ABELIAN GROUPS WITH A VANISHING HOMOLOGY GROUP

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In this paper, we wish to characterize those abelian groups whose integral homology groups vanish in some positive dimension. We obtain a complete characterization provided the dimension in which the homology vanishes is odd; in fact, we prove that the only abelian groups which possess a vanishing homology group in an odd dimension are, up to isomorphism, subgroups of $\mathbf{Q}^n$, where $\mathbf{Q}$ denotes the additive group of rational numbers. The case of vanishing in an even dimension is much more complicated. We exhibit a class of groups whose homology vanishes in even dimensions and is otherwise very nice, namely the subgroups of $\mathbf{Q}/Z$, and then show that unless we impose further restrictions, there exist abelian groups which possess the homology of subgroups of $\mathbf{Q}/Z$ without being isomorphic to a subgroup of $\mathbf{Q}/Z$.

All groups will be abelian and all homology groups will have integral coefficients.

It is well known that if $F(n)$ denotes the free abelian group of rank n, then $H_*(F(n))$ is isomorphic to the exterior algebra $\Lambda_Z[u_1, \ldots, u_n]$ on n generators, where the dimension of each $u_i = 1$. (It is also true that this is an isomorphism of rings.)

PROPOSITION 1. *If A is torsion free, then $H_{n+1}(A) = 0$ if and only if A is isomorphic to a subgroup of $\mathbf{Q}^n$.*

Proof. It is sufficient to assume that $A \subseteq \mathbf{Q}^n$. Since A is torsion free, any finitely generated subgroup of A is free. If $F(m) \subseteq A$, then tensoring with $\mathbf{Q}$ over Z preserves the inclusion, and therefore $\mathbf{Q}^m \subseteq A \otimes_Z \mathbf{Q} \subseteq \mathbf{Q}^n \otimes_Z \mathbf{Q} \approx \mathbf{Q}^n$, which implies that $m \leqq n$. Since for any group G,

$$H_k(G) \approx \varinjlim H_k(G'),$$

where G' runs through the finitely generated subgroups of G, we have that

$$H_{n+1}(A) \approx \varinjlim H_{n+1}(F(m)).$$

Since $m \leqq n$, this implies that $H_{n+1}(A) = 0$.

Conversely, suppose that $H_{n+1}(A) = 0$ and A is torsion free. Again, every finitely generated subgroup of A is free, and hence

$$H_k(A) \approx \varinjlim (H_k(F(m)), g_*),$$

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where the maps $g_*: H_k(F(m)) \to H_k(F(n))$ are induced from the inclusions $g: F(m) \to F(n)$. An easy computation (see **4**) shows that the induced maps are monomorphisms. Since $H_{n+1}(A) = 0$, we must have that $H_{n+1}(F(m)) = 0$ for all finitely generated subgroups of A. Hence, if $F(m) \subseteq A$, we must have that $m \leqq n$. Since A is torsion free, we have a monomorphism $A \to A \otimes_Z \mathbf{Q}$. Since A contains at most n linearly independent elements over Z, $A \otimes_Z \mathbf{Q} \cong \mathbf{Q}^k$ for some $k \leqq n$, and therefore A is isomorphic to a subgroup of $\mathbf{Q}^k \subseteq \mathbf{Q}^n$.

PROPOSITION 2. *If A is isomorphic to a subgroup of $P = \mathbf{Q}/Z$, then $H_0(A) \approx Z$, $H_{2k+1}(A) \approx A$ and $H_{2k}(A) = A$ for all $k > 0$*

Proof. It is sufficient to assume that $A \subseteq P$, and therefore $A \approx A'/Z$ for some subgroup $A' \subseteq Q$. It follows that there exists a spectral sequence $\{E_{p,q}{}^*\}$ with $E_{p,q}{}^2 \approx H_p(A, H_q(Z))$ and $E_*{}^\infty$ isomorphic to the associated graded group of a suitable filtration of $H_*(A')$; see (**1**; **3**). Since $H_q(Z) = 0$ for $q > 1$, there exists an exact sequence

$$\to H_{n+1}(A') \to H_{n+1}(A) \to H_{n-1}(A) \to H_n(A') \to \ldots .$$

Now, $A' \subseteq Q$, and therefore by Proposition 1, $H_q(A') = 0$ for $q > 1$, and thus we obtain $H_{q+1}(A) \approx H_{q-1}(A)$ for $q > 1$ and $0 \to H_2(A) \to H_0(A) \approx Z$ for $q = 1$. It follows that $H_{2k+1}(A) \approx A$ since $H_1(A) \approx A$ and that $H_{2k}(A) \approx H_2(A)$ is either Z or the zero group. Since A is a torsion group, it is the direct limit of finite groups. Since H_* (finite group) is a torsion group or zero in each positive dimension, it follows that $H_*(A)$ is either torsion or zero in each positive dimension. We conclude that $H_{2k}(A) = 0$ for $k \geqq 1$.

We now turn our attention from exhibiting groups with the desired homology groups to determining how complete our enumeration is. The corollary to Theorem 1 shows that for an odd-dimensional vanishing homology group it is totally complete.

The following lemma concerning induced homomorphisms will be needed. A proof may be found in (**4**).

LEMMA. *Let p be a prime and let $f: Z_p \to Z_h$ be a non-zero homomorphism. Then the induced map $f_*: H_*(Z_p) \to H_*(Z_h)$ is non-zero in every odd dimension.*

THEOREM 1. *If $H_{2k+1}(A) = 0$ for some $k \geqq 1$, then A is torsion free.*

Proof. Suppose that A contains an element of finite order, then A contains a subgroup Z_p for some prime p. Since $H_{2k+1}(Z_p) \neq 0$ and $H_{2k+1}(A) = 0$, there exists a finitely generated subgroup A' of A, containing Z_p, such that if $i: Z_p \to A'$ is the inclusion, then $i_*: H_{2k+1}(Z_p) \to H_{2k+1}(A')$ is the zero map. Suppose that $A' \approx F(n) \times Z_{h_1} \times \ldots \times Z_{h_k}$, then the composition $p_{h_s} \circ i$ is non-zero for some s, where p_{h_s} is the projection $A' \to Z_{h_s}$, since $p_F \circ i: Z_p \to F(n)$ must be zero. By the lemma, $(p_{h_s})_* i_*$ is non-zero in dim $2k + 1$, and therefore i_* is non-zero in dim $2k + 1$. This shows that no such A' can exist, and thus A contains no elements of finite order.

COROLLARY. *If A is an abelian group, then A is isomorphic to a subgroup of* $\mathbf{Q}^{2k}$ *if and only if* $H_{2k+1}(A) = 0$.

The situation with regards to the even dimensions is not as simple, and we can give only a partial solution.

A group is said to satisfy the minimum condition on subgroups if $B_1 > B_2 > \ldots > B_k > \ldots$ being a descending chain of subgroups implies $B_k = B_{k+1} = B_{k+2} = \ldots$ for some k.

THEOREM (Kurosh (**2**)). *An abelian group A satisfies the minimum conditions on subgroups if and only if A has finitely many primary components A_p, and each A_p is the direct sum of a finite number of copies of Z_{p^∞} (p-divisible envelope of Z_p) and cyclic p-groups.*

THEOREM 2. *If A satisfies the minimum condition on subgroups and $H_{2k}(A) = 0$, $H_{2k+1}(A) \approx A$ for $k > 0$, then A is isomorphic to a subgroup of $P = Q/Z$.*

Proof. Since $A \approx \times A_p$, it is sufficient to show that A_p is isomorphic to a subgroup of Z_{p^∞}. By Kurosh's theorem, A_p is isomorphic to a direct sum of finitely many copies of Z_{p^∞} and cyclic p-groups. If there is more than one cyclic p-group in this decomposition, then $H_*(Z_{p^e} \times Z_{p^f})$ is a direct summand of $H_*(A_p)$, which is in turn a direct summand of $H_*(A)$. However, it follows easily from the Künneth formula that $H_{2k}(Z_{p^e} \times Z_{p^f}) \neq 0$, and thus $H_{2k}(A) \neq 0$ which is false; hence, $A_p \approx Z_{p^\infty} \times Z_{p^e}$.

We again use the Künneth formula to show that the condition $H_{2k+1}(A) \approx A$ forces either $A_p \approx Z_{p^e}$ or $A_p \approx Z_{p^\infty}$, and therefore $A_p \subseteq Z_{p^\infty}$.

It is natural to ask whether the conditions $H_{2k}(A) = 0$ and $H_{2k+1}(A) \approx A$ are alone sufficient to force A to be isomorphic to a subgroup of P. The following example shows that this is false and indicates that the characterization for even dimensions is much more complicated than for the odd.

Let $A = \oplus_I Z_{p^\infty}$, where I has cardinality $\aleph_0$. Now,

$$A \approx \varinjlim \bigoplus_F Z_{p^\infty},$$

where F runs through the finite subsets of I and the maps are such that if $F \subseteq F'$, then $\gamma\colon \oplus_F Z_{p^\infty} \to \oplus_{F'} Z_{p^\infty}$ embeds $\oplus_F Z_{p^\infty}$ as a direct summand of $\oplus_{F'} Z_{p^\infty}$. It follows that $H_s(\oplus_F Z_{p^\infty})$ is a direct summand of $H_s(\oplus_{F'} Z_{p^\infty})$, and hence γ_* is a monomorphism. An easy application of the Künneth formula and Proposition 2 shows that $H_{2k+1}(\oplus_F Z_{p^\infty}) \approx \oplus_{H(F)} Z_{p^\infty}$, where $H(F)$ is a finite set containing F. Moreover, if $F \subseteq F'$, then $H(F) \subseteq H(F')$. Let $\{F_i\}$ be a linearly ordered cofinal subsystem of all the finite subsets $\{F\}$ of I. It follows that $\{H(F_i)\}$ is a linearly ordered cofinal subsystem of $\{H(F)\}$. Therefore,

$$H_{2k+1}(A) \approx \varinjlim \bigoplus_{H(F_i)} Z_{p^\infty}.$$

Since all the induced maps are monomorphisms, $H_{2k+1}(A)$ is just the "union" of the groups $\oplus_{H(F_i)} Z_{p^\infty}$. Now, the cardinality of $H_{2k+1}(A)$ cannot exceed $\aleph_0$ since we are taking the "union" of only countably many sets, each containing a countable number of elements. Now, since $H_{2k+1}(A)$ is the limit of divisible p-groups, it is a divisible p-group, and therefore isomorphic to $\oplus_J Z_{p^\infty}$ for some index set J. We have seen that the cardinality of J cannot exceed $\aleph_0$; however, it cannot be finite since $H_{2k+1}(A)$ contains $\oplus_{H(F_i)} Z_{p^\infty}$, and the cardinality of $H(F_i)$ approaches $\aleph_0$ as the cardinality of F_i approaches $\aleph_0$. It follows that $H_{2k+1}(A) \approx A$. Since A is a divisible p-group, $H_{2k}(A) = 0$ for all $k \geqq 1$. Hence, we have produced an abelian group whose homology is like the homology of subgroups of P, but is not itself isomorphic to a subgroup of P.

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