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SUMMING UP THE DYNAMICS OF QUADRATIC HAMILTONIAN SYSTEMS WITH A CENTER

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ABSTRACT. In this work we study the global geometry of planar quadratic Hamiltonian systems with a center and we sum up the dynamics of these systems in geometrical terms. For this we use the algebro-geometric concept of multiplicity of intersection $I_p(P,Q)$ of two complex projective curves $P(x,y,z) = 0$, $Q(x,y,z) = 0$ at a point p of the plane. This is a convenient concept when studying polynomial systems and it could be applied for the analysis of other classes of nonlinear systems.

1. **Introduction.** The general structure of planar quadratic vector fields is not known and attempting to classify all such systems is quite a complex task. Ultimately one would like to obtain the bifurcation diagram of this class. The plane quadratic vector fields form a family *QV* which depends on twelve parameters but due to the group action of affine transformations and positive time rescaling, the class ultimately depends on five parameters. Bifurcation diagrams were constructed for small parts of this class (for example *cf.* [A], [S1], [S2]). In this article we study quadratic Hamiltonian systems with a center. Although such systems are discussed in a number of works in the literature, a satisfactory, geometric analysis of this class is still missing. Indeed, while in [V] and [AL] we see phase portraits of these systems, they are not assembled in a bifurcation diagram so as to allow us to easily see how systems change as parameters vary. In [A], a Ph.D. thesis which appeared in russian and was not published, the bifurcation diagram of all quadratic systems with a center was given, hence in particular for the Hamiltonian case, but these diagrams were not realized in the adequate parameter space which is a four-dimensional real projective space and for the Hamiltonian case, the real projective plane. Doing the analysis over the projective plane puts all the parameters on an equal footing and also yields a more condensed picture: it is very convenient to follow the changes in the systems as parameters vary on a disk, representing the real projective plane when the opposite points on the circumference are identified. In [A] the invariant algebraic curves are mentioned but their role in the dynamics and in the integrability of the systems does not appear in this work. While in [S2] this role is highlighted, in [S2] the Hamiltonian case is not discussed. A goal of this article is to give a more satisfactory analysis for the class of Hamiltonian systems with a center. This is one of the simplest

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classes of nonlinear integrable systems, and it makes a good case study for a geometrical viewpoint. It gives an opportunity to observe basic geometrical properties of systems with a rational (in this case a cubic) first integral. The problem of determining when a polynomial system has a rational first integral is open. Poincaré posed this problem (*cf.* [P91]) and he obtained partial results in [P97].

Our work sums up in geometric terms the dynamics of quadratic Hamiltonian systems with a center. We spell out by using in the discussion the algebro-geometric notion of intersection multiplicity at a point of two complex projective curves, the bifurcation phenomena encountered. The treatment of bifurcation of singular points here could be applied to other classes of polynomial systems, nonintegrable quadratic systems, cubic Hamiltonian systems, etc.

The article is organized as follows: in Section 2 we describe the canonical forms for the systems we consider. In Section 3 we briefly describe the Poincaré compactification which is used for the systems. In Section 4 we discuss the singular points in the finite plane using the intersection multiplicity at a point of two complex projective curves. The singular points at infinity are studied in Section 5. In Section 6 we determine the curves in the parameter space, on which saddle connections occur. In Section 7 we sum up the main facts concerning the global geometry and the dynamics of the quadratic Hamiltonian systems with a centre and we draw their bifurcation diagram.

2. **Canonical forms and symmetry for the Hamiltonian vector fields with a centre. The parameter space.** A center of a planar vector field is an isolated singularity surrounded by closed solution curves. In a quadratic system, a center is necessarily a weak focus *i.e.* a singularity with pure imaginary eigenvalues (*cf.* [B], [DT] and [J]). So let us first consider quadratic systems with a weak focus. Such a system can be brought by affine coordinate transformations and positive time rescaling to the form:

$$\frac{dx}{dt} = -y + kx^2 + mxy + ny^2, \quad \frac{dy}{dt} = x + ax^2 + bxy + cy^2. \tag{2.1}$$

The form (2.1) is invariant under rotations of axes. A rotation of axes of an angle θ brings the system to one of the same form but in coefficients k', m', n', a', b', c' and we have

$$c' = c\cos^3\theta - (b-n)\cos^2\theta\sin\theta + (m+a)\cos\theta\sin^2\theta - k\sin^3\theta.$$

Thus, if $c \neq 0$ we can find θ such that $c' = 0$. So we shall only consider systems:

$$\frac{dx}{dt} = -y + kx^2 + mxy + ny^2, \quad \frac{dy}{dt} = x + ax^2 + bxy. \tag{2.2}$$

A system (2.2) is Hamiltonian if and only if $m = 0 = 2k + b$. So we consider only systems

$$\frac{dx}{dt} = -y + kx^2 + ny^2, \quad \frac{dy}{dt} = x + ax^2 - 2kxy. \tag{2.3}$$

The system corresponding to the parameter $\lambda = (a, k, n)$ has the following Hamiltonian:

$$H_\lambda(x, y) = -\frac{ax^3 - ny^3}{3} + kx^2y - \frac{x^2 + y^2}{2}. \tag{2.4}$$

We study systems (2.3) which are nonlinear *i.e.* $\lambda = (a, k, n) \neq 0$. For $(a, k, n) \neq 0$ the systems can be rescaled, hence the parameter space needed for the bifurcation diagram is actually the real projective plane $P_2(\mathbb{R})$ and not $\mathbb{R}^3$. $P_2(\mathbb{R})$ could be pictured as a disk with opposite points on the circumference identified. We may place $n = 0$ on the circumference of the disk. Since for $n \neq 0$ we can rescale the system, we may assume $n > 0$. Furthermore we observe that the following identity holds for the systems (2.3):

$$H_{(-a,k,n)}(x, y) = H_{(a,k,n)}(-x, y). \tag{2.5}$$

In view of (2.5) it is sufficient to discuss the systems (2.3) in a semidisk corresponding to $a \geq 0$ (or $a \leq 0$), since the case $a < 0$ (resp. $a > 0$) can be recuperated from the symmetry.

3. **The Poincaré compactification.** For systems

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{3.1}$$

with P, Q polynomials with coefficients in $\mathbb{R}$ we use the Poincaré compactification (*cf.* [P81] and [GV]). This is obtained as follows: we identify the (x, y)-plane with the plane $Z = 1$ in $\mathbb{R}^3$ with coordinates (X, Y, Z). We project this plane by a central projection on the sphere $S^2 = \{(X, Y, Z) \in \mathbb{R}^3 \mid X^2 + Y^2 + Z^2 = 1\}$. The point (x, y) is sent on two opposite points of the sphere. The central projection associates to our vector field (3.1) a vector field on the complement of the equator of the sphere. In [GV] it was shown that this vector field can be extended to an analytic vector field $A(S)$ on the whole sphere. Our vector field is diffeomorphic to the vector field obtained by restricting $A(S)$ to the upper hemisphere. A compactification $A_N(S)$ is obtained by considering the restriction of $A(S)$ to the upper hemisphere completed with the equator. Projecting the vector field $A_N(S)$ vertically on the plane $Z = 1$ we obtain a vector field on the disk of radius 1 in the (x, y)-plane. The phase portraits of the systems (2.3) will thus be pictured on disks.

4. **The study of the singular points of the systems (2.3) in the finite plane.** The finite singular points of (3.1) are the intersection points of the curves $P(x, y) = 0$, $Q(x, y) = 0$. For the nonlinear quadratic case (2.3) *i.e.* for $(a, k, n) \neq 0$, these two curves are:

$$P(x, y) = -y + kx^2 + ny^2 = 0, \quad Q(x, y) = x + ax^2 - 2kxy = 0. \tag{4.1}$$

$(0, 0)$ is a common point of the curves (4.1) for all values (a, k, n) but for all values $(a, k, n) \neq 0$ the curves (4.1) have at least one other common point in $\mathbb{R}^2$.

NOTATION 4.1. We denote by $N(a, k, n)$ the number of distinct (finite) singular points of (2.3) for the parameter (a, k, n). Equivalently $N(a, k, n)$ is the number of distinct common points of (4.1).

For two systems (2.3) corresponding to (a, k, n) and (a', k', n'), which are topologically equivalent $N(a, k, n) = N(a', k', n')$. For $(a, k, n) \neq 0$, we can rescale the coefficients in

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(2.3), hence the parameter space is actually the real projective plane $P_2(\mathbb{R})$ and N yields a function on $P_2(\mathbb{R})$, $N[a,k,n]$, which describes part of the the dynamics of the systems. To determine the function N it is convenient to take into consideration *the full projective completions of the curves* (4.1) in the complex projective space $P_2(\mathbb{C})$ *i.e.* the curves over $\mathbb{C}$ given by the homogenized equations corresponding to (4.1):

$$-yz + kx^2 + ny^2 = 0, \quad xz + ax^2 - 2kxy = 0 \tag{4.2}$$

and to consider their intersection points, counted with multiplicities, over $\mathbb{C}$. Roughly speaking the intersection multiplicity $I_p(P,Q)$ of two algebraic curves $P=0$, $Q=0$ at a point p indicates how many points the curves have in common at that point. For example the intersection multiplicity of $y=0$ with $y-x^2=0$ at $(0,0)$ is two, since the line is tangent to the parabola at $(0,0)$. At the same time the intersection multiplicity of $xy=0$ with $y-x^2=0$ at $(0,0)$ is three since $x=0$ and $y=0$ are components of $xy=0$ and $x=0$ intersects the parabola $y-x^2=0$ transversely. For a quick understanding of the concept of intersection multiplicity and of its properties the reader is advised to consult [G] or [Ki]. We look if there are values of the parameters for which the curves in (4.2) have common components. The second curve in (4.2) has two components:

$$x = 0, \quad z + ax - 2ky = 0. \tag{4.3}$$

The first curve in (4.2) is reducible if and only if $k=0$ and then clearly $x=0$ cannot be a component of the first conic in (4.2). If $k=0$ the second line in (4.3) is a component of the first conic in (4.2) if and only if $z+ax=0$ is $z-ny=0$ which only occurs when $a=0=n$ yielding a linear system. So the curves in (4.2) have no common component for all $(a,k,n)\neq(0,0,0)$ and the same holds true for their affine parts (4.1). By Bézout's theorem (*cf.* [Ki]) the number of intersection points of (4.2) in $P_2(\mathbb{C})$, counted with multiplicities, is four. We therefore have $1 \le N(a,k,n) \le 4$. The intersection points of the curves (4.2), counted with multiplicities are given by the intersection points (counted with multiplicities), of each one of the straight lines in (4.3) with the first conic in (4.2). We consider the intersections of each of the following two sets of curves:

$$-yz + kx^2 + ny^2 = 0, \quad x = 0 \tag{4.4}$$

$$-yz + kx^2 + ny^2 = 0, \quad z + ax - 2ky = 0. \tag{4.5}$$

Correspondingly we have the affine curves obtained by letting $z=1$ in (4.4) and (4.5) *i.e.*:

$$-y + kx^2 + ny^2 = 0, \quad x = 0 \tag{4.6}$$

$$-y + kx^2 + ny^2 = 0, \quad 1 + ax - 2ky = 0. \tag{4.7}$$

We shall write $I_p(4.i)$ in place of $I_p(P,Q)$ if the curves are those of $(4.i)$ with $i = 1,2,4,5,6,7$. Clearly, for each intersection point p of either (4.4) or (4.5) we have $I_p(4.i) \le 2$, $i=4,5$. If $p=(x,y)$ we also write $p=[x,y,1]$ identifying p with its image

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$[x, y, 1]$ in the projective plane. $N(a, k, n) = 4$ if and only if the curves (4.2) have no point of intersection "at infinity" (*i.e.* for $z = 0$) and there is no singular point p of (2.3) with $I_p(4.1) > 1$. $N(a, k, n) = 3$ if the curves (4.2) have four distinct points of intersection, one of them a point at infinity for the curves in (4.1) or if (4.2) have three distinct points of intersection, none at infinity for (4.1) and one and only one of them, p, with $I_p(4.2) = 2$. Thus determining N is related to determining the intersection points at infinity of (4.1) and to the intersection multiplicities of the curves (4.2). We first look at the intersection points at infinity for (4.1) (or of (4.2) with $z = 0$) and at their multiplicity of intersection:

PROPOSITION 4.1. *The curves (4.2) intersect "at infinity" (i.e. for $z = 0$) if and only if $nC = 0$, where $C = na^2 + 4k^3$, in which case they have a unique point p of intersection at infinity.*

i) If $n = 0 \neq C$, $p = [0, 1, 0]$ with $I_p(4.2) = 1$ and $N[a, k, 0] \leq 3$.
ii) If $C = 0 \neq n$, $p = [2k, a, 0]$ if $(a, k) \neq 0$, with $I_p(4.2) = 1$, $N[a, k, n] \leq 3$ and $p = [1, 0, 0]$ if $(a, k) = 0$ in which case $I_p(4.2) = 2$, $N[0, 0, 1] = 2$.
iii) If $n = 0 = C$, $p = [0, 1, 0]$, $I_p(4.2) = 2$ and $N[a, 0, 0] = 2$.

PROOF. For (4.4) the points of intersection are $[0, 0, 1]$ and $[0, 1, n]$. Hence the curves in (4.4) intersect for $z = 0$ if and only if $n = 0$. If $n = 0$ we have $I_{[0,1,0]}(4.4) = 1$ ($x = 0$ is not tangent to $-yz + kx^2 = 0$ at $[0, 1, 0]$). The curves in (4.5) have a point of intersection at infinity if and only if $ax - 2ky = 0$ and $kx^2 + ny^2 = 0$ have a common nontrivial solution in $\mathbb{R}^2$. We distinguish the cases $k \neq 0$ and $k = 0$. If $k \neq 0$, $y = ax/(2k)$ and hence $\bigl(k + na^2/(4k^2)\bigr)x^2 = 0$. This equation has a nontrivial solution if and only if $C = 0$. Simple calculations yield the remaining part of the proposition.

PROPOSITION 4.2. *The systems (2.3) have four distinct singular points (real or complex) if and only if $nC\delta(n - 2k) \neq 0$, where $\delta = a^2 - 4kn + 8k^2$. These are: $(0, 0)$, $P_n = (0, 1/n)$ and $P_\pm = (x_\pm, y_\pm)$ where*

$$(4.8)\qquad (x_\pm, y_\pm) = \begin{cases} \left(\frac{(ak - an \pm k\cdot\operatorname{sgn}(a)\cdot\delta^{1/2}}{C}, \frac{4k^2 + a^2 \pm |a|\delta^{1/2}}{2C}\right) & \text{if } a \neq 0 \\ \left(\pm\left(\frac{2k-n}{4k^3}\right)^{1/2}, \frac{1}{2k}\right) & \text{if } a = 0 \end{cases}$$

(If $a \neq 0$, $\operatorname{sgn}(a) = |a|/a$). These points are real if and only if $\delta > 0$ in which case $N(a, k, n) = 4$.

PROOF. If $C \neq 0$, we have $(a, k) \neq 0$. If $a \neq 0$ then replacing $x = (2ky - 1)/a$, $z = 1$ into the first equation of (4.5) we obtain the equation in y

$$(4.9)\qquad (na^2 + 4k^3)y^2 - (a^2 + 4k^2)y + k = 0.$$

If $a = 0$, since $C \neq 0$, we have $k \neq 0$, so replacing $z = 1$ and $y = 1/(2k)$ in the first equation of (4.5) we get the equation

$$(4.10)\qquad kx^2 + \frac{n - 2k}{4k^2} = 0.$$

The discriminant of the equation (4.9) is $a^2\delta$. If $nC\delta(n-2k) \neq 0$ and if $a \neq 0$ we can solve the equation (4.9) for y and if $a = 0 \neq k$ we can solve the equation (4.10) for x obtaining (4.8). The equation (4.9) has real solutions if and only if $\delta > 0$. If $a = 0$, $\delta = -4k(n-2k)$. The equation (4.10) applies in this case and (4.10) has two real solutions if and only if $\delta > 0$.

PROPOSITION 4.3. *If p is a singular point of (2.3) for $\lambda = (a, k, n)$, then $I_p(4.1) \leq 3$. There exists a singular point p_0 of (2.3) such that $I_{p_0}(4.1) \geq 2$ if and only if $\delta(n-2k) = 0$. If $\delta = 0 \neq n - 2k$ or if $\delta \neq 0 = n - 2k$ and $k \neq 0$, then $N(a, k, n) = 3$, we have two singular points p with $I_p(4.1) = 1$ and one p_0 with $I_{p_0}(4.1) = 2$. If $\delta = 0 = n - 2k$, then $N[a, k, n] = N[0, 1, 2] = 2$, the two singular points being $(0, 0)$ and $P_n = (0, 1/n)$ with $I_{(0,0)} = 1$ and $I_{P_n}(4.2) = 3$.*

PROOF. Since $(0, 0)$ is a nonsingular point of both curves in (4.1), and the curves are transversal at $(0, 0)$, $I_{(0,0)}(4.1) = 1$ and hence for any other singular point p of (2.3) we must have $I_p(4.1) \leq 3$. We could have $I_p(4.1) > 2$ only if a point p of intersection of the curves (4.1) is at the same time the point of intersection of the lines in (4.3) (*i.e.* if $p = [0, 1, 2k]$) and in addition one of the lines in (4.3) is tangent at p to the first conic in (4.2). Clearly $x = 0$ cannot be tangent to the first curve in (4.2). $p = [0, 1, 2k]$ lies on the first conic of (4.2) so $I_p(4.1) \geq 2$. The tangent at p of the first conic in (4.2) is: $2(n-k)y - z = 0$. This line coincides with $z + ax - 2ky = 0$ if and only if $a = 0 = n - 2k$ in which case $\delta = 0$, $p = P_n$ and $I_p(4.2) = I_{P_n}(4.1) = 3$. For all other singular points p we have $I_p(4.1) \leq 2$. We have equality if either $[0, 1, 2k]$ lies on the first curve in (4.2) *i.e.* when $n - 2k = 0$ but $a \neq 0$ *i.e.* when $\delta \neq 0$ or when a point p on the second line in (4.3), $p \neq [0, 1, 2k]$, is the point at which this line is tangent to the first curve in (4.2). Since for $(a, k) = 0$, the line $z + ax - 2ky = 0$ is the line at infinity, we must consider $(a, k) \neq 0$. $z + ax - 2ky = 0$ is tangent to the conic in (4.5) if and only if $\delta = 0$.

COROLLARY 4.1. *The bifurcation set for the systems (2.3) due to a change in the number of (finite) singular points of (2.3) is the set defined by the equation $nC\delta(n-2k) = 0$ i.e.*

$$B = \{[a, k, n] \in P_2(\mathbb{R}) \mid nC\delta(n-2k) = 0\} = B_n \cup B_C \cup B_\delta \cup B_{n-2k} \tag{4.11}$$

where we put

$$B_\nu = \{[a, k, n] \in P_2(\mathbb{R}) \mid \nu = 0\}, \quad \nu \in \{n, C, \delta, (n-2k)\}. \tag{4.12}$$

To study the singular points we use the matrix of the linearized system at a point (x, y):

$$L(x, y) = \begin{pmatrix} H_{yx} & H_{yy} \\ -H_{xx} & -H_{xy} \end{pmatrix} = \begin{pmatrix} 2kx & 2ny - 1 \\ -(1 + 2ax - 2ky) & -2kx \end{pmatrix} \tag{4.13}$$

and its characteristic polynomial

$$p(\lambda) = \lambda^2 + \operatorname{Det} L(x, y) \tag{4.14}$$

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where $\operatorname{Det} L(x, y)$ is the determinant of $L(x, y)$. A singular point (x, y) for which $\operatorname{Det} L(x, y) \neq 0$ is either a weak focus or a saddle according to whether $\operatorname{Det} L(x, y) > 0$ or $\operatorname{Det} L(x, y) < 0$. Since the system is Hamiltonian, if $\operatorname{Det} L(x, y) > 0$, the singular point is a center.

OBSERVATION 4.1. Each singular point (x_0, y_0) of the system (2.3) is also a double point of the Hamiltonian level curve

$$H(x, y) - H(x_0, y_0) = 0, \tag{4.15}$$

the equations of the tangent lines at (x_0, y_0) to the curve (4.15) being given by

$$\begin{aligned} H_{xx}(x_0, y_0)(x - x_0)^2 + 2H_{xy}(x_0, y_0)(x - x_0)(y - y_0) \\ + H_{yy}(x_0, y_0)(y - y_0)^2 = 0. \end{aligned} \tag{4.16}$$

We describe the nature of the singular points starting with $P_n = (0, 1/n)$, $n \neq 0$.

PROPOSITION 4.4. *Suppose $n \neq 0$.*

i) If $n - 2k \neq 0$ the singular point $P_n = (0, 1/n)$ is a saddle or a center according to whether or not $(2k - n)n < 0$ or $(2k - n)n > 0$.

ii) If $n - 2k = 0$, the singular point $P_n = (0, 1/n)$ is a cusp if $\delta \neq 0$ (i.e. $a \neq 0$) and it is a saddle if $\delta = 0$ (i.e. $a = 0$).

PROOF. i) Follows from (4.14) and the fact that $\operatorname{Det} L(0, 1/n) = (2k - n)/n$.

ii) If $n = 2k$, $\operatorname{Det}(P_n) = \operatorname{Det}(0, 1/n) = 0$. The level curve of the Hamiltonian passing through P_n is:

$$\begin{aligned} H(x, y) - H(0, 1/n) = -\frac{ax^3}{3} + \frac{x^2(ny - 1)}{2} \\ + \frac{(ny - 1)^2(2ny + 1)}{6n^2} = 0 \end{aligned} \tag{4.17}$$

and the equation (4.17) yields two coincident tangent lines at $P_n = (0, 1/n)$:

$$\left(y - \frac{1}{n}\right)^2 = 0. \tag{4.18}$$

For $a \neq 0$, $ny - 1 = 0$ is not a component of (4.17) and therefore $P_n = (0, 1/n)$ is a cusp for (4.17) which yields a cusp for the system. If $a = 0$ the equation (4.17) can be written

$$H(x, y) - H\left(0, \frac{1}{n}\right) = (ny - 1)\left(\frac{x^2}{2} + \frac{y^2}{3} - \frac{y}{6n} - \frac{1}{6n^2}\right) = 0. \tag{4.19}$$

$ny - 1 = 0$ is tangent to the second component of (4.19) and P_n is a topological saddle. ∎

PROPOSITION 4.5. *If $n\delta C(n - 2k) \neq 0$ and $\delta > 0$ we have:*

i) The points $P_\pm$ are both saddles if and only if $nC < 0$ or $n(n - 2k) < 0$.

ii) One of the points $P_\pm$ is a center and the other a saddle if and only if $nC > 0$ and $n(n - 2k) > 0$.

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PROOF. Using (4.8) we get

$$\operatorname{Det} L(P_\pm) = \begin{cases} -\pm \operatorname{sgn}(a)x_\pm \cdot \delta^{1/2} & \text{if } a \neq 0 \\ -4k^2(x_\pm)^2 & \text{if } a = 0. \end{cases} \tag{4.20}$$

It suffices to prove the proposition for $n > 0$. Proof of i): Assume $C < 0$. We need $\operatorname{sgn}\bigl(\operatorname{Det}(P_\pm)\bigr)$ and straightforward calculations for cases $a \neq 0$ and $a = 0$ yield:

(4.21)

$$\operatorname{sgn}\bigl(\operatorname{Det}(P_\pm)\bigr) = \begin{cases} -\pm \operatorname{sgn}(C)\cdot \operatorname{sgn}\{\operatorname{sgn}(k-n)\times|k-n|\,|a| \pm \operatorname{sgn}(k)\cdot|k|\delta^{1/2}\} & \text{if } a \neq 0 \\ -1 & \text{if } a = 0. \end{cases}$$

If $C < 0$, $na^2 < -4k^3$. For $n > 0$, $k < 0$. So $\operatorname{sgn}(k-n) = \operatorname{sgn}(k) = -1$ and hence $\operatorname{sgn}\bigl(\operatorname{Det}(P_\pm)\bigr) = -\pm \operatorname{sgn}\{|k-n|\,|a| \pm |k|\cdot\delta^{1/2}\}$. Clearly $\operatorname{sgn}\bigl(\operatorname{Det}(P_+)\bigr) = -1$ and hence P_+ is a saddle. For P_- we compare $|k-n|\,|a|$ with $|k|\cdot\delta^{1/2}$. Squaring these, since $n-2k > 0$ we have $(k-n)^2a^2 < k^2\delta$. Hence $\operatorname{sgn}\bigl(\operatorname{Det}(P_-)\bigr) = -1$ and P_- is a saddle. If $C > 0$ and $n-2k > 0$, $k-n < -k$. If $k > 0$, $\operatorname{sgn}(k-n) = -1$. Then P_+ is a center and P_- is a saddle. If $k < 0$, $\operatorname{sgn}(k-n) = -1$, P_+ is a center and P_- is a saddle.

THEOREM 4.1. *Consider a nonlinear system (2.3) i.e. for $\lambda = (a,k,n) \neq 0$. We have:*

$$N(a,k,n) = \begin{cases} 4 & \textit{iff } nC\delta(n-2k) \neq 0 \textit{ and } \delta > 0. \\ 3 & \textit{iff only one of the equations } n = 0, \delta = 0, C = 0, \\ & \quad n-2k = 0 \textit{ is satisfied.} \\ 2 & \textit{iff } \delta < 0 \textit{ or two distinct ones of the equations } n = 0, \\ & \quad \delta = 0, C = 0, n-2k = 0 \textit{ are satisfied.} \end{cases} \tag{4.22}$$

When $N(a,k,n) = 4$ we have a center and three saddles if and only if $nC < 0$ and two centers and two saddles if and only if $nC > 0$.

PROOF. Let $B_\cap = \{p \in P_2(\mathbb{R}) \mid \exists \nu_1, \nu_2 \in \{n, C, \delta, n-2k\}, \nu_1 \neq \nu_2, p \in B_{\nu_1} \cap B_{\nu_2}\}$. Then

$$B_\cap = \{[1,0,0], [0,1,2], [0,0,1]\}. \tag{4.23}$$

More precisely we have:

i) $[a,k,n]$ satisfies $\delta = 0 = C$ if and only if $[a,k,n] = [0,0,1]$.

ii) $[a,k,n]$ satisfies $\delta = 0 = n-2k$ if and only if $[a,k,n] = [0,1,2]$.

In both cases i) and ii) the system has only two singular points: $(0,0)$ and P_n.

iii) $[a,k,n]$ satisfies $C = 0 = n-2k$ if and only if $[a,k,n] = [1,0,0]$. In this case we have only two singular points: $(0,0)$ and $(-1/a, 0)$.

iv) The equations $n = 0 = C$, or $n = 0 = n-2k$ hold only for $[a,k,n] = [1,0,0]$.

In both cases we have only two singular points: $(0,0)$ and $(-1/a, 0)$. $\delta = 0 = n$ cannot occur for a real nonlinear system. We look at the number of singular points for systems in $B_\nu - B_\cap$, with $\nu \in \{n, C, \delta, n-2k\}$. We have: On $B_\delta - B_\cap$, the singular points are:

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$(0,0)$, $P_+ = P_-$, $P_n = (0, 1/n)$. On $B_n - B_\cap$, there are three singular points: $(0,0)$ and $P_\pm$. On $B_C - B_\cap$, $ak \neq 0$, (4.9) is a first degree equation in y yielding only one singular point: $P_f = (x_f, y_f)$. When we approach $C = 0$, one of the two singular points $P_\pm$ runs to infinity and on $C = 0$ we are left with only one of them in the finite plane, point which we denote $P_f = (x_f, y_f)$. We have:

$$P_f = (x_f, y_f) = \left(-\frac{a^2 + 2k^2}{a(a^2 + 4k^2)}, \frac{k}{a^2 + 4k^2}\right). \tag{4.24}$$

In this case the singular points are: $(0,0)$, P_f, P_n. On $B_{n-2k} - B_\cap$ the singular points are: $(0,0)$, $P_n = P_+$ and P_-. Propositions 4.3 and 4.4 yield the remaining part of the proof.

We consider now the nature of the singular points P_+ and P_-, and of P_f, if $nC(n-2k) = 0$ with $\delta > 0$.

PROPOSITION 4.6. *If $nC(n-2k) = 0$ and $\delta > 0$, for the singular points $P_\pm$ (if $C \neq 0$), respectively P_f (if $C = 0$) we distinguish the following possibilities:*

I. *For systems with $[a, k, n]$ in $B_n - B_\cap$ we have $n - 2k \neq 0$ and the singular points $P_\pm$ are topological saddles.*
II. *If $[a, k, n] \in B_C - B_\cap$, P_f is a topological saddle.*
III. *If $[a, k, n] \in B_{n-2k} - B_\cap$, P_- is a topological saddle and $P_+ = P_n$ and it is a cusp.*
IV. *If $[a, k, n] \in B_\cap$ we have two singular points $(0,0)$ and P_n except for $[a, k, n] = [1, 0, 0]$ when the singular points are $(0,0)$ and $P_f = (-1/a, 0)$ which is a topological saddle.*

PROOF. I. For this case we use (4.20) and (4.21).

To prove II. we use (4.24) obtaining

$$\operatorname{Det} L(P_f) = -\frac{a^2 + 4k^2}{a^2}. \tag{4.25}$$

To prove III. we first obtain:

$$(x_\pm, y_\pm) = \begin{cases} \left(\frac{[a \pm (a^2 + 8k^2)^{1/2}]}{4k^2}, \frac{[a^2 + 4k^2 \pm (a^2 + 8k^2)^{1/2}]}{8k^3}\right) & \text{for } a \neq 0 \\ \left(\pm\left[\frac{(2k-n)}{4k^3}\right]^{1/2}, \frac{1}{2k}\right) & \text{for } a = 0. \end{cases} \tag{4.26}$$

This yields

$$\operatorname{Det} L(x_\pm, y_\pm) = \begin{cases} - \pm x_\pm (a^2 + 8k^2)^{1/2} & \text{for } a \neq 0 \\ \frac{2k-n}{k} & \text{for } a = 0. \end{cases} \tag{4.27}$$

To prove V. we use $\operatorname{Det} L(P_f) = \operatorname{Det} L(-1/a, 0) = -1$.

5. **The study of the singular points at infinity.** To study the singular points at infinity it suffices to use two charts: one obtained by projecting the hemisphere $X \geq 0$ of S^2 on the plane $X = 1$ (we denote the coordinates Y, Z in this plane by u, z) and another one by projecting the hemisphere $Y \geq 0$ on the plane $Y = 1$ (we denote the coordinates X, Z by v, z). Projecting the plane $Z = 1$ (with coordinates x, y) on the upper hemisphere

and then on the two planes $X = 1$ and $Y = 1$ we pass from coordinates (x, y) to (u, z), respectively to (v, z): $z = (1/x)$, $u = y/x$ and $z = 1/y$, $v = x/y$. We obtain in this way, after time rescaling, the two systems of equations:

$$\frac{dz}{d\tau} = z(-k + uz - nu^2), \quad \frac{du}{d\tau} = (1 + u^2)z + a - 3ku - nu^3 \tag{5.1}$$

$$\frac{dz}{d\tau} = -z(-2kv + vz + av^2), \quad \frac{dv}{d\tau} = -(1 + v^2)z + n + 3kv^2 - av^3. \tag{5.2}$$

The singular points of (5.1) having the form $(z, u) = (0, u)$ are given by the equation in u

$$G(u) = nu^3 + 3ku - a = 0. \tag{5.3}$$

We recall an algebraic result needed to discuss the number of real roots of this equation:

PROPOSITION 5.1. *All the roots of the equation $a_0u^3 + a_1u^2 + a_2u + a_3 = 0$ are real if and only if its discriminant $D = a_1^2a_2^2 - 4a_0a_2^3 - 4a_1^3a_3 - 27a_0^2a_3^2 + 18a_0a_1a_2a_3$ is positive or zero. The equation has three distinct real roots in the case $D > 0$. The equation has a single real root if and only if $D < 0$. In particular for $G(u) = 0$ we have*

$$D = -27Cn. \tag{5.4}$$

When $nC \neq 0$, the sign of D depends on the sign of nC. The matrix of the linearized system for (5.1) at singular points of the form $(z, u) = (0, u)$ is given by

$$A(0, u) = \begin{pmatrix} -nu^2 - k & 0 \\ u^2 + 1 & -3nu^2 - 3k \end{pmatrix} \tag{5.5}$$

with eigenvalues

$$\lambda_1 = -3(nu^2 + k) = -G'(u), \quad \lambda_2 = -(nu^2 + k) = -\frac{G'(u)}{3}. \tag{5.6}$$

We have

$$\lambda_1\lambda_2 = 3(nu^2 + k)^2 = \frac{[G'(u)]^2}{3}. \tag{5.7}$$

Clearly, the singular points $(0, u)$ are of node type whenever u is a simple root of $G(u)$. The nature of the singular points is given by the following proposition:

THEOREM 5.1. *I. If $nC \neq 0$, there are three singular points at infinity or one according to whether $nC < 0$ or $nC > 0$. In both situations the singular points at infinity are nodes.*

II.(a) If $C = 0 \neq n$, then either $(a, k) \neq (0, 0)$ and we have two singular points at infinity, a node and a singular point with an elliptic region, or $(a, k) = (0, 0)$ and the only singular point at infinity is $u = 0 = z$ which is of node type.

II.(b) If $n = 0 \neq C$ we have two singular points at infinity: $z = 0 = v$ which is with an elliptic region and $z = 0$, $u = a/(3k)$ which is a node.

III. If $n = 0 = C$, there is only one singularity at infinity which is $z = 0 = v$ and it is a node.

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PROOF. I. Follows from Proposition 5.1, from (5.3), (5.4) and (5.7). All nondegenerate (*i.e.* such that $G'(u) \neq 0$) singular points $(0, u)$ are nodes due to formula (5.7). It remains to consider the singular points $(0, u)$ with $G'(u) = 0$ and if $n = 0$, the singular point $z = 0 = v$.

II.(a) Assume first that $ak \neq 0 = C$. We may assume $n > 0$ and because of the symmetry (2.5) we may assume $a > 0$. In this case since $C = 0$ we have $a^2 n = -4k^3$ and hence $k < 0$. We also have $a = -2k(-k/n)^{1/2}$ and $G(u) = n\big(u+(-k/n)^{1/2}\big)^2\big(u-2(-k/n)^{1/2}\big)$. Hence the singular points at infinity are $(z, u) = (0, u)$ with $u = -(-k/n)^{1/2}$, or $u = 2(-k/n)^{1/2}$. By (5.7), $z = 0$, $u = 2(-k/n)^{1/2}$ is a node. The point $z = 0$, $u = -(-k/n)^{1/2}$, has as its linear part a nilpotent 2×2 matrix and the standard blow up technique [BM] or application of results in [ALGM, Chapter IX], yields a point with an elliptic region (*cf.* [PS]). If $a = k = 0 \neq n$, the first integral is $H(x, y) = -(x^2 + y^2)/2 + ny^3/3$. All the curves $H_\lambda(x, y) - K = 0$, K a constant, have only one point at infinity: $z = 0 = u$. Clearly the projective completions of the curves $H_\lambda(x, y) - K = 0$ are all tangent to the line at infinity at this point. Only two singular real curves are in this family, one passing through $(0, 0)$ and another one passing through the singular point $(0, 1/n)$ which is $H_\lambda(x, y) - H(0, 1/n) = 0$. This curve can be traced easily; it is a nodal cubic with only one point at infinity. The phase portraits are thus clear and the point at infinity in this case is a node.

II.(b) Two singular points at infinity are present: $z = 0 = v$ and $(z, u) = (0, u)$ with $u = a/(3k)$. The analysis for $z = 0 = v$ is done by standard blow up techniques (*cf.* [BM] or [ALGM]) and the point turns out to have an elliptic region. The point $z = 0$, $u = a/3k$ is a node.

III. $C = 0 = n \neq a$. In this case the first integral is $H_\lambda(x, y) = -ax^3/3 - (x^2 + y^2)/2$. We have only two singular real Hamiltonian level curves, one with an isolated singular point at the origin, the other one passing through the singular point $(-1/a, 0)$ which is $H_\lambda(x, y) - H_\lambda(-1/a, 0) = -ax^3/3 - (x^2 + y^2)/2 + 1/6a^2 = 0$. This curve (as well as each of the other Hamiltonian level curves) is tangent to the line at infinity at the point at infinity which is in the direction of y-axis. Translating the origin at the singular point $(-1/a, 0)$ by $X = x + 1/a$, $Y = y$ the curve $H_\lambda(x, y) - H_\lambda(-1/a, 0) = 0$ becomes $-2aX^3 + 3X^2 - 3Y^2 = 0$ which is clearly a nodal cubic yielding a homoclinic loop and this cubic has only one point at infinity which is a node for the system and the phase portrait is (except for the orientation of the integral curves) is as indicated in Figure 2.

6. The saddle-connections.

These will occur when the Hamiltonian level curve passing through a saddle, will also pass through another saddle. This cubic curve is reducible, in view of Observation 4.1, with a straight line component passing through the two saddles.

DEFINITION 1. We consider a polynomial vector field

$$X = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y} \tag{6.1}$$

with P and Q polynomials with real coefficients. An algebraic invariant curve of (6.1) is a curve $f(x,y) = 0$, with f a polynomial with real or complex coefficients such that for some polynomial g over $\mathbb{R}$ or over $\mathbb{C}$ we have

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = f \cdot g. \tag{6.2}$$

An algebraic solution of a polynomial differential equation $P(x,y)\,dy - Q(x,y)\,dx = 0$ is an irreducible invariant algebraic curve for (6.1).

We look for invariant lines for the systems (2.3). $f(x,y) = rx+sy+t = 0$ is an invariant line for a quadratic field (6.1) if for some $g(x,y) = r'x + s'y + t'$ we have:

$$rP(x,y) + sQ(x,y) = (rx + sy + t)(r'x + s'y + t'). \tag{6.3}$$

Straightforward calculations yield:

PROPOSITION 6.1. *The systems (2.3) admit invariant straight lines if and only if (a,k,n) satisfy one of the following two conditions:*

i) $an \neq 0$ and (a,k,n) is on the curve (6.4) of saddle connections whose equation is:

$$a^2n - (n+k)^2(n-2k) = 0. \tag{6.4}$$

A systems (2.3) with (a,k,n) on the curve (6.4) has only one invariant line whose equation is: $\mathcal{L}(x,y) = anx - n(n+k)y + n + k = 0$.

ii) If $a = 0 \neq k$, the line $f(x,y) = -2ky + 1 = 0$ is invariant and if $n + k = 0$, the lines $\pm(3)^{1/2}nx - ny + 1 = 0$ are also invariant.

In the case i), $(n+k)(n-2k) \neq 0$ and $P_n = (0, 1/n)$ lies on the invariant line $\mathcal{L}(x,y) = 0$. Let $\mathcal{C}(P_n) = H(x,y) - H(0,1/n) = 0$. Calculations yield the factorization for $\mathcal{C}(P_n)$:

$$\mathcal{L}(x,y)[-2an(n+k)x^2 - 2an^2y^2 + 2n(n+k)(2k-n)xy + (2k-n)(n+k)x + any + a] = 0. \tag{6.5}$$

The conic component intersects $\mathcal{L}(x,y) = 0$ at P_n and at another singular point q. q cannot be the origin or a point at infinity. Hence $q = P_{\pm}$ and when $C = 0$ the line contains P_f.

The affine type of the conic component is determined by its corresponding quadratic form and the value of the determinant Δ of its associated matrix:

$$\begin{pmatrix} -2na(n+k) & n(n+k)(2k-n) \\ n(n+k)(2k-n) & -2n^2a \end{pmatrix}. \tag{6.6}$$

Since on (6.4), $a^2n = (n+k)(2k-n)^2$, simple calculations yield that on (6.4) we have:

$$\Delta = 3n^2(n+k)^2(n-2k)(n+2k). \tag{6.7}$$

For the nature of the conic component it suffices to assume $n > 0$ and its affine type is easily obtained.

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7. **The bifurcation diagram of quadratic Hamiltonian systems with a centre.** As seen in Section 2, the parameter space for nonlinear systems (2.3) is the real projective plane $P_2(\mathbb{R})$. Since the systems (2.3) are Hamiltonian, they have no limit cycle. We sum up the dynamics of quadratic Hamiltonian systems with a center as follows:

THEOREM 7.1. *$\lambda = (a, k, n)$ is a bifurcation point for the family (2.3) if and only if $[a, k, n]$ lies on the projective curve $\mathcal{C}: nC\delta(n-2k)a[an-(n+k)(n-2k)] = 0$ and if λ is on $a = 0$ then $\delta > 0$. Let $\mathrm{Sing}(\mathcal{C})$ be the set of singular points of the projective curve $\mathcal{C}$. The codimension one stratum is made of points in $\mathcal{C} \setminus \mathrm{Sing}(\mathcal{C})$ and they are grouped as follows:*

I. *$N[a, k, n] = 3$. This occurs if and only if $nC\delta(n-2k) = 0$ in which case we have a bifurcation of singular points.*

II. *$N[a, k, n] = 4$. This occurs if and only if $a[an-(n+k)(n-2k)] = 0$ and if $a = 0$ then $\delta > 0$. This is a saddle-connection bifurcation.*

The codimension two stratum is made of the points in

$$\mathrm{Sing}(\mathcal{C}) = \{[1, 0, 0], [0, 0, 1], [0, 1, 2], [0, 1, -1], [0, 1, 0], [\pm\sqrt{2}, 1, -2]\}.$$

For these points $N[a, k, n] = 2$.

We note that for $a = 0$, the systems are symmetric with respect to the y-axis. We spell out the types of bifurcation points in the codimension one and two strata and the characteristics of the systems as follows:

THEOREM 7.2. *Consider the points in the codimension one stratum $= \mathcal{C} \setminus \mathrm{Sing}(\mathcal{C})$. These are of the following types:*

I.(a) If $nC = 0$, one of the finite singular points disappears at infinity becoming a singular point at infinity (a point with an elliptic region). The only other singular point at infinity is a node. The finite singular points are a center and two saddles.

I.(b) If $\delta(n-2k) = 0$ there exists a unique singular point p of (2.3) where two finite singular points coalesce at a finite singular point p. The singularities are: the point p which is a cusp, a center, a saddle and only one singular point at infinity, a node.

In the codimension two stratum $= \mathrm{Sing}(\mathcal{C})$ we have:

- *i) $\mathrm{Sing}(nC = 0) = \{[1, 0, 0], [0, 0, 1]\}$. At one of these points a multiple singular point of (2.3) disappears at infinity. This corresponds to the equations $P = 0$ and $Q = 0$ having a point p at infinity with $I_p(4.5) = 2$.*
- *ii) $\mathrm{Sing}\big(\delta(n-2k) = 0\big) = \{[0, 1, 2]\}$. At $[0, 1, 2]$ we have a coincidence of three singular points, $I_p(P, Q) = 3$. In both cases i) and ii) we have $N(\lambda) = 2$ (a center and a saddle) and we have only one singular point at infinity, a node.*
- *iii) $\mathrm{Sing}\big(a[a^2n-(n+k)^2(n-2k)] = 0\big) = \{[0, 1, -1], [0, 1, 2]\}$. At $[0, 1, -1]$ we have three invariant lines which are saddle connections. ($[0, 1, 2]$ was already discussed above.)*
- *iv) $\{\pm 2, 1, -2\}$. At these points the situation is similar to the case $[0, 1, 0]$ where we have a center and two saddles and two singular points at infinity: a node and point with an elliptic region.*

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These theorems follow from the results of the preceeding sections. The six bifurcation curves components of $\mathcal{C}$ appear on Figure 1 where we represent the projective plane as a disk with $n = 0$ on the circumference, the opposite points being identified. Due to the symmetry of the Hamiltonian *i.e.* $H_{(-a,k,n)}(x, y) = H_{(a,k,n)}(-x, y)$ it is only necessary to draw the bifurcation diagram for systems (2.3) with $a \leq 0$, *i.e.* on the semi-disk with $a \leq 0$. The ellipse $\delta = 0$ divides the semi-disk in two regions: its interior where there are two singular points and its exterior where there are four finite singular points generically. The curve $C = 0$ divides the semi-disk corresponding to $a \leq 0$ in two regions: one with three infinite singular points generically and one with one infinite singular point. Placing also the saddle connection (6.4) as well as the bifurcation line $n - 2k = 0$, and all the phase portraits (except for the orientation of the integral curves which we leave out but which can easily be drawn) obtained by using the results in previous sections, we obtain the bifurcation diagram pictured in Figure 2. We have ten topologically distinct phase portraits, eight of which are located on the bifurcation lines and three in the generic situation:

(a) With two singular cycles which are homoclinic loops;
(b) with just one homoclinic loop and one singular point at infinity;
(c) with one homoclinic loop, three other singular points which are saddles through which pass three distinct irreducible singular cubic solutions and with three singular points at infinity.

For fixed $\lambda = (a, k, n)$ the solution curves lie on curves of the form:

$$F_{(K,\lambda)}(x, y) = H_\lambda(x, y) - K = 0. \tag{7.1}$$

where K is a constant. All the curves (7.1) for fixed λ and variable K pass through the same points at infinity which are given by the cubic terms of $F_{(K,\lambda)}(x, y)$.

THEOREM 7.3. *All finite singular points of a nonlinear quadratic Hamiltonian system with a center (2.3) are ordinary double points for the level curves of the Hamiltonian passing through them except for* $\delta(n - 2k) = 0$. *In this case there is a unique singular point p of (2.3) for which* $F_{(H(p),\lambda)}(x, y) = 0$ *is irreducible having a cusp at p or this curve is reducible with a line component tangent to the conic component at p. All singular points at infinity are nonsingular points for the projective completions of the Hamiltonian level curves (7.1) passing through them. These are points where the curves (7.1) are transversal to the line at infinity except if* $nC = 0$ *when we have curves in (7.1) which are tangent to the line at infinity at the elliptic singular point at infinity of (2.3). For bifurcation points* λ *on only one of the curves: (6.4),* $a = 0$, *one of the singular curves in the family* $F_{(K,\lambda)}(x, y) = 0$ *is reducible with an irreducible conic component and a line component which is not tangent to the conic. If* λ *belongs to both (6.4) and* $a = 0$, *then we have a cubic in (7.1) which either has three line components or a line and an irreducible conic component which are tangent.*

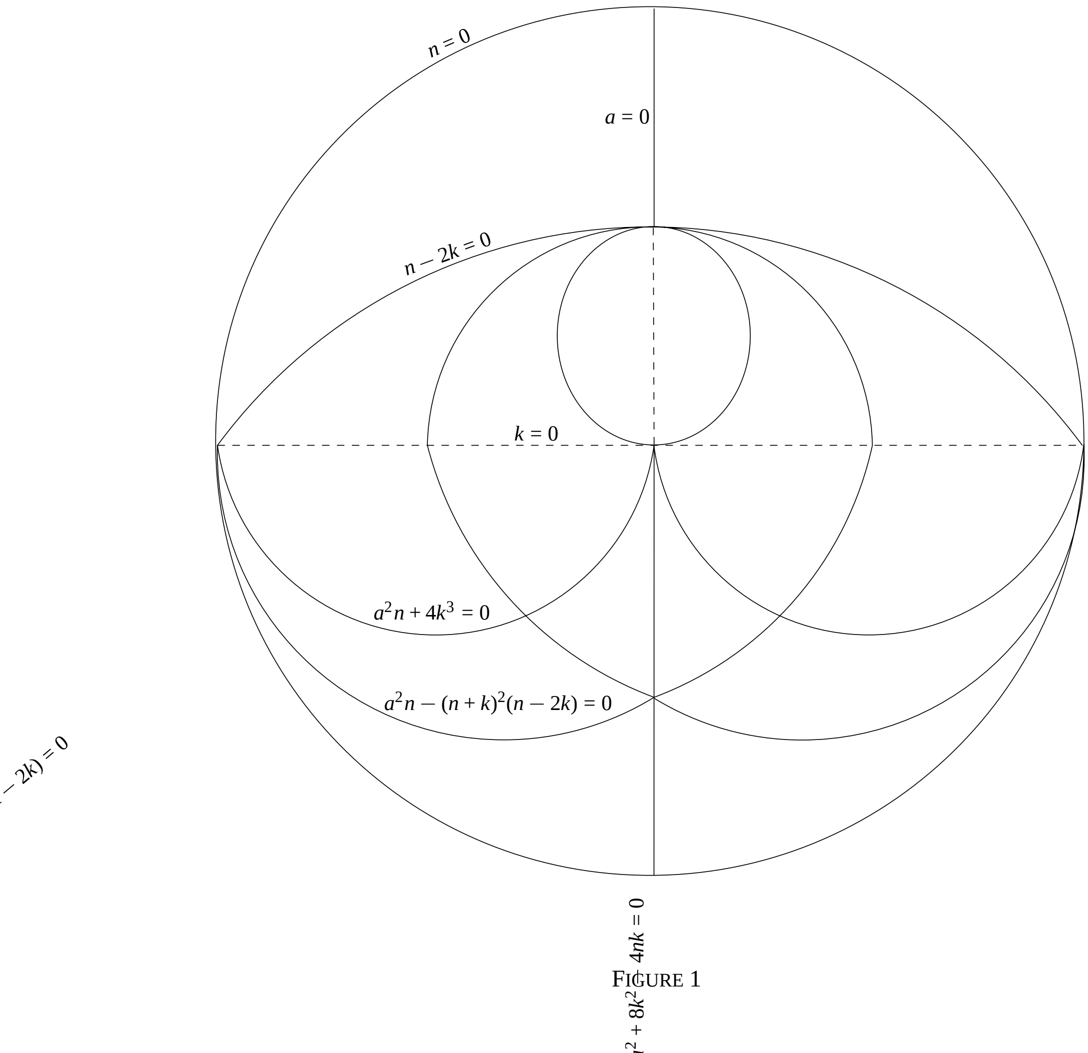

FIGURE 1

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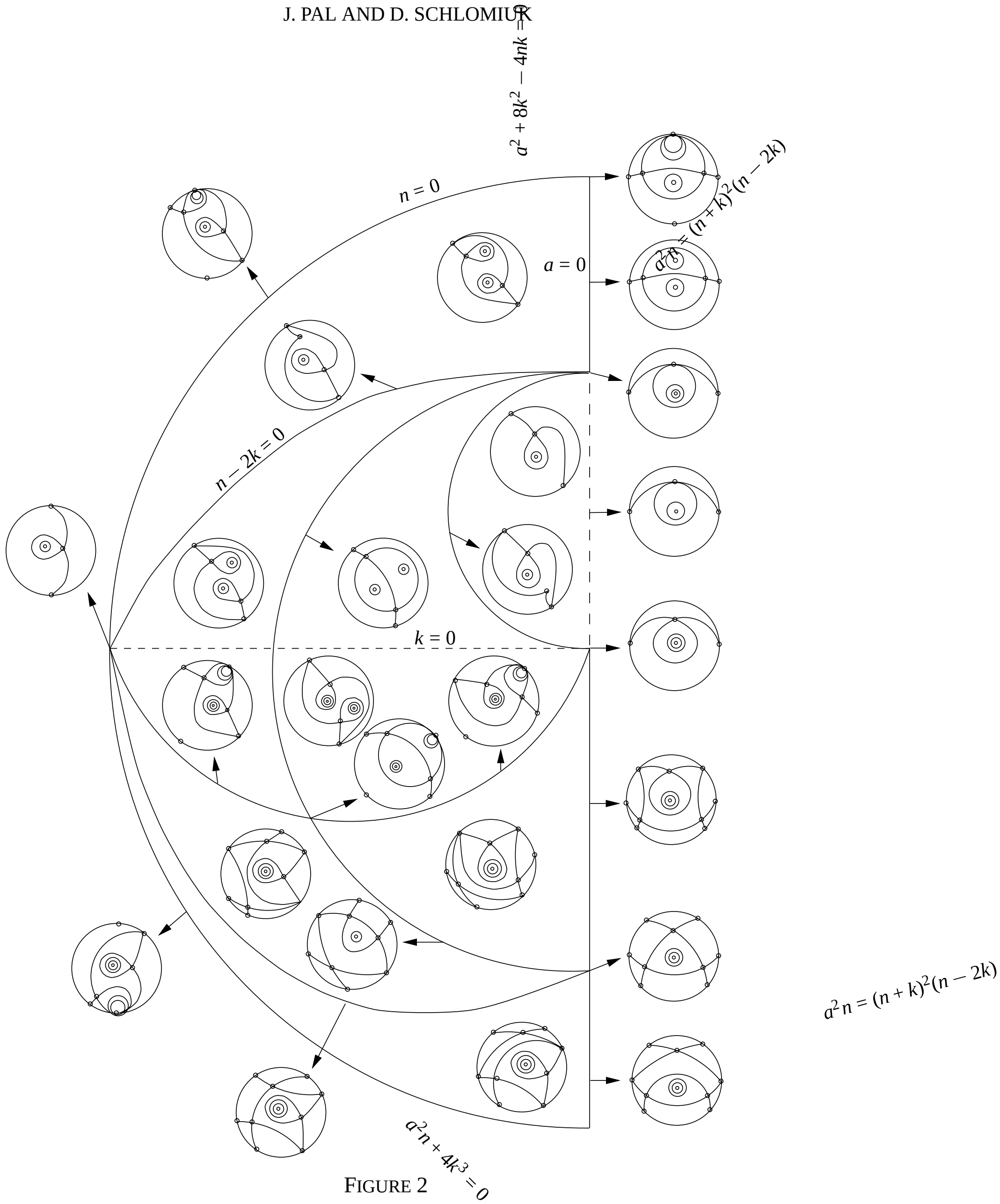

FIGURE 2

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