Canad. Math. Bull. Vol. 15 (1), 1972

BOUNDS FOR CHARACTERISTIC VALUES OF POSITIVE DEFINITE MATRICES

BY

P. A. BINDING, W. D. HOSKINS AND P. J. PONZO

Introduction. We consider the problem of determining the best possible bounds on the eigenvalues of an nth order positive definite matrix B, when the determinant (D) and trace (T) are given. A large variety of bounds on the eigenvalues are known when different information concerning B is available (see, for example, [1], [2]). Since D and T simply provide the geometric mean and arithmetic mean of the positive, real eigenvalues of B, the solution to the problem involves certain inequalities satisfied by these means (see [3] for such inequalities in a more general setting). A related problem in which the largest and smallest eigenvalue are known, and inequalities involving D and T are obtained, is described in [4].

Eigenvalue Bounds. We suppose that B is a positive definite matrix whose order (n), determinant (D) and trace (T) are known. We propose finding the least upper bound and greatest lower bound on the eigenvalues of B, in terms of n, D, T assuming $D^{1/n} \neq T/n$ (for otherwise all these eigenvalues are equal to $D^{1/n}=T/n$).

Let $0<x\leq\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_{n-2}\leq y$ be the eigenvalues of B. Then

$$x+\sum_{1}^{n-2}\lambda_i+y = T \tag{1}$$

and

$$x\left(\prod_{1}^{n-2}\lambda_i\right)y = D. \tag{2}$$

Substituting the extreme values of the λ_i into (1) and (2) yields

$$(n-1)x+y \leq T \leq x+(n-1)y. \tag{3}$$

$$x^{n-1}y \leq D \leq xy^{n-1}. \tag{4}$$

Shading in Fig. 1 illustrates the region in the x, y plane defined by (3) and (4).

Some points within the shaded region are not admissible. For example, the point S corresponds to $\sum_1^{n-2}\lambda_i=(n-2)y$ (since S lies on $x+(n-1)y=T$) and, simultaneously, $\prod_1^{n-2}\lambda_i=x^{n-2}$ (since S lies on $x^{n-1}y=D$). This requires all λ_i to equal x and y, simultaneously, which is impossible (except for the trivial case when $x=\lambda_1=\lambda_2=\cdots=\lambda_{n-2}=y$, which can be easily recognized by the fact that $D^{1/n}=T/n$). Hence S is not an admissible point in the x, y space of minimum and maximum eigenvalues.

We proceed to obtain the admissible set (x, y).

Received by the editors December 3, 1970 and, in revised form, April 22, 1971.

https://doi.org/10.4153/CMB-1972-011-6 Published online by Cambridge University Press

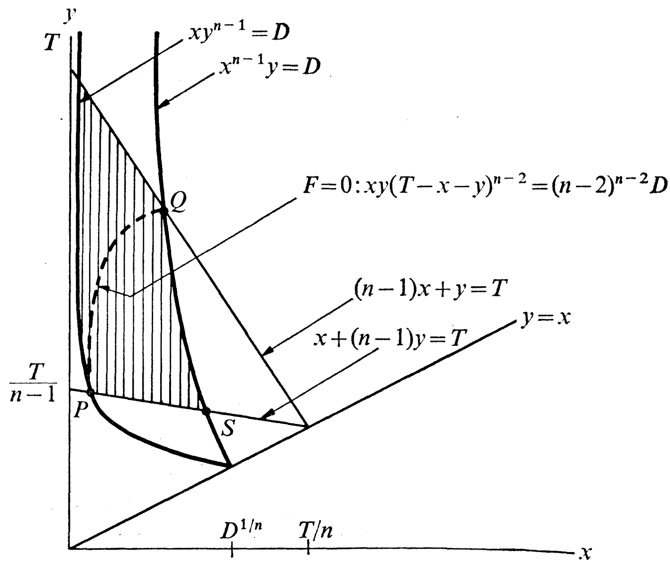

FIGURE 1.

Note that

$$(T-x-y)/(n-2) = \text{arithmetic mean of the } \lambda_i = A,$$

$$(D/xy)^{1/n-2} = \text{geometric mean of the } \lambda_i = G,$$

and since $A \geq G$ for positive λ_i, then $(T-x-y)/(n-2) \geq (D/xy)^{1/n-2}$ provides an additional restriction on (x, y). The contour

$$F(x, y) = xy(T-x-y)^{n-2} - D(n-2)^{n-2} = 0$$

is easily shown to pass through P and Q, the points of intersection of the bounding contours described by (3) and (4). Further

$$(5) \qquad \frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\left(\frac{y}{x}\right) \frac{T-[(n-1)x+y]}{T-[(x+(n-1)y]} = \begin{cases} \infty \text{ at } P \\ 0 \text{ at } Q \end{cases}.$$

The curve $F=0$ is shown dashed in Fig. 1.

Note that x_p, the abscissa at P, and y_Q, the ordinate at Q, are respectively the smaller and larger of the two positive real roots of

$$(6) \qquad \lambda = T-(n-1)(D/\lambda)^{1/(n-1)}$$

which may also be written

$$(7) \qquad \lambda = D[(n-1)/(T-\lambda)]^{n-1}.$$

We summarize the above.

THEOREM 1. *The smaller and larger of the two positive real roots of eq.* (6) (*or eq.* (7)) *provide, respectively, the greatest lower bound and least upper bound of the eigenvalues of the positive definite $n \times n$ matrix B, where $D=\det(B)$ and $T=\operatorname{trace}(B)$.*

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Proof. Since x_P and y_Q are respectively lower and upper bounds on the eigenvalues of B we need only show that each is attained for certain positive definite matrices. It is easy to see that, for any positive definite matrix with only two eigenvalues x and y, the smaller having multiplicity $(n-1)$, then $(n-1)x+y=T$ and $x^{n-1}y=D$. The solution (x, y) gives the coordinates of Q. Similarly, P is attained by any positive definite matrix with only two eigenvalues whose larger eigenvalue has multiplicity $(n-1)$.

The following lemmata are easily established.

LEMMA 1. *Each member of the sequence* $\{x_m\}_0^\infty$, *where* $x_0=0$ *and* $x_{m+1}=D[(n-1)/(T-x_m)]^{n-1}$, *provides a lower bound on the eigenvalues of* B, *and* $\lim x_m=x_p$.

Proof. The sequence is monotone increasing and has x_p, the smaller root of eq. (7), as a limiting value. Since x_p is the greatest lower bound then each member of the sequence is a lower bound.

LEMMA 2. *Each member of the sequence* $\{y_m\}_0^\infty$, *where* $y_0=T$ *and* $y_{m+1}=T-(n-1)(D/y_m)^{1/(n-1)}$, *provides an upper bound on the eigenvalues of* B, *and* $\lim y_m=y_Q$.

Proof. The sequence decreases monotonically to y_Q, the larger root of eq. (6), and y_Q is the least upper bound.

LEMMA 3. *The maximum separation of eigenvalues of* B (*i.e.* $\max(y-x)$) *is attained for that point on the contour* $F=0$ *where* $dy/dx=1$ (*from eq.* (5)).

LEMMA 4. *The maximum ratio of eigenvalues of* B (*i.e.* $\max(y/x)$) *is attained on* $F=0$ *at the point where* $dy/dx=y/x$ (*obtained from eq.* (5)).

LEMMA 5. *If* D *and* T *are the determinant and trace of* B^2, *and* B *is an* $n\times n$ *Hermitian matrix, not necessarily positive definite, but with nonvanishing determinant, then Theorem* 1 *provides "best" bounds on the squares of the eigenvalues of* B.

THEOREM 2. *In the case when* B *is just an arbitrary matrix, but still has a given nonvanishing determinant, then if instead of the trace of* B, *the trace of* B^*B *is available, then* x, y *provide, respectively, least upper bounds and greatest lower bounds for the squares of the moduli of the eigenvalues of* B.

The proof follows by considering the inequalities,

$$\text{spectral radius } (B) \le \sqrt{\text{spectral radius } (B^*B)}$$

and

$$\text{spectral radius } (B^{-1}) \le \sqrt{\text{spectral radius } (B^{-1*}B^{-1})}$$

in conjunction with the bounds for x, y previously established.

To determine the minimum values of $(y-x)$, or of (y/x), requires establishing the complete set of admissible points (x, y). It was noted above that the point S, for example, is not admissible.

It is shown in the appendix that if (as above) A and G refer to the means of the λ_i, then the problem of minimizing G (given A) results in $n-2$ configurations, depending upon the value of A.

Since

$$mx+(n-2-m)y \le (n-2)A \le (m-1)x+(n-1-m)y$$

for some $m=1, 2, \ldots, n-2$, then

$$G^{n-2} \ge x^{m-1}y^{n-2-m}[(n-2)A-(m-1)x-(n-2-m)y].$$

Putting $(n-2)A=T-x-y$ and $G^{n-2}=D/xy$ yields $n-2$ constraints on (x, y):

$$F_m(x, y) = x^m y^{n-1-m}[T-mx-(n-1-m)y] \le D$$

when

$$(m+1)x+(n-1-m)y \le T \le mx+(n-m)y \quad (m = 1, 2, \ldots, n-2).$$

The bounding contours $F_m=D$ satisfy

$$\begin{aligned} dy/dx &= -n/(n-1-m) \quad (y/x)[T-(m+1)x-(n-1-m)y]/[T-mx-(n-m)y] \\ &= \infty \text{ on } T = mx+(n-m)y \\ &= 0 \text{ on } T = (m+1)x+(n-1-m)y. \end{aligned}$$

These contours are shown in Fig. 2.

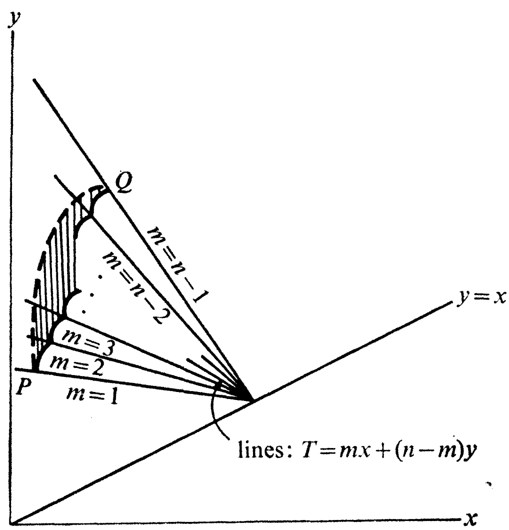

FIGURE 2.

The boundary of the shaded region in Fig. 2 may be described as follows:

If we set $\lambda_1=\lambda_2=\cdots=\lambda_{n-2}=\lambda$ and let λ vary from x to y, then eqs. (1) and (2) provide a parametric description of a point (x, y) which moves from Q to P, along $F=0$. At P, all $(n-2)$ "middle" eigenvalues are equal to y. If λ_1 is now moved from y to x, eqs. (1) and (2) yield a point (x, y) which moves along the lowest contour $F_1=D$ in Fig. 2. If, successively, $\lambda_2, \lambda_3, \ldots, \lambda_{n-2}$ are moved from y to x, the solution (x, y) of eqs. (1) and (2) moves, in turn, along each of $F_2=D$, $F_3=D, \ldots, F_{n-2}=D$ until, with all middle eigenvalues returned to the value x, Q is again reached. Clearly every point on the boundary of the admissible region is attained for positive definite matrices whose smallest and largest eigenvalues have the multiplicities associated with the corresponding branch of the boundary.

EXAMPLE. $B=\{b_{ij}\}$ and is of order $n\times n$, with

$$\begin{aligned} b_{ij} &= 4, \qquad i=j, \\ &= 1, \quad |i-j| = 1, \\ &= 0, \quad |i-j| \neq 1, i \neq j. \end{aligned}$$

Thus $2<\lambda_k<6$,

$$T = 4n$$

and

$$D = \frac{1}{2\sqrt{3}}\{(2+\sqrt{3})^{n+1}-(2-\sqrt{3})^{n+1}\}.$$

Particular values for D and T are:

n	D	T
2	15	8
3	56	12
4	209	16
5	780	20
6	2911	24

Theorem 1 gives the following bounds,

Order n	$\max_i \lambda_i$	Upper bound	$\min_i \lambda_i$	Lower bound
2	5	5	3	3
3	5.4142	5.7685	2.5886	2.4625
4	5.6180	6.4218	2.3820	2.1022
5	5.7321	7.0075	2.2680	1.8333
6	5.8019	7.5485	2.1981	1.6205

Note that if the diagonal elements of B are known (as well as D and T) then the minimum and maximum eigenvalues satisfy

$$x_P \leq x \leq \min_i b_{ii} \quad \text{and} \quad \max_i b_{ii} \leq y \leq y_Q,$$

so that one has an indication of how good the bounds are, as given by Theorem 1.

https://doi.org/10.4153/CMB-1972-011-6 Published online by Cambridge University Press

APPENDIX. In what follows we deal with the product and sum of the middle eigenvalues, rather than the geometric and arithmetic means. Also, for convenience, we let $N=n-2$.

We wish to minimize $\lambda_1\lambda_2\ldots\lambda_N=P$, where the λ_i are numbers lying in $0<x\leq\lambda_i\leq y$, under the condition that $\lambda_1+\lambda_2+\cdots+\lambda_N=S$ is fixed. We associate with each λ_i a unit mass located on the real line at the place λ_i. The prescribed sum S fixes the centre of mass of the N point masses, hence the position of a fulcrum which "balances" the mass distribution.

For any initial distribution, lying between x and y, the balance will be maintained if any two masses are moved apart by an equal amount. However, the product of the λ_i will be reduced (since $(\lambda_1-d)(\lambda_2+d)<\lambda_1\lambda_2$ for $\lambda_2<\lambda_1$ and $d>0$). We then proceed to separate pairs of mass points, continuing the separation of each until one of the pair reaches either x or y. This set of operations maintains the balance (hence the sum S), but reduces continuously the product P, and terminates when all point masses (with perhaps the exception of one mass point) are located at either x or y. The final sum is given by $S=mx+(N-1-m)y+\lambda$ where m mass points are eventually located at x, $N-1-m$ at y and a single mass point at λ where $x\leq\lambda\leq y$. Thus, of for some $m=0,1,2,\ldots,N-1$, the prescribed sum S satisfies $(m+1)x+(N-1-m)y\leq S\leq mx+(N-m)y$, the product P is minimized when m of the λ_i are at x, $N-1-m$ are at y and one is at $S-mx-(N-1-m)y$. This gives $P\geq x^m y^{N-1-m}[S-mx-(N-1-m)y]$.

If $G=(\lambda_1\lambda_2\cdots\lambda_N)^{1/N}$ is the geometric mean, and $A=(\lambda_1+\lambda_2+\cdots+\lambda_N)/N$ the arithmetic mean of the λ_i, with $0<x\leq\lambda_i\leq y$, then the above result may be reworded as follows:

If, for some $m=0,1,2,\ldots,N-1$ we have $[(m+1)x+(N-1-m)y]/N\leq A\leq[mx+(N-m)y]/N$ then

$$G^N \geq x^m y^{N-1-m}[NA-mx-(N-1-m)y]$$

REFERENCES

1. Bodewig, *Matrix calculus*, North Holland, Amsterdam (1965), 69–70.

2. Shi Zhong-Ci and Wang Bo-Ying, *Bounds for the determinant, characteristic roots and condition number of certain types of matrices*, Chinese Math. **7** (1965), 21–40.

3. Cargo and Shisha, *Bounds on ratios of means*, J. Res. Nat. Bur. Standards Vol. **66B** (1962), 169–170.

4. Mond and Shisha, *Inequalities*, Vol. **II**, Academic Press, New York (1970), 241–249.

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