Canad. Math. Bull. Vol. **18** (2), 1975

DECOMPOSITION OF PROJECTIONS ON ORTHOMODULAR LATTICES[1]

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1. **Introduction.** The set of projections in the BAER*-semigroup of hemimorphisms on an orthomodular lattice L can be partially ordered such that the subset of closed projections becomes an orthocomplemented lattice isomorphic to the underlying lattice L. The set of closed projections is identical with the set of Sasaki-projections on L (Foulis [2]). Another interesting class of (in general non-closed) projections, first investigated by Janowitz [4], are the symmetric closure operators. They map onto orthomodular sublattices where Sasaki-projections map onto segments of the lattice L.

In this paper we consider products of Sasaki-projections with symmetric closure operators. A necessary and sufficient condition is given for such a product to be a projection on L. Then we prove that every projection Ψ on L can be represented as the product of a Sasaki-projection with a symmetric closure operator. This decomposition of Ψ is not unique. However, the Sasaki-projection is uniquely determined by Ψ and among the symmetric closure operators decomposing Ψ there is a smallest one.

2. **Preliminaries.** An *orthomodular lattice* L is an orthocomplemented lattice which satisfies the condition $x \leq y\ (x, y \in L) \Rightarrow x \vee (x' \wedge y) = y$. A sublattice M of L which is closed under the orthocomplementation of L is itself an orthomodular lattice; we say $-M$ is an *orthomodular sublattice* of L. A segment $[x; y]$ is a sublattice of L and becomes an orthomodular lattice by means of the mapping $z \in [x; y] \rightarrow z^{\#} := (x \vee z') \wedge y \in [x; y]$ as orthocomplementation (this orthocomplementation is meant if we consider a segment as an orthomodular lattice). For basic results concerning orthomodular lattices see [1, p. 52; 3].

A mapping $\Xi : L \rightarrow L$ is a *hemimorphism* provided (i) $\Xi o = o$ and $\Xi(x \vee y) = \Xi x \vee \Xi y$, (ii) there exists another mapping Ξ^* with $\Xi^* o = o$ and $\Xi^*(x \vee y) = \Xi^* x \vee \Xi^* y$ such that $\Xi(\Xi^* x)' \leq x'$ and $\Xi^*(\Xi x)' \leq x'$. Clearly Ξ^* is a hemimorphism too and is called *adjoint* hemimorphism of Ξ. A given hemimorphism has exactly one adjoint hemimorphism. The set of hemimorphisms of L is an involution semigroup (with zero) with "function composition" as multiplication and the

Received by the editors September 20, 1972 and, in revised form, June 15, 1973.

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https://doi.org/10.4153/CMB-1975-050-0 Published online by Cambridge University Press

mapping $\Xi \to \Xi^*$ as involution. A hemimorphism Ψ for which $\Psi^* = \Psi \circ \Psi = \Psi$ is valid, is called *projection* on L. The set of projections on L is a poset by means of the ordering relation $\Psi_1 \le \Psi_2 :\Leftrightarrow \Psi_1 \circ \Psi_2 = \Psi_1$. The mapping $z \in L \to \phi_a z := (a' \vee z) \wedge a \in L$ is a projection on L. Mappings such as ϕ_a $(a \in L)$ are usually called Sasaki-*projections* [6; 5]. Further properties of Sasaki-projections are proved in [3]. Let Ξ be a hemimorphism, then the mapping $\Xi \to \Xi' := \phi_{(\Xi^* 1)'}$ makes the involution semigroup of hemimorphisms into a BAER*-semigroup [2].

A subset A of a lattice L is called *weakly meet-complete* (*weakly join-complete*) whenever $E_0(A; z) := \{x \mid z \le x;\ x \in A\}$ $(E^0(A; z) := \{x \mid x \le z;\ x \in A\})$ has a smallest (largest) element for every $z \in L$. Notice that a weakly meet- (join-) complete subset contains the largest (1) (smallest (o)) element if it exists in L. Furthermore $\bigwedge E_0(A; z)$ $(\bigvee E^0(A; z))$ exists in L for all $z \in L$, whenever A is a weakly meet- (join-) complete subset of L. A weakly meet-complete subset of an orthocomplemented lattice which is closed under orthocomplementation is also weakly join-complete and vice versa.

A *closure operator* Γ on a lattice L is a mapping $\Gamma: L \to L$ such that (i) $z \le \Gamma z$, (ii) $z_1 \le z_2 \Rightarrow \Gamma z_1 \le \Gamma z_2$ and (iii) $\Gamma(\Gamma z) = \Gamma z$. The range ΓL of a closure operator Γ is a weakly meet-complete subset of L and $\Gamma z = \bigwedge E_0(\Gamma L; z)$. This implies that a closure operator is uniquely determined by its range. Every weakly meet-complete subset A of a lattice is the range of a closure operator, namely $\Gamma\{A\}z := \bigwedge E_0(A; z)$. A closure operator Γ on an orthomodular lattice L is called *symmetric*, whenever $\Gamma z = z$ implies $\Gamma z' = z'$ [4].

3. **The decomposition theorem.** In the following, L denotes always an orthomodular lattice.

THEOREM 1. *A mapping $\Psi: L \to L$ is a projection if and only if Ψ satisfies the following conditions:*

(i) $z_1 \le z_2 \Rightarrow \Psi z_1 \le \Psi z_2$,

(ii) $\Psi(\Psi z) = z$ *and*

(iii) $\Psi(\Psi z)' \le z'$ $(z, z_1, z_2 \in L)$.

Proof. The crucial point in the proof is to show that a mapping satisfying (i), (ii) and (iii) preserves joins of elements of L [4].

LEMMA 2. *Let Γ be a closure operator on L. Γ is a symmetric closure operator if and only if it is a projection.*

Proof. Suppose Γ is a symmetric closure operator. Since $\Gamma(\Gamma z) = \Gamma z$ and $z \le \Gamma z$, it follows that $\Gamma(\Gamma z)' = (\Gamma z)' \le z'$. Conversely, suppose that Γ is a projection. If z is an element such that $\Gamma z = z$, we get $z' \le \Gamma z' = \Gamma(\Gamma z)' \le z'$. Hence $\Gamma z' = z'$. QED

THEOREM 3. *A subset $A \subseteq L$ is the range of a symmetric closure operator if and only if A is a weakly meet-complete, orthomodular sublattice of L.*

https://doi.org/10.4153/CMB-1975-050-0 Published online by Cambridge University Press

Proof. Let A be the range of a symmetric closure operator Γ. From the definition it follows immediately that A is a weakly meet-complete subset of L closed under orthocomplementation. By Lemma 2 Γ is also a projection, hence $\Gamma L=A$ is also closed under the join operation. This proves that A is an orthomodular sublattice of L. Conversely, suppose that A is a weakly meet-complete orthomodular sublattice of L. Clearly, A is the range of a closure operator Γ. If $\Gamma z=z$, then $z\in A$; but being an orthomodular sublattice, it follows that $z'\in A$ and thus $\Gamma z'=z'$. QED

LEMMA 4. *A projection Ψ is a symmetric closure operator on an orthomodular lattice L, provided $\Psi 1=1$.*

Proof. Let Ψ be a projection with $\Psi 1=1$. From $\Psi(\Psi z)'=\Psi(\Psi(\Psi z))'\leq(\Psi z)'$ we get by orthomodularity of L

$$\Psi(\Psi z)' = (\Psi z)' \wedge (\Psi z \vee \Psi(\Psi z)').$$

But

$$\Psi z \vee \Psi(\Psi z)' = \Psi(\Psi z \vee (\Psi z)') = \Psi 1 = 1,$$

thus $\Psi(\Psi z)'=(\Psi z)'$. Since $\Psi(\Psi z)'\leq z'$, it follows that $(\Psi z)'\leq z'$ and finally $z\leq\Psi z$. This result together with theorem 1 shows that Ψ is a closure operator; hence by lemma 2 a symmetric closure operator. QED

LEMMA 5. *The restriction of a projection Ψ on L to the segment $[o;\Psi 1]$ makes Ψ into a symmetric closure operator on the orthomodular lattice $[o;\Psi 1]$.*

Proof. By monotony of the projection Ψ, we get $\Psi z\leq\Psi 1$ for all $z\in L$. Consequently, the restriction of Ψ to $[o;1_\#]$ $(1_\#:=\Psi 1)$, denoted by $\Psi_\#$, maps this segment into itself. Clearly, $\Psi_\#$ is monotone and idempotent; furthermore $\Psi_\#(\Psi_\# z)^\#=\Psi[(\Psi z)'\wedge\Psi 1]\leq\Psi(\Psi z)'\wedge\Psi 1\leq z'\wedge\Psi 1=z^\#$ for $z\in[o;1_\#]$. Thus, by theorem 1, $\Psi_\#$ is a projection on $[o;1_\#]$. But we also have $\Psi_\# 1_\#=\Psi(\Psi 1)=\Psi 1=1_\#$, hence, by lemma 4, $\Psi_\#$ is a symmetric closure operator on the orthomodular lattice $[o;1_\#]$. QED

THEOREM 6. *Let Ψ be a projection on L. Then $\Psi L\cup(\Psi L)'$ (where $(\Psi L)'=\{z\mid z'\in\Psi L\}$) is the range of a symmetric closure operator.*

Proof. By theorem 3, we have to show that $\Psi L\cup(\Psi L)'$ is a weakly meet-complete, orthomodular sublattice of L. Since $\Psi_\#$ is a symmetric closure operator on $[o;1_\#]$ (lemma 5), $\Psi L=\Psi_\#\,[o;1_\#]$ is, by theorem 3, a weakly meet-complete, orthomodular sublattice of $[o;1_\#]$ and therefore also a weakly join-complete subset of $[o;1_\#]$.

Since $E^0(\Psi L;z)=E^0(\Psi L;z\wedge\Psi 1)$ for all $z\in L$, it follows that ΨL is also a weakly join-complete subset of L.

Suppose now that $z\notin[o;1_\#]$. Then there is no $x\in\Psi L$ such that $z\leq x$. This implies the equality $E_0(\Psi L\cup(\Psi L)';z)=E_0((\Psi L)';z)$.

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As remarked above $E^0(\Psi L; z')$ has a largest element, hence $E_0(\Psi L \cup (\Psi L)'; z) = E_0((\Psi L)'; z) = [E^0(\Psi L; z')]'$ has a smallest one. If $z \in [o; 1_\#]$, then $E_0(\Psi L; z)$ has a smallest element, say a. Suppose further that there is a $y \in (\Psi L)'$ such that $z \leq y$. Then $z \leq a \wedge y = a \wedge (y')' = a \wedge (y')' \wedge \Psi 1 = a \wedge y'^{\#} \in \Psi L$; thus $a \leq a \wedge y \leq y$. This implies that a is also the smallest element of $E_0(\Psi L \cup (\Psi L)'; z)$. Hence weakly meet-completeness of $\Psi L \cup (\Psi L)'$ is proved.

It remains to show that this subset is an orthomodular sublattice of L.

ΨL is a sublattice of L, thus, whenever $z_1, z_2 \in \Psi L$, resp. $z_1, z_2 \in (\Psi L)'$, it follows immediately that $z_1 \vee z_2 \in \Psi L \subseteq \Psi L \cup (\Psi L)'$, resp. $z_1 \vee z_2 = (z_1' \wedge z_2')' \in (\Psi L)' \subseteq \Psi L \cup (\Psi L)'$. If $z_1 \in \Psi L$ and $z_2 \in (\Psi L)'$, then $z_1 \vee z_2 \geq z_2 \geq (\Psi 1)'$. Thus $z_1 \vee z_2 = (z_1' \wedge \Psi 1)' \vee z_2 = (z_1^{\#})' \vee z_2 \in (\Psi L)'$, since $z_1^{\#}, z_2' \in \Psi L$ and consequently $z_1^{\#} \wedge z_2' \in \Psi L$. Thus $\Psi L \cup (\Psi L)'$ is closed under the formation of joins. Of course this subset is also closed under orthocomplementation of L. QED

Lemma 7. *The product of two projections is a projection if and only if the projections commute.*

Proof. Let Ψ_1, Ψ_2 and $\Psi_1 \circ \Psi_2$ be projections. Then $\Psi_1 \circ \Psi_2 = (\Psi_1 \circ \Psi_2)^* = \Psi_2^* \circ \Psi_1^* = \Psi_2 \circ \Psi_1$. Conversely, if two projections Ψ_1, Ψ_2 commute then $\Psi_1 \circ \Psi_2 = (\Psi_1 \circ \Psi_2)^* = (\Psi_1 \circ \Psi_2) \circ (\Psi_1 \circ \Psi_2)$. QED

Theorem 8. *Let Γ be a symmetric closure operator on L and $a \in L$. Then $\phi_a \circ \Gamma$ is a projection if and only if $\Gamma a = a$.*

Proof. Let $\Gamma a = a$. From $\phi_a z \leq a$ we get, using monotony of Γ, $\Gamma(\phi_a z) \leq \Gamma a = a$. Thus $\phi_a[\Gamma(\phi_a z)] = \Gamma(\phi_a z)$ for all $z \in L$ or equivalently $\phi_a \circ \Gamma \circ \phi_a = \Gamma \circ \phi_a$. Now $\Gamma \circ \phi_a = \phi_a \circ \Gamma \circ \phi_a = (\phi_a \circ \Gamma \circ \phi_a)^* = (\Gamma \circ \phi_a)^* = \phi_a \circ \Gamma$ thus, by lemma 7, $\phi_a \circ \Gamma$ is a projection. Conversely let $\phi_a \circ \Gamma$ be a projection. Hence $\phi_a \circ \Gamma = \Gamma \circ \phi_a$ and in particular $\Gamma(\phi_a a) = \phi_a(\Gamma a)$. Thus $\Gamma a = \phi_a(\Gamma a) \leq a$. Γ being a closure operator, we get $a \leq \Gamma a$ and finally $a = \Gamma a$. QED

Lemma 9. *Let Γ_1, Γ_2 be symmetric closure operators and $a \in L$ such that $\phi_a \circ \Gamma_i$ $(i=1, 2)$ are projections. Then $\phi_a \circ \Gamma_1 = \phi_a \circ \Gamma_2$ if and only if $\Gamma_1 L \cap [o; a] = \Gamma_2 L \cap [o; a]$.*

Proof. Let $\phi_a \Gamma_1 = \phi_a \circ \Gamma_2$. Clearly $z \in \Gamma_i L \cap [o; a]$ if and only if $\Gamma_i z = z \leq a$ $(i=1, 2)$. If $z \in \Gamma_1 L \cap [o; a]$, then $a \geq z = (\phi_a \circ \Gamma_1)z = (\phi_a \circ \Gamma_2)z = (\Gamma_2 \circ \phi_a)z = \Gamma_2 z$. Thus $z \in \Gamma_2 L \cap [o; a]$ or equivalently $\Gamma_1 L \cap [o; a] \subseteq \Gamma_2 L \cap [o; a]$. In a similar way we get $\Gamma_1 L \cap [o; a] \supseteq \Gamma_2 L \cap [o; a]$. Conversely, let $\Gamma_1 L \cap [o; a] = \Gamma_2 L \cap [o; a]$. Since $\phi_a z \leq a$ and $a \in \Gamma_i L$ $(i=1, 2)$ (theorem 8), we get, using basic propertie of closure operators, $(\Gamma_1 \circ \phi_a)z = \Gamma_1(\phi_a z) = \bigwedge E_0(\Gamma_1 L; \phi_a z) = \bigwedge E_0(\Gamma_1 L \cap [o; a]; \phi_a z) = \bigwedge E_0(\Gamma_2 L \cap [o; a]; \phi_a z) = \bigwedge E_0(\Gamma_2 L; \phi_a z) = \Gamma_2(\phi_a z) = (\Gamma_2 \circ \phi_a)z$ for all $z \in L$. QED

Theorem 10. *Every projection Ψ on an orthomodular lattice L can be represented as the product of a Sasaki-projection and a symmetric closure operator.*

Among the symmetric closure operators which decompose Ψ in this way, there exists

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a smallest one: $\Gamma\{\Psi L \cup (\Psi L)'\}$. *The Sasaki-projection in this decomposition is uniquely determined:* $\phi_{\Psi 1}$.

Explicitly:

$$\Psi = \phi_{\Psi 1} \circ \Gamma\{\Psi L \cup (\Psi L)'\}$$

or

$$\Psi z = \bigwedge \{x \mid [(\Psi 1)' \vee z] \wedge \Psi 1 \leq x;\, x \in \Psi L \cup (\Psi L)'\}.$$

Proof. (i) By theorem 6, $\Psi L \cup (\Psi L)'$ is the range of a symmetric closure operator, namely $\Gamma := \Gamma\{\Psi L \cup (\Psi L)'\}$. By theorem 8, $\phi_{\Psi 1} \circ \Gamma$ is a projection since $\Psi 1 \in \Psi L \subseteq \Psi L \cup (\Psi L)'$.

We now prove the equality $\Psi = \phi \circ \Gamma$ $(\phi := \phi_{\Psi 1})$. Since $\Psi z \in \Psi L \cup (\Psi L)'$ and $\Psi z \leq \Psi 1$, we get $(\phi \circ \Gamma \circ \Psi)z = \phi(\Gamma(\Psi z)) = \phi(\Psi z) = \Psi z$ for all $z \in L$. Thus $\Psi \leq \phi \circ \Gamma$ and $\Psi \leq \phi$. Γ being a mapping of L onto $\Psi L \cup (\Psi L)'$, we have two possibilities, either $\Gamma z = \Psi x$ or $\Gamma z = (\Psi y)'$ for suitable x and y. In the first case we get $(\Psi \circ \phi \circ \Gamma)z = (\phi \circ \Psi)(\Gamma z) = (\phi \circ \Psi)(\Psi x) = \phi(\Psi(\Psi x)) = \phi(\Psi x) = (\phi \circ \Gamma)z$. In the latter case we have $(\Psi \circ \phi \circ \Gamma)z = (\Psi \circ \phi)(\Psi y)' = \Psi(\phi(\Psi y)') = \Psi(\Psi y)^{\#}$ because $\phi(\Psi y)' = (\Psi y)' \wedge \Psi 1 = (\Psi y)^{\#}$. $\Psi_{\#}$ being a symmetric closure operator on $[o; 1_{\#}]$ and $\Psi_{\#}[o; 1_{\#}] = \Psi L$, it follows that $(\Psi y)^{\#} \in \Psi L$; thus $\Psi(\Psi y)^{\#} = (\Psi y)^{\#}$. Now again $(\Psi y)^{\#} = \phi(\Psi y)' = (\phi \circ \Gamma)z$. This proves that $(\Psi \circ \phi \circ \Gamma)z = (\phi \circ \Gamma)z$ for all $z \in L$ or equivalently $\phi \circ \Gamma \leq \Psi$.

(ii) Suppose that $\Psi = \phi_{a_1} \circ \Gamma_1 = \phi_{a_2} \circ \Gamma_2$, then by theorem 8 $\Gamma_2 a_2 = a_2$ and furthermore $a_1 \geq (\phi_{a_1} \circ \Gamma_1)a_2 = (\phi_{a_2} \circ \Gamma_2)a_2 = \phi_{a_2} a_2 = a_2$. A similar argument leads to $a_2 \geq a_1$. Hence $a_1 = a_2$, which proves that the Sasaki-projection in this decomposition of Ψ is uniquely determined.

(iii) Let $\tilde{\Gamma}$ be a symmetric closure operator on L such that $\Psi = \phi \circ \tilde{\Gamma}$. Since by (i) of the proof also $\phi \circ \Gamma\{\Psi L \cup (\Psi L)'\} = \Psi$ we get by lemmata 9 and 4 $\tilde{\Gamma} L \cap [o; \Psi 1] = [\Psi L \cup (\Psi L)'] \cap [o; \Psi 1] = \Psi L$. Thus $\Psi L \subseteq \tilde{\Gamma} L$ and since $\tilde{\Gamma}$ is a symmetric closure operator, hence $(\tilde{\Gamma} L)' = \tilde{\Gamma} L$, we get $\Psi L \cup (\Psi L)' \subseteq \tilde{\Gamma} L$. Consequently $\tilde{\Gamma}(\Gamma\{\Psi L \cup (\Psi L)'\}z) = \Gamma\{\Psi L \cup (\Psi L)'\}z$ for all $z \in L$. Hence $\Gamma\{\Psi L \cup (\Psi L)'\} \leq \tilde{\Gamma}$.

QED

Note added in Proof: Similar results have been obtained by M. F. Janowitz in "Equivalence Relations induced by Baer*—semigroups", Journal of Natural Sciences and Mathematics 11, 83–102 (1972). See also T. S. Blyth and M. F. Janowitz "Residuation Theory", Pergamon Press (1972).

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