (x, y')),$$

with equality for $x' = x$. Thus (1) becomes

$$\bigvee_{y'} h(z, (x, y')) = f(z, x),$$

and $qh = g$ leads similarly to

$$\bigvee_{x'} h(z, (x', y)) = g(z, y).$$

The meet of these two equations gives

$$(2) \quad \bigvee_{x',y'} h(z, (x, y')) \wedge h(z, (x', y)) = f(z, x) \wedge g(z, y).$$

Now

$$h(z, (x, y')) \wedge h(z, (x', y)) \leqq \delta((x, y'), (x', y))$$
$$\leqq \delta((x, y'), (x, y))$$

and hence

$$h(z, (x, y')) \wedge h(z, (x', y)) \leqq h(z, (x, y)),$$

in which equality occurs for $x' = x$, $y' = y$. Thus (2) becomes

$$h(z, (x, y)) = f(z, x) \wedge g(z, y)$$

and simple calculations show that this h is indeed a morphism satisfying $ph = f$, $qh = g$. Note that if $f = \overline{f_0}$ and $g = \overline{g_0}$ then

$$(f, g) = \overline{(f_0, g_0)}.$$

The $\mathscr{A}$-valued set $\hat{1}$ with set part $\{*\}$ and $\epsilon(*) = 1$ is easily seen to be terminal in $\mathscr{S}(\mathscr{A})$.

For each $\mathscr{A}$-valued set (X, δ), define $P(X, \delta)$ to be the set of all functions $\alpha:X \to \mathscr{A}$ such that

$$\alpha(x) \leqq \epsilon(x) \quad \text{and}$$

$$\alpha(x) \wedge \delta(x, y) \leqq \alpha(y) \quad \text{for all } x, y \text{ in } X.$$

For α in $P(X, \delta)$, define $\delta_\alpha : X \times X \to \mathscr{A}$ by

$$\delta_\alpha(x, y) = \delta(x, y) \wedge \alpha(x).$$

Then (X, δ_α) is an $\mathscr{A}$-valued set with $\epsilon_\alpha = \alpha$ and 1_X represents a morphism from (X, δ_α) to (X, δ).

(2.3) *A morphism* $f:(X, \delta) \to (Y, \delta)$ *is mono if and only if*

$$f(x, y) \wedge f(x', y) \leqq \delta(x, x') \quad \textit{for all } x, x' \textit{ in } X \textit{ and } y \textit{ in } Y.$$

Proof. Define ξ in $P(X \times X \times Y, \delta)$ by

$$\xi(x, x', y) = f(x, y) \wedge f(x', y)$$

and let $g, g':(X \times X \times Y, \delta_\xi) \to (X, \delta)$ be the morphisms represented by the first and second projections respectively; then $fg = fg'$. Thus if f is mono we have $g = g'$ and (2.1) (c) leads to the above inequality. Suppose that this inequality is given to hold and let $g, g':(Z, \delta) \to (X, \delta)$ be morphisms such that $fg = fg'$. Then for all z, x, y we have

$$\begin{aligned} g(z, x) \wedge f(x, y) &\leqq \bigvee_{x'} g'(z, x') \wedge f(x', y) \wedge f(x, y) \\ &\leqq \bigvee_{x'} g'(z, x') \wedge \delta(x, x') = g'(z, x). \end{aligned}$$

Taking the join of this over all y gives

$$g(z, x) \wedge \epsilon(x) \leqq g'(z, x),$$

that is, $g(z, x) \leqq g'(z, x)$. The reverse inequality is obtained similarly and hence $g = g'$ as required.

(2.4) *A morphism* $f:(X, \delta) \to (Y, \delta)$ *is epi if and only if*

$$\bigvee_x f(x, y) = \epsilon(y) \quad \textit{for all } y \textit{ in } Y.$$

Proof. Define β in $P(Y, \delta)$ by

$$\beta(y) = \bigvee_x f(x, y).$$

Let Y_0 and Y_1 be disjoint copies of Y and let $(Y_0 \cup Y_1, \delta^*)$ be the $\mathscr{A}$-valued set with

$$\delta^*(y_0, y_0') = \delta(y_0, y_0'),$$

$$\delta^*(y_1, y_1') = \delta(y_1, y_1'),$$

$$\delta^*(y_0, y_1) = \delta^*(y_1, y_0) = \delta_\beta(y_0, y_1)$$

where the subscripts show which copy of Y the y_i concerned comes from. Let

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$$g, g':(Y, \delta) \to (Y_0 \cup Y_1, \delta^*)$$

be the morphisms represented by the functions $Y \rightrightarrows Y_0$ and $Y \rightrightarrows Y_1$ respectively; then $gf = g'f$. Thus if f is epi we have $g = g'$ and hence $\epsilon(y) \leqq \delta_\beta(y, y)$ for all y in Y by (2.1) (c) so that $\epsilon(y) = \beta(y)$ as required. That this condition is sufficient for f to be epi is shown by a calculation very similar to the one used in the second half of the preceding proof.

(2.5) *If $f:(X, \delta) \to (Y, \delta)$ is mono and epi then it is iso.*

Proof. Define $g: Y \times X \to \mathscr{A}$ by $g(y, x) = f(x, y)$. Then (2.3) and (2.4) show that g is a morphism from (Y, δ) to (X, δ). Furthermore

$$(gf)(x, x') = \bigvee_y f(x, y) \wedge f(x', y) = \bigvee_y f(x, y) \wedge \delta(x, x')$$

$$= \epsilon(x) \wedge \delta(x, x') = \delta(x, x')$$

for all x, x' in X, so that $gf = 1_{(X,\delta)}$. Interchanging f and g, as we may, we obtain $fg = 1_{(Y,\delta)}$ too.

(2.6) *Let $f:(Z, \delta) \to (X, \delta)$ be a morphism and let α be in $P(X, \delta)$; define α_f in $P(X, \delta)$ by*

$$\alpha_f(x) = \bigvee_z f(z, x),$$

γ in $P(Z, \delta)$ by

$$\gamma(z) = \bigvee_x f(z, x) \wedge \alpha(x),$$

and g: $(Z, \delta_\gamma) \to (X, \delta_\alpha)$ by

$$g(z, x) = f(z, x) \wedge \alpha(x).$$

Then

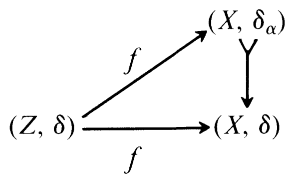
(a) *1_X represents a mono $(X, \delta_\alpha) \rightarrowtail (X, \delta)$;*

(b)

$$\begin{array}{ccc} (Z, \delta_\gamma) & \xrightarrow{\;g\;} & (X, \delta_\alpha) \\ \downarrow & & \downarrow \\ (Z, \delta) & \xrightarrow[\;f\;]{} & (X, \delta) \end{array}$$

is a pullback;

(c) *f factors through $(X, \delta_\alpha) \to (X, \delta)$ if and only if $\alpha_f \leqq \alpha$ and if this is so then the above diagram becomes*



(d) *taking* $\alpha = \alpha_f$ *in the last diagram gives the epi/mono factorization of* f.

The proof of this is straightforward.

COROLLARY. *The subobjects of* (X, δ) *are bijective with the elements of* $P(X, \delta)$.

Another way to obtain a subobject of (X, δ) is to take a subset Y of X, restrict δ to $Y \times Y$, and consider the mono $(Y, \delta) \rightarrowtail (X, \delta)$ represented by the inclusion of Y into X. It is not the case in general that every subobject of (X, δ) can be obtained in this way (it is if (X, δ) is a sheaf in the sense of Section 3) but what is useful is the fact that $(Y, \delta) \rightarrowtail (X, \delta)$ may be an isomorphism even when Y is a proper subset of X. If $(Y, \delta) \rightarrowtail (X, \delta)$ is an isomorphism then Y is said to be an *adequate domain* for (X, δ) ([**12**]; if in addition (Y, δ) is ample then Y is an ample domain for (X, δ)). It follows from (2.4) and (2.5) that Y is an adequate domain for (X, δ) if and only if

$$\bigvee_{y \in Y} \delta(x, y) = \epsilon(x) \quad \text{for all } x \text{ in } X.$$

(2.7) *In* (2.6), *let* f *be represented by* f_0. *Then*

(a) $\gamma(z) = \epsilon(z) \wedge \alpha(f_0(x))$ *for all* z *in* Z *and* g *is represented by* f_0;

(b) $f_0(Z)$ *is an adequate domain for the image* (X, δ_{α_f}) *of* f, *and if* f_0 *preserves* ϵ (*in particular if* $\epsilon(z) = 1$ *for all* z *in* Z) *then the epi/mono factorization of* f *may be taken to be*

$$(Z, \delta) \twoheadrightarrow (f_0(Z), \delta) \rightarrowtail (X, \delta)$$

where the epi part is represented by f_0;

(c) *if the epi part of* f *is split and* (Z, δ) *is ample then* $f_0(Z)$ *is an ample domain for* (X, δ_{α_f}).

Proof. (a) is clear. For $f_0(Z)$ to be an adequate domain for (X, δ_{α_f}) we need

$$\bigvee_z \delta_{\alpha_f}(f_0(z), x) = \alpha_f(x) \quad \text{for each } x \text{ in } X$$

and this is easily verified, as is the fact that if f_0 preserves ϵ then

$$\alpha_f(x) = \epsilon(x) \quad \text{for all } x \text{ in } f_0(Z).$$

To obtain (c), suppose that $g:(X, \delta_{\alpha_f}) \to (Z, \delta)$ is such that fg is the identity on (X, δ_{α_f}) and let h be any morphism to (X, δ_{α_f}); then $gh = \overline{k_0}$ for some k_0 since (Z, δ) is ample and therefore

$$h = fgh = \bar{f}_0\bar{k}_0 = \overline{f_0k_0}.$$

Define the *power object* $\mathscr{P}(X, \delta)$ of an $\mathscr{A}$-valued set (X, δ) to be $(P(X, \delta), \delta)$ where

$$\delta(\alpha, \beta) = \bigwedge_x \alpha(x) \leftrightarrow \beta(x) \quad \text{for all } \alpha, \beta \text{ in } P(X, \delta)$$

(notice that then $\epsilon(\alpha) = 1$ for all α in $P(X, \delta)$), define ϵ in $P((X, \delta) \times \mathscr{P}(X, \delta))$ by $\in (x, \alpha) = \alpha(x)$ and let

$$\mathscr{E}(X, \delta) \rightarrowtail (X, \delta) \times \mathscr{P}(X, \delta)$$

be the associated mono. The next result shows that $\mathscr{P}(X, \delta)$ is ample; it is essentially, but with some sharpening, what follows from the two theorems on pages I-35 and I-43 of Scott [**13**] (note that every complete d gives a definite v).

(2.8) *For all $\mathscr{A}$-valued sets (X, δ) and (Y, δ), every morphism*

$$h:(Y, \delta) \to \mathscr{P}(X, \delta)$$

is represented by a unique function h_0 such that

$$h_0(y)(x) \leqq \epsilon(y) \quad \textit{for all } x \textit{ in } X \textit{ and } y \textit{ in } Y.$$

Proof. Given h, define h_0 by

$$h_0(y)(x) = \bigvee_\beta h(y, \beta) \wedge \beta(x)$$

where β runs over $P(X, \delta)$; then

$$h_0(y)(x) \leqq \epsilon(y).$$

By (2.1)(a),

h_0 represents h

$\Leftrightarrow h(y, \alpha) \leqq \delta(h_0(y), \alpha)$ for all y in Y and α in $P(X, \delta)$

$\Leftrightarrow h(y, \alpha) \leqq (\bigvee_\beta h(y, \beta) \wedge \beta(x)) \leftrightarrow \alpha(x)$ for all x in X, y in Y, and α in $P(X, \delta)$

$$\Leftrightarrow \begin{cases} h(y, \alpha) \wedge \bigvee_\beta h(y, \beta) \wedge \beta(x) \leqq \alpha(x) \quad \text{and} \\ h(y, \alpha) \wedge \alpha(x) \leqq \bigvee_\beta h(y, \beta) \wedge \beta(x) \text{ for all such } x, y \text{ and } \alpha; \end{cases}$$

the first of these follows from

$$h(y, \alpha) \wedge h(y, \beta) \leqq \delta(\alpha, \beta)$$

and the second is obvious. The uniqueness of h_0 is an easy consequence of (2.1) (c).

(2.9) *$\mathscr{S}(\mathscr{A})$ is a topos.*

Proof. We already know from (2.2) and (2.6) that $\mathscr{S}(\mathscr{A})$ has finite limits. Therefore by [**8**] (see also [**6**, p. 43]), it is sufficient to show that for every mono

$$(X \times Y, \delta_\gamma) \rightarrowtail (X \times Y, \delta)$$

in $\mathscr{S}(\mathscr{A})$ there exists a unique $h:(Y, \delta) \rightarrow \mathscr{P}(X, \delta)$ such that for some g

$$\begin{array}{ccc} (X \times Y, \delta_\gamma) & \xrightarrow{\quad g \quad} & \mathscr{E}(X, \delta) \\ \big\downarrow & & \big\downarrow \\ (X \times Y, \delta) & \xrightarrow[1 \times h]{} & (X, \delta) \times \mathscr{P}(X, \delta) \end{array}$$

is a pullback, the *power object pullback.* By (2.8), any such h is represented by an h_0 satisfying

$$h_0(y)(x) \leqq \epsilon(y).$$

But then by (2.7) (a),

$$\gamma(x, y) = \epsilon(x) \wedge \in(y) \wedge \in (h_0(y), x) = h_0(y)(x) \quad \text{for all } x$$
and y

which shows both that h_0 exists and that it is unique, and thus the same is true for h.

One way to construct the object $(\widetilde{X, \delta})$ of 'partial elements' of an $\mathscr{A}$-valued set (X, δ) is as follows (see [**6**, p. 28]): (i) the mono

$$\Delta:(X, \delta) \rightarrowtail (X \times X, \delta)$$

is represented by $(1_X, 1_X)$; (ii) the corresponding element of $P(X \times X, \delta)$ is δ; (iii) the singleton morphism

$$\{\ \}:(X, \delta) \rightarrow \mathscr{P}(X, \delta)$$

which corresponds to Δ by the power object pullback is represented by d_0 where

$$d_0(x)(y) = \delta(x, y) \quad \text{for all } x, y \text{ in } X;$$

(iv) the mono

$$(1, \{\ \}):(X, \delta) \rightarrow (X, \delta) \times \mathscr{P}(X, \delta)$$

is represented by $(1_X, d_0)$; (v) the corresponding element of $P((X, \delta) \times \mathscr{P}(X, \delta))$ is θ where

$$\theta(x, \alpha) = \bigvee_y \delta(x, y) \wedge \delta(\alpha, d_0(y))$$
$$= \bigvee_y \delta(x, y) \wedge \bigwedge_z \alpha(z) \leftrightarrow \delta(y, z)$$

$$= \epsilon(x) \wedge \bigwedge_z \alpha(z) \leftrightarrow \delta(x, z);$$

(vi) the morphism h from $\mathscr{P}(X, \delta)$ to itself which corresponds to the mono $(1, \{\ \})$ by the power object pullback is represented by h_0 where

$$h_0(\alpha)(x) = \theta(x, \alpha) \text{ for all } \alpha \text{ in } P(X, \delta) \text{ and } x \text{ in } X;$$

(vii) $(\widetilde{X, \delta})$, the image of h, may be taken by (2.7) (b) to be $(h_0(P(X, \delta)), \delta)$ (note that this is ample by (2.7) (c) since $\mathscr{P}(X, \delta)$ is ample and h, being idempotent, has its epi part split ([**3**]; in fact h_0 is idempotent)); (viii) $h_0(P(X, \delta)) = S(X, \delta)$ where $S(X, \delta)$ consists of those α's in $P(X, \delta)$ such that

$$\alpha(x) \wedge \alpha(y) \leqq \delta(x, y) \quad \text{for all } x, y \text{ in } X$$

(it is easily verified that $h_0(\alpha)$ is in $S(X, \delta)$ for all α in $P(X, \delta)$ and that $h(\alpha) \leqq \alpha$ with equality if α is in $S(X, \delta)$); and finally (ix) $(\widetilde{X, \delta}) \cong (S(X, \delta), \delta)$. As noted, this representation of $(\widetilde{X, \delta})$ is ample and for it, the mono

$$(X, \delta) \rightarrowtail (\widetilde{X, \delta})$$

is represented by d_0.

3. $\mathscr{S}(\mathscr{A})$ and sheaves. We first describe how the category $\mathrm{Sh}(\mathscr{A})$ of (set-valued) sheaves on $\mathscr{A}$ may be regarded as a full subcategory of $\mathscr{S}(\mathscr{A})$. Since each $\mathscr{A}$-valued set is easily seen to be isomorphic to a sheaf on $\mathscr{A}$, $\mathscr{S}(\mathscr{A})$ is thus equivalent to $\mathrm{Sh}(\mathscr{A})$ (and in particular $\mathscr{S}(2)$ is equivalent to the category $\mathscr{S}$ of sets).

Recall that a sheaf on $\mathscr{A}$, where $\mathscr{A}$ is considered as a category carrying the canonical 'topology', is a functor $F:\mathscr{A}^{\mathrm{op}} \to \mathscr{S}$ with the property that if x_i in $F(a_i)$, i in I, are such that

$$x_i|_{a_i \wedge a_j} = x_j|_{a_i \wedge a_j} \quad \text{for all } i \text{ and } j$$

then there is a unique x in $F(\bigvee_i a_i)$ with $x|_{a_i} = x_i$ for all i ($x|_a$ denotes the element $F(k)(x)$ of $F(a)$ where x is in $F(b)$, $a \leqq b$, and k is the unique morphism in $\mathscr{A}$ from a to b). If F is a sheaf on $\mathscr{A}$, let $|F|$ = the union of the $F(a)$'s, which we assume to be disjoint, and for x, y in $|F|$ let $\delta_F(x, y)$ = the largest c in $\mathscr{A}$ for which $x|_c = y|_c$. Then $(|F|, \delta_F)$ is an $\mathscr{A}$-valued set, from which F is recoverable since

$$F(a) = \{x \in |F| : \epsilon_F(x) = a\},$$

and $x|_a$ = the unique y in $|F|$ such that

$$\delta_F(x, y) = \epsilon_F(y) = a.$$

We may thus identify sheaves on $\mathscr{A}$ with certain $\mathscr{A}$-valued sets and it is convenient to refer to the $\mathscr{A}$-valued sets which arise in this way as being themselves sheaves on $\mathscr{A}$: they will now be characterized.

Let (X, δ) be any $\mathscr{A}$-valued set. If x, y in X satisfy $\delta(x, y) = \epsilon(y)$ write $y \leqq x$ (this is a preorder on X) and if $\delta(x, y) = \epsilon(x) \wedge \epsilon(y)$ write $x \| y$ and say that x and y are *compatible*. It is easy to see that (X, δ) is a sheaf if and only if (i) $\leqq$ is a partial order, (ii) for each x in X and $a \leqq \epsilon(x)$ there exists $y \leqq x$ with $\epsilon(y) = a$, and (iii) every (pairwise) compatible family of elements in X has a join in $(X, \leqq)$. Suppose that (X, δ) is a sheaf; then the following further facts are also easily checked: the element y in (ii) is unique, being just $x|_a$ of course; $(X, \leqq)$ is a meet semilattice with smallest element 0; a join $\bigvee_i x_i$ exists in $(X, \leqq)$ if and only if the x_i's are compatible; this is the case for chains, so that $(X, \leqq)$ admits the application of Zorn's Lemma; for every α in $P(X, \delta)$, $y \leqq x$ implies

$$\alpha(y) \leqq \alpha(x), \ \alpha(x|_a) = \alpha(x) \wedge a,$$

$$\alpha(x \wedge y) = \delta_\alpha(x, y), \ \alpha(0) = 0, \quad \text{and}$$

$$\alpha(\bigvee_i x_i) = \bigvee_i \alpha(x_i) \quad \text{whenever } \bigvee_i x_i \text{ exists.}$$

Now a morphism in $\mathrm{Sh}(\mathscr{A})$ from a sheaf (X, δ) to a sheaf (Y, δ) is just a function $f_0 : X \to Y$ such that

$$\epsilon(f_0(x)) = \epsilon(x) \quad \text{and}$$

$$f_0(x|_a) = f_0(x)|_a$$

for all x in X and $a \leqq \epsilon(x)$ or, equivalently, such that

$$\delta(x, x') \leqq \delta(f_0(x), f_0(x'))$$

for all x, x' in X with equality if $x = x'$. Thus f_0 represents a morphism in $\mathscr{S}(\mathscr{A})$ and that we thereby obtain a bijection between morphisms from (X, δ) to (Y, δ) in $\mathrm{Sh}(\mathscr{A})$ and in $\mathscr{S}(\mathscr{A})$ follows from:

(3.1) *For any sheaf* (X, δ) *and* $\mathscr{A}$*-valued set* (Z, δ), *every morphism* $f:(Z, \delta) \to (X, \delta)$ *in* $\mathscr{S}(\mathscr{A})$ *is represented by a unique* ϵ*-preserving function* f_0.

Proof. Define $f_0 : Z \to X$ by

$$f_0(z) = \bigvee_x x|_{f(z,x)}$$

(the terms here are compatible; incidentally, $f_0(z)$ could also be defined as the unique x in X such that $f(z, x) = \epsilon(z) = \epsilon(x)$). Using the fact that $\delta(-, x)$ is in $P(X, \delta)$ we obtain

$$\delta(f_0(z), x) = \bigvee_{x'} \delta(x', x) \wedge f(z, x') = f(z, x)$$

for all z and x so that f_0 represents f; also

$$\epsilon(f_0(z)) = \bigvee_x f(z, x) = \epsilon(z) \quad \text{for all } z,$$

that is, f_0 preserves ϵ. Let f_0 be any function with these properties; then

$$\delta(f_0(z), x) = f(z, x) \quad \text{for all } z \text{ and } x$$

and therefore

$$f_0(z) = \bigvee_x x|_{\delta(f_0(z),x)} = \bigvee_x x|_{f(z,x)}$$

so that f_0 is unique.

That every $\mathscr{A}$-valued set (X, δ) is isomorphic to a sheaf may be seen by taking the element $\sigma = \alpha_{\overline{d}_0}$ of $P(S(X, \delta), \delta)$ corresponding to the mono

$$(X, \delta) \rightarrowtail (S(X, \delta), \delta)$$

represented by the function

$$d_0:X \rightarrow S(X, \delta)$$

(see the end of Section 2): then (X, δ) is isomorphic to $(S(X, \delta), \delta_\sigma)$ by the results of Section 2 and it is not difficult to verify that $(S(X, \delta), \delta_\sigma)$ is a sheaf (the following facts concerning $(S(X, \delta), \delta_\sigma)$ are useful:

$$\sigma(\alpha) = \bigvee_x \alpha(x), \ \delta_\sigma(\alpha, \beta) = \bigvee_x \alpha(x) \wedge \beta(x),$$

$\alpha \leqq \beta$ if and only if

$$\alpha(x) \leqq \beta(x) \text{ for all } x \text{ in } X,$$

and $\alpha||\beta$ if and only if

$$\alpha(x) \wedge \beta(y) \leqq \delta(x, y) \text{ for all } x, y \text{ in } X).$$

It may also be verified that (X, δ) is itself a sheaf if and only if d_0 is bijective, in which case (X, δ) and $(S(X, \delta), \delta_\sigma)$ are exact copies of each other.

(3.2) *Let (X, δ) be a sheaf. Then a subset Y of X is an ample domain for (X, δ) if and only if Y contains the set M of maximal elements of X.*

Proof. Suppose that Y is an ample domain for (X, δ). Then $(Y, \delta) \rightarrowtail (X, \delta)$ is an isomorphism and the isomorphism inverse to it is represented by some function $r_0:X \rightarrow Y$. By (2.1) (c) and (d), the fact that

$$(X, \delta) \xrightarrow{\overline{r_0}} (Y, \delta) \rightarrowtail (X, \delta) = 1_{(X, \delta)}$$

gives

$$\epsilon(x) \leqq \delta(r_0(x), x)$$

(that is, $x \leqq r_0(x)$) for all x in X and taking x to be in M shows that $M \subseteq Y$. Suppose conversely that $M \subseteq Y$ is given and let $r_0:X \rightarrow Y$ be such that $x \leqq r_0(x)$ for all x in X. Then

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$$\delta(x, y) \leqq \delta(r_0(x), r_0)(y)) \quad \text{and}$$

$$\delta(r_0(x), x) = \epsilon(x) \quad \text{for all } x, y \text{ in } X,$$

so that r_0 represents a right, and therefore two-sided, inverse to $(Y, \delta) \rightarrowtail (X, \delta)$. Thus Y is an adequate domain for (X, δ) and that it is an ample domain follows from the ampleness of (X, δ) by (2.7) (c).

COROLLARY. *An $\mathscr{A}$-valued set (X, δ) is ample if and only if $d_0(X)$ contains the set of maximal elements of the associated sheaf $(S(X, \delta), \delta_\sigma)$.*

As an aside, we mention briefly the étale approach to sheaves on $\mathscr{A}$. Let (X, δ) be an $\mathscr{A}$-valued set. Then the unique morphism $(X, \delta) \rightarrow \hat{1}$ gives rise to a morphism $\mathscr{P}(\hat{1}) \rightarrow \mathscr{P}(X, \delta)$ which by (2.8) is represented by a unique function

$$h_0 : P(\hat{1}) \rightarrow P(X, \delta);$$

$P(\hat{1})$ may be identified with $\mathscr{A}$ and it is easily checked that h_0 is then given by

$$h_0(a)(x) = a \wedge \epsilon(x).$$

$P(X, \delta)$ is a CHA under the pointwise order and $h_0 : \mathscr{A} \rightarrow P(X, \delta)$ is a $\wedge$, $\bigvee$-preserving function of a particular type which we may call étale (if $p : T \rightarrow S$ is a local homeomorphism of topological spaces then the inverse image function from the open set lattice of S to that of T is étale). Conversely, every étale function h_0 from $\mathscr{A}$ to a CHA $\mathscr{B}$ determines an $\mathscr{A}$-valued set and this way $\mathscr{S}(\mathscr{A})$ and therefore Sh($\mathscr{A}$) become equivalent to $\mathscr{E}^{\text{op}}/\mathscr{A}$ where $\mathscr{E}$ is the category of CHA's and étale functions ([**4**] in effect deals with the particular case in which $\mathscr{A}$ is boolean and only injective $\mathscr{A}$-valued sets are considered, such $\mathscr{A}$-valued sets corresponding to étale, there called analytic, embeddings $\mathscr{A} \rightarrow \mathscr{B}$). In the CHA analogue of the situation described in [**6**, p.10], the equivalence

$$\text{Sh}(\mathscr{A}) \cong \mathscr{E}^{\text{op}}/\mathscr{A}$$

arises by restriction from adjoint functors

$$\mathscr{S}^{\mathscr{A}^{\text{op}}} \leftrightarrows \mathscr{H}^{\text{op}}/\mathscr{A}$$

where $\mathscr{H}$ is the category of CHA's and all $\wedge$, $\bigvee$-preserving functions.

4. Injectives and injective hulls in $\mathscr{S}(\mathscr{A})$. Throughout this section, (X, δ) is a sheaf, M is the set of maximal elements of X, and Γ is the set of global elements of X, where x is *global* if and only if $\epsilon(x) = 1$; clearly $\Gamma \subseteq M$. (Some of the definitions and results go through for arbitrary $\mathscr{A}$-valued sets but they can then seem rather artificial; for example one can have $(X, \delta) \cong (Y, \delta)$ yet $(X, \tilde{\delta}) \ncong (Y, \tilde{\delta})$.)

Define $\tilde{\delta}:X \times X \to \mathscr{A}$ by

$$\tilde{\delta}(x, y) = (\epsilon(x) \vee \epsilon(y)) \to \delta(x, y)$$
$$= (\epsilon(x) \to \delta(x, y)) \wedge (\epsilon(y) \to \delta(x, y)).$$

It is easily verified that an equivalent definition is obtained by transfering the δ function in $(S(X, \delta), \delta)$ across to X via the bijection

$$d_0:X \to S(X, \delta)$$

and this shows that, like $(S(X, \delta), \delta)$, $(X, \tilde{\delta})$ is an ample $\mathscr{A}$-valued set isomorphic to $(\widetilde{X, \delta})$. Since $\tilde{A}$ is injective in any topos (see [**3**, Proposition 2.23]), it follows that $(X, \tilde{\delta})$ is injective.

(4.1) $(M, \tilde{\delta})$ *is ample and injective.*

Proof. We show that there exists a function $r_0:X \to M$ such that

(i) $x \leqq r_0(x)$ for all x in X, and

(ii) $\tilde{\delta}(x, y) \leqq \tilde{\delta}(r_0(x), r_0(y))$ for all x, y in X.

Suppose that we have shown the existence of such an r_0; then r_0 represents a morphism

$$\overline{r_0}:(X, \tilde{\delta}) \to (M, \tilde{\delta})$$

which, since $r_0(m) = m$ for all m in M, is left inverse to

$$(M, \tilde{\delta}) \rightarrowtail (X, \tilde{\delta});$$

the fact that $(X, \tilde{\delta})$ is ample and injective then implies the same for $(M, \tilde{\delta})$ (for the ampleness one uses (2.7) (c)).

To show that an r_0 with the required properties does exist, let $\mathscr{R}$ be the set of all functions $r_0:X \to X$ satisfying (i) and (ii). Then under the pointwise order, $\mathscr{R}$ satisfies the hypotheses of Zorn's Lemma. To see this let, $\{r_i\}_{i \in I}$ be a non-empty chain in $\mathscr{R}$ and let

$$r_0 = \bigvee_i r_i;$$

then r_0 certainly satisfies (i). Also, since each r_i satisfies (ii), we have

$$\epsilon(r_i(x)) \wedge \tilde{\delta}(x, y) \leqq \delta(r_i(x), r_i(y)) \quad \text{for all } i;$$

taking the join of this over i and using on the right-hand side the fact that the r_i's form a chain, we obtain

$$\epsilon(r_0(x)) \wedge \tilde{\delta}(x, y) \leqq \delta(r_0(x), r_0(y)),$$

and similarly with x and y interchanged, whence r_0 satisfies (ii). Also $\mathscr{R}$ is non-empty since 1_X is in $\mathscr{R}$. Let r_0 be a maximal member of $\mathscr{R}$. We want to show that $r_0(X) \subseteq M$. Let x_0 be in X. Then $r_0(x_0) \leqq$ some element of M,

$r_0(x_0) \leqq m_0$ say. For each x in X put

$$m_0(x) = m_0|_{\epsilon(m_0) \wedge \tilde{\delta}(x_0, x)}.$$

Then

$$\epsilon(r_0(x)) \wedge \epsilon(m_0(x)) \leqq \epsilon(m_0(x)) = \delta(m_0(x), m)$$

and

$$\epsilon(r_0(x)) \wedge \epsilon(m_0(x)) \leqq \epsilon(r_0(x)) \wedge \tilde{\delta}(r_0(x_0), r_0(x))$$

$$= \delta(r_0(x_0), r_0(x)) \leqq \epsilon(m_0(x)) = \delta(m_0(x), m_0);$$

hence

$$\epsilon(r_0(x)) \wedge \epsilon(m_0(x)) \leqq \delta(r_0(x), m_0(x)),$$

that is, $r_0(x)$ and $m_0(x)$ are compatible. Let

$$r_1(x) = r_0(x) \vee m_0(x) \quad \text{for all } x;$$

we claim that r_1 is in $\mathscr{R}$. Clearly r_1 satisfies (i). Now for each x, y in X,

$$\epsilon(m_0(x)) \wedge \tilde{\delta}(x, y) = \epsilon(m_0(y)) \wedge \tilde{\delta}(x, y) = a$$

say, and

$$m_0(x)|_a = m_0(y)|_a$$

(each $= m_0|_a$). Hence

$$a \leqq \delta(m_0(x), m_0(y))$$

and this shows that m_0 satisfies (ii). Using the fact that r_0 satisfies (ii), together with the inequality

$$\tilde{\delta}(x, y) \wedge \tilde{\delta}(x', y') \leqq \tilde{\delta}(x \vee x', y \vee y'),$$

we see that r_1 satisfies (ii). Since $r_1 \geqq r_0$, the maximality of r_0 gives $r_1 = r_0$, so that $m_0(x) \leqq r_0(x)$ for each x in X. Taking $x = x_0$ gives

$$m_0 = m_0(x_0) \leqq r_0(x_0)$$

and thus $r_0(x_0) = m_0 \in M$ as desired.

(4.2) *The following are equivalent*:
(i) (X, δ) *is injective*,
(ii) $M = \Gamma$,
(iii) Γ *is an ample domain for* (X, δ).

Proof. The equivalence of (ii) and (iii) follows from (3.2). Suppose that (i) holds; then $(X, \delta) \rightarrowtail (X, \tilde{\delta})$ has a left inverse, represented by r_0 say, and $r_0(X)$ is an ample domain for (X, δ) by (2.7) (c); but $\Gamma \supseteq r_0(X)$ $(\epsilon(r_0(x)) \geqq \tilde{\epsilon}(x) = 1$ for all x), whence (iii). That (ii) implies (i) follows from (3.2), (4.1), and the fact that $\tilde{\delta}(m, n) = \delta(m, n)$ for m, n in Γ.

In order to describe injective hulls we need the fact that

$$\widetilde{\delta}(m, n) = (\epsilon(m) \wedge \epsilon(n)) \rightarrow \delta(m, n) \quad \text{for } m, n \text{ in } M$$

and the following terminology seems to be useful here: if m and x are elements of X satisfying the condition: for all $y \leqq x$, $y||m$ implies $y \leqq m$, say that m is *maximal with respect to* x.

(4.3) *The following are equivalent*:
(i) *m is maximal with respect to x*,
(ii) $\widetilde{\delta}(m, n) = \epsilon(m) \rightarrow \delta(m, x)$,
(iii) $(\epsilon(m) \wedge \epsilon(x)) \rightarrow \delta(m, x) = \epsilon(x) \rightarrow \delta(m, x)$.

Proof. We have

$$\begin{aligned}\text{(ii)} &\Leftrightarrow \epsilon(m) \rightarrow \delta(m, x) \leqq \epsilon(x) \rightarrow \delta(m, x)\\ &\Leftrightarrow \epsilon(x) \wedge (\epsilon(m) \rightarrow \delta(m, x)) \leqq \delta(m, x)\\ &\Leftrightarrow \epsilon(x) \wedge ((\epsilon(m) \wedge \epsilon(x) \rightarrow \delta(m, x)) \leqq \delta(m, x) \Leftrightarrow \text{(iii)}.\end{aligned}$$

To prove that (i) implies (ii), let

$$a = \epsilon(x) \wedge (\epsilon(m) \rightarrow \delta(m, x))$$

and let $y = x|_a$. Then

$$\epsilon(m) \wedge a = \epsilon(m) \wedge \epsilon(x) \wedge \delta(m, x) = \delta(m, x)$$

and hence

$$\delta(m, y) = \delta(m, x) \wedge a = \epsilon(m) \wedge a = \epsilon(m) \wedge \epsilon(y),$$

that is $y||m$. If (i) holds, this gives $y \leqq m$ so that

$$a = \epsilon(y) \leqq \epsilon(m) \quad \text{and} \quad \delta(m, x) = \epsilon(m) \wedge a = a,$$

which implies (ii) by the equivalence at the beginning of this proof. Suppose that (ii) holds and let $y \leqq x$, $y||m$. Then

$$\delta(m, y) = \epsilon(m) \wedge \epsilon(y)$$

and hence

$$\begin{aligned}\epsilon(y) &\leqq \epsilon(x) \wedge (\epsilon(m) \rightarrow \delta(m, y)) \leqq \epsilon(x) \wedge (\epsilon(m) \rightarrow \delta(m, x))\\ &\leqq \delta(m, x) \leqq \epsilon(m)\end{aligned}$$

so that $\delta(m, y) = \epsilon(y)$, and $y \leqq m$ as required for (i).

(4.4) *An element m of X is in M if and only if it is maximal with respect to each x in X.*

Proof. It is enough to note that m is in M if and only if for all y in X, $y||m$ implies $y \leqq m$ (for the necessity of this condition consider $m \vee y$, and for the sufficiency take y to be in M and $\geqq m$).

As an immediate consequence of (4.3) and (4.4) we have

(4.5) *If m and n are in M then*

$$\tilde{\delta}(m, n) = (\epsilon(m) \wedge \epsilon(n)) \rightarrow \delta(m, n).$$

(4.6) *The injective hull of* (X, δ) *is given by* $(M, \tilde{\delta})$.

Proof. Since $(X, \delta) \cong (M, \delta)$ by (3.2) and $(M, \tilde{\delta})$ is injective by (4.1), it is enough to prove that the mono

$$u:(M, \delta) \rightarrowtail (M, \tilde{\delta})$$

represented by 1_M is essential. So let

$$f:(M, \tilde{\delta}) \rightarrow (Y, \delta);$$

we require that fu mono implies f mono. Now

$$(fu)(m, y) = \epsilon(m) \wedge f(m, y)$$

for all m in M and y in Y by (2.1) (d); thus we require that

$$\epsilon(m) \wedge f(m, y) \wedge \epsilon(n) \wedge f(n, y) \leqq \delta(m, n)$$

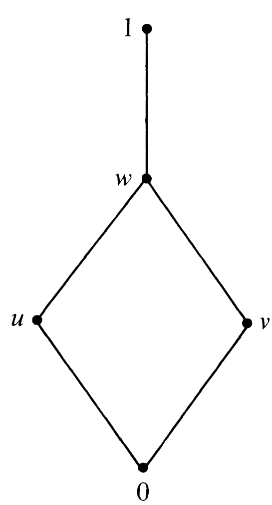
for all m, n in M and y in Y implies

$$f(m, y) \wedge f(n, y) \leqq \tilde{\delta}(m, n)$$

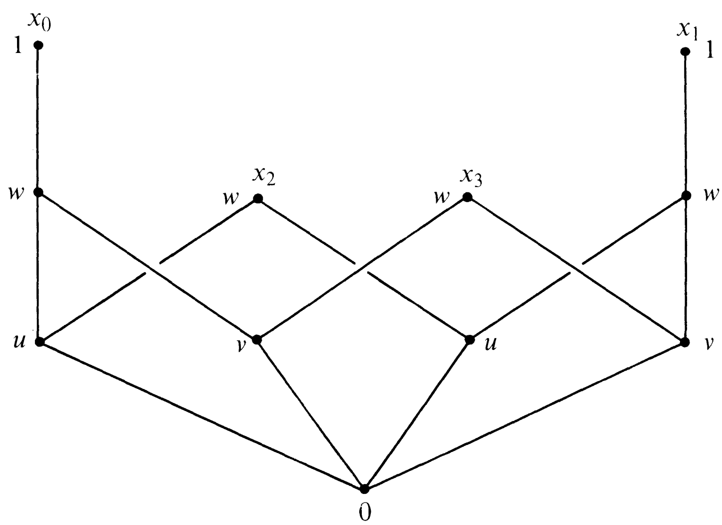
for all such m, n and y. But this is immediate from (4.5).

(Note. The theorem on page I-40 of [**13**] implicitly involves the construction of injective hulls, in the boolean case.)

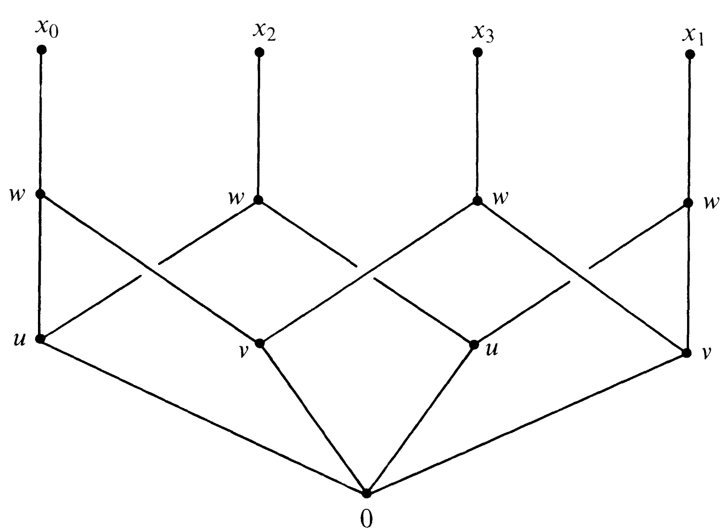
We conclude with a simple example of an injective hull. Let $\mathscr{A}$ be



and consider the $\mathscr{A}$-valued set ($\{x_0, x_1\}$, δ) where δ is the usual Kronecker δ (it is easy to see that ($\{x_0, x_1\}$, δ) $\cong$ 1 + 1). The associated sheaf (X, δ) is



where the values of ϵ are as shown. The two maximal elements x_2 and x_3 are not global and so (X, δ) is not injective (since $\Gamma = \{x_0, x_1\}$ is an adequate domain for (X, δ), this shows that we cannot replace "ample" by "adequate" in (4.2) (iii)). The injective hull $(M, \tilde{\delta})$ of (X, δ), or rather the whole sheaf associated with it, is



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