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# ON MODULES OVER COMMUTATIVE RINGS

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**Abstract**

Our main purpose is to extend several results of interest that have been proved for modules over integral domains to modules over arbitrary commutative rings $R$ with identity. The classical ring of quotients $Q$ of $R$ will play the role of the field of quotients when zero-divisors are present. After discussing torsion-freeness and divisibility (Sections 2–3), we study Matlis-cotorsion modules and their roles in two category equivalences (Sections 4–5). These equivalences are established via the same functors as in the domain case, but instead of injective direct sums $\oplus Q$ one has to take the full subcategory of $Q$-modules into consideration. Finally, we prove results on Matlis rings, i.e. on rings for which $Q$ has projective dimension 1 (Theorem 6.4).

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## 1. Introduction

Throughout $R$ will denote a commutative ring with 1, and all the modules are unital $R$-modules.

There is no universally accepted definition of torsion-freeness for commutative rings with divisors of zero. According to the two most frequently used definitions, a module $M$ is torsion-free if $\mathrm{Tor}_1^R(A, M) = 0$ either for all $A = R/Rr$ with regular, i.e. non-zero-divisor $r \in R$, or for all $A$ of weak dimension less than or equal to 1. The former definition is used, for example, by Bazzoni and Herbera [4], while the second definition is more widely accepted as it gives a more powerful tool in proofs (see, for example, Dauns and Fuchs [7]). A third version is used by Göbel and Trlifaj [11] where for $R/Rr$ in the Tor above any $r \in R$ is admitted. Our main concern here is with the first version (this is what we will mean by torsion-freeness in this paper), which is in line with the classical ring of quotients $Q$ of the ring $R$. In fact, in several cases, $Q$ will play a role similar to the field of quotients in the domain case. However, one should not expect too much from $Q$, as most important features are missing: $Q$ is in general not injective, and injective torsion-free modules are no longer direct sums of indecomposable modules. It came as a pleasant surprise that in spite of these critical deficiencies several important theorems in the domain case can be verified even if the rings have divisors of zero.

We start by discussing the relevant properties of torsion-freeness (Section 2), as well as the accompanying notion of divisibility and $h$-divisibility (Section 3). Most of the results are either well known or easy to prove, but for the sake of completeness we give sketchy proofs. We then focus our attention on Matlis-cotorsion modules. Recall that an $R$-module $M$ is called *Matlis-cotorsion* if it satisfies

$$\mathrm{Ext}^1_R(Q, M) = 0.$$

It is *Enochs-cotorsion* if $\mathrm{Ext}^1_R(F, M) = 0$ for all flat $R$-modules $F$, and *Warfield-cotorsion* if $\mathrm{Ext}^1_R(A, M) = 0$ for all torsion-free $A$. The implications Warfield-cotorsion $\Rightarrow$ Enochs-cotorsion $\Rightarrow$ Matlis-cotorsion are obvious.

In Section 4, we prove the equivalence of the category of $h$-reduced torsion modules to the category of $h$-reduced adjusted Matlis-cotorsion modules (for definitions see below). In the next section we quote a more useful category equivalence due to Matlis [19]: the equivalence of the category of $h$-reduced torsion-free Matlis-cotorsion modules to the category of $h$-divisible torsion modules (Theorem 5.1).

Our final concern is a generalization of Matlis domains to rings with divisors of zero. We define a *Matlis ring* as a ring for which the projective dimension of $Q$ is less than or equal to 1. Results by Angeleri *et al.* [1] generalize well-known results from the domain case even to not necessarily commutative rings that admit left Ore quotient rings. Here we give a new proof of the theorem that $R$ is a Matlis ring if and only if divisible $R$-modules are $h$-divisible if and only if the quotient $K = Q/R$ is a direct sum of countably presented modules (generalizing results by Matlis [17], Hamsher [12] and Lee [14]).

For extensions of several more results on modules over domains to modules over commutative rings, depending on the finitistic dimensions of the rings of quotients, we refer to Fuchs [8].

The symbol $M^\flat = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ will denote the *character module* of the $R$-module $M$. We keep in mind that $M^\flat$ is always pure-injective. We will have occasion to apply the isomorphism

$$\mathrm{Ext}^1_R(A, \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Tor}^R_1(A, B), \mathbb{Q}/\mathbb{Z})$$

or, in a shorter form,

$$\mathrm{Ext}^1_R(A, B^\flat) \cong (\mathrm{Tor}^R_1(A, B))^\flat,$$

valid for all $R$-modules $A, B$ (see Cartan and Eilenberg [6, page 120]). In addition, we will make use of the isomorphism

$$\mathrm{Ext}^1_R(C \otimes_R A, B) \cong \mathrm{Ext}^1_R(C, \mathrm{Hom}_R(A, B)) \tag{1.1}$$

that holds for the $R$-modules $A, B, C$ provided that the conditions

$$\mathrm{Ext}^1_R(A, B) = 0 \quad \text{and} \quad \mathrm{Tor}^R_1(C, A) = 0 \tag{1.2}$$

are satisfied (see Fuchs and Lee [9]).

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Our notation is standard; in particular, we write p.d. for projective, i.d. for injective, and w.d. for weak (i.e. flat) dimension. For unexplained notation and facts we refer to Fuchs and Salce [10] and to Göbel and Trlifaj [11].

## 2. Torsion-freeness

We repeat that $R$ will always denote a commutative ring with identity; most of our results are known if $R$ is an integral domain. $R^\times$ will stand for the set of all regular (i.e. non-zero-divisor) elements in $R$.

Throughout, we write $Q$ for the classical ring of quotients of $R$. It is the direct limit of the projective $R$-modules $r^{-1}R$ (for $r \in R^\times$). Therefore, $Q$ is a flat $R$-module; it is obviously divisible (see below). Evidently, $K = Q/R$ is the direct limit of the cyclically presented modules $r^{-1}R/R$ with $r \in R^\times$. We have w.d.$K = 1$.

A module $T$ is *torsion* if for each $x \in T$ there exists an $r \in R^\times$ such that $rx = 0$. Evidently, submodules of torsion modules are torsion, and the class of torsion modules is closed under extensions. In any module $M$, the set of elements annihilated by some $r \in R^\times$ form a submodule, the *torsion submodule* $t(M)$.

Lemma 2.1. *The following are equivalent for an $R$-module $M$:*

(i) *$M$ is torsion;*
(ii) *$Q \otimes_R M = 0$;*
(iii) *$M \cong \mathrm{Tor}_1^R(K, M)$ (naturally).*

Proof. (i) ⇒ (ii) If $M$ is torsion, then for every $x \in M$, there exists an $r \in R^\times$ such that $rx = 0$. Then for every $q \otimes x \in Q \otimes_R M$, $q \otimes x = (q/r) \otimes rx = 0$.

(ii) ⇒ (iii) This follows from the exact sequence

$$0 \to \mathrm{Tor}_1^R(K, M) \to R \otimes_R M \to Q \otimes_R M \to K \otimes_R M \to 0 \tag{2.1}$$

induced by $0 \to R \to Q \to K \to 0$ (recall $Q$ is flat).

(iii) ⇒ (i) Condition (iii) means that the map $R \otimes_R M \to Q \otimes_R M$ in (2.1) is trivial. Then $1 \otimes x = 0$ $(x \in M)$ in $Q \otimes_R M$, which can happen only if there exists an $r \in R^\times$ such that $rx = 0$ (and 1 is divisible by $r$ in $Q$). □

We call an $R$-module $M$ *torsion-free* if $t(M) = 0$, i.e. if it satisfies the following condition: if $rx = 0$ for an $r \in R^\times$ and some $x \in M$, then $x = 0$. Equivalently, $\mathrm{Hom}_R(R/Rr, M) = 0$ holds for all $r \in R^\times$. From the definition it is evident that submodules of torsion-free modules are torsion-free. A torsion-free module may contain elements with nonzero annihilators; e.g. $R$ itself is such a torsion-free module if it has zero-divisors.

Lemma 2.2.

(i) *For every module $M$, there is an exact sequence*

$$0 \to t(M) \to M \to M/t(M) \to 0$$

*where $t(M)$ is torsion and $M/t(M)$ is torsion-free.*

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(ii) *For each $M$, there is a natural isomorphism*

$$t(M) \cong \mathrm{Tor}_1^R(K, M).$$

Proof. (i) We have already observed that $t(M)$ is a submodule. It is an easy exercise to show that $M/t(M)$ is torsion-free.

(ii) Clearly, $\mathrm{Tor}_1^R(K, M)$ is the kernel of the natural map between the first two tensor products in (2.1), and evidently, exactly the torsion elements in $M$ are annihilated by $Q$ in $Q \otimes_R M$. □

Lemma 2.3. *The following are equivalent for an $R$-module $M$:*

(i) *$M$ is torsion-free;*
(ii) $\mathrm{Tor}_1^R(R/Rr, M) = 0$ *for all $r \in R^\times$;*
(iii) $\mathrm{Tor}_1^R(K, M) = 0$*;*
(iv) *the natural map $M \to Q \otimes_R M$ is a monomorphism.*

Proof. (i) ⇔ (ii) $\mathrm{Tor}_1^R(R/Rr, M) = 0$ means that the natural map $Rr \otimes_R M \to R \otimes_R M \cong R$ is monic. Clearly, $r \otimes x = 0$ $(x \in M)$ in $R \otimes_R M$ if and only if $rx = 0$.

(i) ⇔ (iii) Follows from Lemma 2.2.

(iii) ⇔ (iv) It is obvious that (iv) holds if and only if (iii) holds. □

The following lemma will be needed later on; purity is used in the usual sense (P. M. Cohn).

Lemma 2.4. *A pure extension of a torsion-free module by a torsion module is splitting.*

Proof. Let $0 \to M \to N \xrightarrow{\phi} T \to 0$ be a pure-exact sequence such that $M$ is torsion-free and $T$ is torsion. For every $Rs \leq T$, there is an epimorphism $\psi : R/Rr \to Rs$ for some $0 \neq r \in R$. As $R/Rr$ is finitely presented, purity ensures that the map $\psi$ lifts to $R/Rr \to N$, thus every $s \in T$ is the image of some $x \in t(N)$ under $\phi$. This means that $\phi|_{t(N)} : t(N) \to T$ is an epimorphism. As $t(N) \cap M = 0$, we have $\mathrm{Ker}\,\phi|_{t(N)} = 0$, i.e. $t(N)$ maps isomorphically upon $T$. Consequently, $N \cong t(N) \oplus M$. □

Example 2.5. To illustrate the above in the special case when the ring $R = S \oplus S$ is the direct sum of two copies of a domain $S$, note that $Q = V \oplus V$ is the classical ring of quotients of $R$ where $V$ denotes the quotient field of $S$. The regular elements in $R$ are $r = (s_1, s_2)$ with $0 \neq s_i \in S$. The module $(V, 0)$ is torsion-free: its annihilator ideal $(0, S)$ contains no regular element; it is an $h$-divisible module (see definition below), the smallest $h$-divisible module containing $(S, 0)$. It is isomorphic to $Q \otimes_R (S, 0)$. The module $(V/S, 0)$ is $h$-divisible torsion.

## 3. Divisibility

An $R$-module $D$ is called *divisible* if for each $r \in R^\times$ and for each $d \in D$, the equation $rx = d$ is solvable for $x$ in $D$. Alternatively, the map $x \mapsto rx$ is surjective in $D$ for all $r \in R^\times$. Define $D$ as *$h$-divisible* if every homomorphism $\phi : R \to D$ extends to some

$\psi : Q \to D$. It is easy to see that this is equivalent to saying that $D$ is an epimorphic image of a direct sum of copies of $Q$. Thus $Q$ is a generator of the subcategory of $h$-divisible $R$-modules.

It is obvious that $h$-divisible modules are divisible, and injective modules are $h$-divisible. Both divisibility and $h$-divisibility are preserved under epimorphic images. It also follows that an extension of a divisible module by another one is also divisible (this fails to hold in general for $h$-divisible modules). Hence in every module $M$, there is a unique maximal divisible submodule $dM$ such that $M/dM$ has no divisible submodule not equal to 0. We say that a module $M$ is *$h$-reduced* if $dM = 0$, i.e. $\mathrm{Hom}_R(Q, M) = 0$.

Following the pattern of [1] and [4], we define a divisible module $\partial_R$ that is a generator of the category of all divisible $R$-modules. $\partial_R$ is generated by the ordered $k$-tuples

$$(r_1, \ldots, r_k)_R \quad \text{for all nonunits } r_i \in R^{\times} \ (k < \omega),$$

including $(\emptyset)_R$ (that generates a submodule $\cong R$). The defining relations are

$$r_1(r_1)_R = (\emptyset)_R \quad \text{and} \quad r_k(r_1, \ldots, r_k)_R = (r_1, \ldots, r_{k-1})_R \quad \text{if } k > 1.$$

It follows immediately that $\partial_R$ generates the subcategory of divisible $R$-modules. Evidently, $\mathrm{p.d.}\partial_R = 1$. It is worthwhile recording the following lemma (though it will not be needed in this paper).

Lemma 3.1. *Let $n$ be any integer $\geq 1$.*

(i) $\mathrm{Tor}_n^R(\partial, A) = 0$ *holds for all torsion-free modules $A$.*
(ii) $\mathrm{Ext}_R^n(\partial, D) = 0$ *for all divisible modules $D$.*

Proof. See, for example, the proof of [10, Lemma 1.3, page 249]. □

Let us point out that $h$-divisible modules are quite abundant: each module is contained in an $h$-divisible module (e.g. in its injective envelope). There are other useful such embeddings, such as the one that preserves weak dimension.

Lemma 3.2. *Every module can be embedded in an $h$-divisible module such that the cokernel is a direct sum of copies of $K$.*

Proof. Given a module $M$, we can construct a push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \oplus R & \longrightarrow & \oplus Q & \longrightarrow & \oplus K & \longrightarrow & 0 \\ & & \alpha\downarrow & & \downarrow\gamma & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & D & \longrightarrow & \oplus K & \longrightarrow & 0 \end{array}$$

where $\alpha$ is chosen as a surjective map, so the same holds for $\gamma$. Thus $D$ is $h$-divisible, whence the claim follows. □

We now give characterizations of divisibility and $h$-divisibility.

Lemma 3.3. *The following are equivalent for an $R$-module $D$:*

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(i) *$D$ is divisible;*
(ii) *$D$ is injective relative to the exact sequence*

$$0 \to Rr \to R \to R/Rr \to 0 \quad \textit{for each } r \in R^{\times};$$

(iii) *$D$ satisfies*

$$\mathrm{Ext}^1_R(R/Rr, D) = 0 \quad \textit{for each } r \in R^{\times}.$$

Proof. The given exact sequence induces the exact sequence

$$0 \to \mathrm{Hom}_R(R/Rr, D) \to \mathrm{Hom}_R(R, D) \to \mathrm{Hom}_R(Rr, D) \to \mathrm{Ext}^1_R(R/Rr, D) \to 0.$$

Hence it follows that (i)–(iii) are equivalent. □

Lemma 3.4.

(a) *For an $h$-divisible $D$, there is a natural isomorphism*

$$\mathrm{Ext}^1_R(K, D) \cong \mathrm{Ext}^1_R(Q, D).$$

(b) *$D$ is $h$-divisible if it satisfies* $\mathrm{Ext}^1_R(K, D) = 0$.

Proof. (a) The standard exact sequence $0 \to R \to Q \to K \to 0$ induces the exact sequence

$$\mathrm{Hom}_R(Q, D) \to \mathrm{Hom}_R(R, D) \to \mathrm{Ext}^1_R(K, D) \to \mathrm{Ext}^1_R(Q, D) \to 0. \tag{3.1}$$

The map between the Homs is surjective if $D$ is $h$-divisible, thus $\mathrm{Ext}^1_R(K, D) \cong \mathrm{Ext}^1_R(Q, D)$ naturally.

(b) This is evident from (3.1). □

We shall in due course prove that the Exts in (a) vanish whenever $R$ is a Matlis ring (Lemma 3.10).

Some useful facts about the relations between certain modules and their character modules are presented in the next theorem.

Theorem 3.5.

(i) *$M$ is a torsion module if and only if $M^\flat$ is $h$-reduced.*
(ii) *$M$ is torsion-free if and only if $M^\flat$ is $h$-divisible.*
(iii) *$M$ is divisible exactly if $M^\flat$ is torsion-free.*

Proof. (i) Recall that $M$ is torsion exactly if $Q \otimes_R M = 0$. Therefore, we have 0 on the left-hand side of the isomorphism $\mathrm{Hom}_R(Q, M^\flat) \cong (Q \otimes_R M)^\flat$ if and only if $M$ is torsion.

(ii) $M$ is torsion-free means that $\mathrm{Tor}^R_1(K, M) = 0$. The isomorphism $\mathrm{Ext}^1_R(K, M^\flat) \cong (\mathrm{Tor}^R_1(K, M))^\flat$ shows (see (1.1)) that $M^\flat$ is $h$-divisible if and only if $M$ is torsion-free (Lemma 3.4(b)).

(iii) $M$ is divisible if and only if $(R/Rr) \otimes_R M = 0$ for all $r \in R^{\times}$. The proof is as in (i), using $R/Rr$ in place of $Q$. □

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The following lemma is an analogue of a result in Lee [16] proved for domains (in which case $A$ need not be torsion).

Lemma 3.6. *Assume* $0 \to D \to M \to A \to 0$ *is an exact sequence where* $D$ *is divisible and* $A$ *is torsion of* w.d. $\leq 1$. *Then* $D$ *is pure in* $M$.

Proof. Consider the exact sequence of character modules

$$0 \to A^\flat \to M^\flat \to D^\flat \to 0.$$

Here w.d.$A \leq 1$ implies i.d.$A^\flat \leq 1$, and $D$ divisible implies that $D^\flat$ is torsion-free (Theorem 3.5). By the same theorem and Theorem 4.4, $A^\flat$ is Warfield-cotorsion (it is $h$-reduced), so the displayed sequence splits. This is known to be equivalent to the purity of $D$ in $M$. □

From Lemma 2.3 it follows that for a torsion-free module $M$, the natural map $M \to Q \otimes_R M$ is monic. In this case, $Q \otimes_R M$ is the $h$-divisible hull of $M$, i.e. the smallest $h$-divisible module containing $M$. It is likewise torsion-free.

Lemma 3.7. *The following statements hold over every commutative ring:*

(i) *divisible torsion-free modules are h-divisible;*
(ii) *if* $D$ *is h-divisible torsion-free, then* $Q \otimes_R D \cong D$;
(iii) *the torsion submodule of an h-divisible module is also h-divisible.*

Proof. (i) Let $D$ be a divisible torsion-free $R$-module and $0 \to D \to M \to K \to 0$ an exact sequence. By Lemma 3.6, this sequence is pure-exact, and therefore by Lemma 2.4 it is splitting. Hence $\mathrm{Ext}^1_R(K, D) = 0$, and $D$ is $h$-divisible by Lemma 3.4.

(ii) In the exact sequence $0 \to \mathrm{Tor}^R_1(K, D) \to D \to Q \otimes_R D \to K \otimes_R D \to 0$ obtained from $0 \to R \to Q \to K \to 0$, both $\mathrm{Tor}^R_1(K, D)$ and $K \otimes_R D$ are 0 since $D$ is torsion-free divisible and $K$ is torsion. Hence the result follows.

(iii) Since $K$ is a torsion module, the image of any extension of a map $\phi : R \to t(M)$ to $\bar{\phi} : Q \to M$ must also be contained in $t(M)$. □

The following lemma deals with divisible torsion-free modules; these are precisely the $Q$-modules (viewed as $R$-modules), as is clear from (i) and (ii) in the last lemma.

Lemma 3.8. *Assume* $D$ *is a divisible torsion-free module. Then*

(i) $\mathrm{Hom}_R(Q, D) \cong D$;
(ii) $\mathrm{Ext}^1_R(Q, D) = 0$;
(iii) $\mathrm{Ext}^1_R(T, D) = 0$ *for all torsion modules* $T$.

Proof. By the proof of Lemma 3.7(i), $\mathrm{Ext}^1_R(K, D) = 0$. Claims (i)–(ii) follow from Lemma 3.4 and from the exact sequence (3.1).

(iii) First of all, we observe that torsion-free divisible modules are $h$-divisible; see Lemma 3.7(i). Then we refer to the isomorphism

$$\mathrm{Ext}^1_R(T \otimes_R Q, D) \cong \mathrm{Ext}^1_R(T, \mathrm{Hom}_R(Q, D))$$

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that holds for all torsion-free $h$-divisible $D$ and for every module $T$, since conditions (1.2) are satisfied. Now $\mathrm{Hom}_R(Q, D) \cong D$ by (i), and the tensor product vanishes whenever $T$ is torsion. Thus the isomorphism reduces to the desired equality $\mathrm{Ext}^1_R(T, D) = 0$. □

We point out that (ii) holds for all $h$-divisible modules $D$ provided $R$ is a Matlis ring (see Section 6). Indeed, if p.d.$Q \leq 1$, then $\mathrm{Ext}^1_R(Q, D) = 0$ implies that the same is true if $D$ is replaced by an epic image.

Lemma 3.9. *Let $D$ be an $h$-divisible torsion-free module.* Hom, Ext, *and the tensor product are $h$-divisible torsion-free modules if $D$ is one of the arguments in them.*

Proof. Multiplication by $r \in R^\times$ is an automorphism of $D$, which induces automorphisms on the indicated functors. □

The following result is concerned with modules of p.d. $\leq 1$.

Lemma 3.10.

(i) *If an $R$-module $M$ satisfies $\mathrm{Ext}^1_R(M, D) = 0$ for all $h$-divisible modules $D$, then* p.d.$M \leq 1$.
(ii) *If* p.d.$K = 1$, *then* $\mathrm{Ext}^1_R(K, D) = 0$ *for all $h$-divisible modules $D$.*
(iii) *Claim (ii) holds for $Q$ (in place of $K$) as well.*

Proof. (i) Given any module $N$, let $0 \to N \to E \to D \to 0$ be an exact sequence where $E$ is injective. As $D$ is $h$-divisible, we have $\mathrm{Ext}^1_R(M, D) = 0$ by hypothesis. Thus the sequence $0 \to \mathrm{Ext}^2_R(M, N) \to \mathrm{Ext}^2_R(M, E) = 0$ is exact, whence p.d.$M \leq 1$ is evident.

(ii) For an $h$-divisible $D$, there is an exact sequence $0 \to N \to E \to D \to 0$ with $E \cong \oplus Q$. Hence we obtain the exactness of $0 \to \mathrm{Ext}^1_R(K, D) \to \mathrm{Ext}^2_R(K, N)$ as $\mathrm{Ext}^1_R(K, E) = 0$ by Lemma 3.8(iii). Here $\mathrm{Ext}^2_R$ is 0, whenever p.d.$K = 1$, so $\mathrm{Ext}^1_R(K, D) = 0$. The proof of (iii) is the same. □

Observe that in the first part of the last proof, it suffices to assume that the equality $\mathrm{Ext}^1_R(M, D) = 0$ holds for those $D$ that are epic images of injectives.

## 4. Matlis-cotorsion modules

In this section, we consider cotorsion modules. By Lemma 3.8, all torsion-free divisible $R$-modules are Matlis-cotorsion. From Lemma 3.10(ii) it follows that all $h$-divisible modules are Matlis-cotorsion if and only if p.d.$K = 1$ (i.e. $R$ is a Matlis ring in the sense defined below). In view of this, we are now focusing our attention on the $h$-reduced Matlis-cotorsion modules.

Lemma 4.1. *If $M$ is an $h$-reduced Matlis-cotorsion module, then*

$$\mathrm{Ext}^1_R(D, M) = 0$$

*for all $h$-divisible torsion-free modules $D$.*

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Proof. We refer to the isomorphism (1.1) that will now use with the following cast of characters: $A = Q$, $B = M$ and $C = D$. As evidently $\mathrm{Tor}_1^R(Q, D) = 0$, and $\mathrm{Ext}_R^1(Q, M) = 0$ holds by definition, we obtain

$$\mathrm{Ext}_R^1(D \otimes_R Q, M) \cong \mathrm{Ext}_R^1(D, \mathrm{Hom}_R(Q, M)).$$

In view of $D \otimes_R Q \cong D$ and $\mathrm{Hom}_R(Q, M) = 0$, the claim follows. □

Lemma 4.2. *Let $M$ be an $h$-reduced $R$-module. Then $\overline{M} = \mathrm{Ext}_R^1(K, M)$ is an $h$-reduced Matlis-cotorsion module containing $M$ such that $\overline{M}/M$ is torsion-free divisible.*

Proof. From the standard exact sequence $0 \to R \to Q \to K \to 0$ we derive the exact sequence

$$0 \to M \to \mathrm{Ext}_R^1(K, M) \to \mathrm{Ext}_R^1(Q, M) \to 0.$$

By Lemma 3.9, $\mathrm{Ext}_R^1(Q, M)$ is $h$-divisible torsion-free, establishing the second part of the statement. Hence we get the long exact sequence

$$0 \to \mathrm{Hom}_R(Q, \mathrm{Ext}_R^1(K, M)) \to \mathrm{Hom}_R(Q, \mathrm{Ext}_R^1(Q, M)) \to$$

$$\to \mathrm{Ext}_R^1(Q, M) \to \mathrm{Ext}_R^1(Q, \mathrm{Ext}_R^1(K, M)) \to \mathrm{Ext}_R^1(Q, \mathrm{Ext}_R^1(Q, M)) = 0.$$

As the map between Hom and Ext is a natural isomorphism, we obtain both equalities, $\mathrm{Hom}_R(Q, \mathrm{Ext}_R^1(K, M)) = 0$ and $\mathrm{Ext}_R^1(Q, \mathrm{Ext}_R^1(K, M)) = 0$, completing the proof. □

Now suppose $T$ is an $h$-reduced torsion module. We then have the exact sequence $0 \to T \to \mathrm{Ext}_R^1(K, T) \to \mathrm{Ext}_R^1(Q, T) \to 0$ where the middle term is $\overline{T}$. Note that $T$ is the torsion submodule of $\overline{T}$. Moreover, $\overline{T}$ is a module without nonzero torsion-free summand, since a torsion-free summand must be $h$-divisible, and Lemma 4.2 tells us that $\overline{T}$ is $h$-reduced. This observation leads to the following theorem (we call an $h$-reduced module *adjusted* if all of its torsion-free epic images are divisible).

Theorem 4.3. *There is a category equivalence between the category $\mathcal{T}$ of $h$-reduced torsion modules $T$ and the category $\mathcal{M}$ of $h$-reduced adjusted Matlis-cotorsion modules $M$. It is given by the functors*

$$\mathcal{T} \to \mathcal{M} : T \mapsto \mathrm{Ext}_R^1(K, T) = \overline{T}, \quad \mathcal{M} \to \mathcal{T} : M \to \mathrm{Tor}_1^R(K, M) = tM.$$

Proof. The preceding lemma and the remark before the theorem prove that $\overline{T} \in \mathcal{M}$. That $M \in \mathcal{M}$ implies $t(M) \in \mathcal{T}$ is trivial. Finally, we have $\mathrm{Tor}_1^R(K, \mathrm{Ext}_R^1(K, T)) \cong T$, since $T$ is the torsion submodule of $\mathrm{Ext}_R^1(K, T)$, and $\mathrm{Ext}_R^1(K, \mathrm{Tor}_1^R(K, M)) \cong M$, since $\mathrm{Ext}_R^1(K, M) = \mathrm{Ext}_R^1(K, t(M))$ if $M$ is adjusted. □

For the proof of Proposition 5.3, we will need the following result on Warfield-cotorsion modules. It is of interest in its own right, as it shows that the $h$-reduced Warfield-cotorsion modules can be characterized in the same fashion as in the domain case.

Theorem 4.4. *An $h$-reduced $R$-module $M$ is Warfield-cotorsion if and only if:*

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(i) $\mathrm{Ext}^1_R(Q, M) = 0$, *i.e. it is Matlis-cotorsion, and*
(ii) i.d.$M \leq 1$.

Proof. Assuming $M$ is Warfield-cotorsion, (i) is obvious as $Q$ is torsion-free. In order to verify (ii), start with a presentation $0 \to H \to F \to N \to 0$ of an arbitrary $R$-module $N$, where $F$ is free. We obtain

$$0 = \mathrm{Ext}^1_R(H, M) \to \mathrm{Ext}^2_R(N, M) \to \mathrm{Ext}^2_R(F, M) = 0,$$

where the first Ext vanishes because $H$ is torsion-free. Hence (ii) holds too.

Conversely, let $M$ be an $h$-reduced $R$-module satisfying (i)–(ii). Consider the exact sequence $0 \to R \to Q \to K \to 0$ and its induced exact sequence $0 = \mathrm{Tor}^R_1(K, A) \to A \to Q \otimes_R A \to K \otimes_R A \to 0$ for any torsion-free $R$-module $A$. This sequence induces the exact sequence

$$\mathrm{Ext}^1_R(Q \otimes_R A, M) \to \mathrm{Ext}^1_R(A, M) \to \mathrm{Ext}^2_R(K \otimes_R A, M) = 0,$$

where the last Ext is 0 as a consequence of (ii). Also the first Ext vanishes, because w.d.$Q = 0$ along with condition (i) implies that the isomorphism

$$\mathrm{Ext}^1_R(Q \otimes_R A, M) \cong \mathrm{Ext}^1_R(A, \mathrm{Hom}_R(Q, M))$$

(see (1.1)) is correct, and due to the $h$-reducedness of $M$, its right-hand side vanishes. We conclude that also $\mathrm{Ext}^1_R(A, M) = 0$, completing the proof. □

In concluding this section, let us point out that with a slight variance of the proof of Theorem 4.4 (replacing the arbitrary $N$ by a module of w.d. $\leq 1$ and the torsion-free $A$ by a flat module), we get the following analogous result.

Theorem 4.5. *An $h$-reduced $R$-module $M$ is Enochs-cotorsion if and only if*
(i) $\mathrm{Ext}^1_R(Q, M) = 0$, *and*
(ii) $\mathrm{Ext}^2_R(A, M) = 0$ *for all modules $A$ of* w.d. $\leq 1$. □

## 5. The general Matlis category equivalence

Let $M$ be $h$-reduced torsion-free and Matlis-cotorsion. Consider the exact sequence

$$0 \to M \to Q \otimes_R M \to D \to 0, \tag{5.1}$$

where (as we will see in a moment) $D$ is $h$-divisible torsion. If we associate $D$ to $M$, then we get a functor from the category $\mathcal{C}$ of $h$-reduced torsion-free Matlis-cotorsion $R$-modules to the category $\mathcal{D}$ of $h$-divisible torsion $R$-modules. For an integral domain $R$, this is known as the Matlis category equivalence [18] (that generalizes the Harrison category equivalence in abelian groups [13]). Since it is not well known that Matlis [19] proved that this is a genuine category equivalence even if $R$ admits divisors of zero, we give a short proof.

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THEOREM 5.1. *For every commutative ring $R$, there is a category equivalence between the category $\mathcal{C}$ of $h$-reduced torsion-free Matlis-cotorsion $R$-modules $M$ and the category $\mathcal{D}$ of $h$-divisible torsion $R$-modules $D$. The correspondences are given by the functors*

$$\mathcal{C} \to \mathcal{D} : M \mapsto D \cong K \otimes_R M, \quad \mathcal{D} \to \mathcal{C} : D \mapsto M \cong \mathrm{Hom}_R(K, D).$$

*The basic relation between corresponding modules is shown in (5.1).*

PROOF. First, assume that $M$ is $h$-reduced torsion-free. By Lemma 2.3(iv), the natural map $M \to Q \otimes_R M$ is an injection, so we can form the exact sequence

$$0 \to M \to Q \otimes_R M \to K \otimes_R M \to 0 \tag{5.2}$$

which we get directly from the exact sequence $0 \to R \to Q \to K \to 0$. Evidently, $D \cong K \otimes_R M$ holds by the naturality of the maps in (5.2), thus $D$ in (5.1) is $h$-divisible torsion. Since $0 \to M \to \overline{M} = \mathrm{Ext}^1_R(K, M) \to A \to 0$ (with a divisible torsion-free $A$) implies $0 \to K \otimes_R M \to K \otimes_R \overline{M} \to 0$, it is clear that in (5.2) we may suppose that $M$ is chosen to be Matlis-cotorsion. This makes it unique (as the next paragraph will show).

Conversely, let $D$ be any $h$-divisible torsion module. There is an $h$-divisible torsion-free module $E \cong \mathrm{Hom}_R(Q, D)$ fitting in the exact sequence $0 \to M \to E \to D \to 0$ with an $h$-reduced torsion-free $M \cong \mathrm{Hom}_R(K, D)$ (this is an exact sequence because of Lemma 3.4(a)). Hence we get the induced exact sequence

$$0 = \mathrm{Tor}^R_1(Q, D) \to Q \otimes_R M \to Q \otimes_R E \cong E \to Q \otimes_R D = 0,$$

thus $E \cong Q \otimes_R M$. Furthermore,

$$0 = \mathrm{Hom}_R(K, E) \to \mathrm{Hom}_R(K, D) \to \mathrm{Ext}^1_R(K, M) \to \mathrm{Ext}^1_R(K, E) = 0,$$

whence $\mathrm{Hom}_R(K, D) \cong \overline{M}$. This establishes the claim. □

The middle term in the Matlis exact sequence (5.2) is an essential extension of $M$, and if $R$ is a domain, then it is the injective envelope of $M$. Thus (5.2) is then also a short injective resolution of $M$. This is no longer true for rings with zero-divisors, but there is still a remarkable connection between the two exact sequences, as demonstrated by the next theorem (which was proved more generally for torsion theories in [20, Proposition 2.4, page 202]).

THEOREM 5.2. *The middle term $Q \otimes_R M$ in the Matlis exact sequence (5.2) is the maximal extension of the torsion-free Matlis-cotorsion $M$ by a torsion module that is contained in an injective envelope of $M$.*

PROOF. Let $0 \to M \to E \to D \to 0$ be an exact sequence, where $E$ is an injective envelope of the torsion-free $M$. As $E$ is both torsion-free and divisible, we have 0s at both ends of the induced exact sequence

$$0 = \mathrm{Tor}^R_1(K, E) \to \mathrm{Tor}^R_1(K, D) \to K \otimes_R M \to K \otimes_R E = 0.$$

Hence $\mathrm{Tor}^R_1(K, D) \cong K \otimes_R M$, thus $K \otimes_R M$ is isomorphic to the torsion submodule of $D$ (Lemma 2.2). This establishes the claim. □

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We proceed to special cases of the Matlis category equivalence that are particularly important. To explore some of them, we need two definitions. An $R$-module $M$ is said to be *weak-injective* if $\mathrm{Ext}^1_R(A, M) = 0$ for all $A$ of weak dimension less than or equal to 1 (Lee [15]). *Perfect rings* were defined by Bass [2] as rings over which flatness and projectivity are equivalent. If $Q$ is perfect, then the projective and flat dimensions of $Q$-modules are either 0 or $\infty$.

Proposition 5.3. *In the above category equivalence, we have the following correspondences:*

*$h$-reduced torsion-free Warfield-cotorsion $\leftrightarrow$ injective torsion;*

*and if $Q$ is a perfect ring, then also*

*$h$-reduced torsion-free Enochs-cotorsion $\leftrightarrow$ weak-injective torsion.*

Proof. (i) Let $M$ be an $h$-reduced torsion-free Matlis-cotorsion module. If $K \otimes_R M$ is injective, then for a torsion-free module $F$ we have

$$\mathrm{Ext}^1_R(F, M) = \mathrm{Ext}^1_R(F, \mathrm{Hom}_R(K, K \otimes_R M)) \cong \mathrm{Ext}^1_R(F \otimes_R K, K \otimes_R M)) = 0$$

where (1.1) was applied (the hypotheses (1.2) for (1.1) being satisfied by the injectivity of $K \otimes_R M$ and $\mathrm{Tor}^R_1(F, K) = 0$). Hence $M$ is Warfield-cotorsion.

Conversely, assume $M$ is $h$-reduced torsion-free and Warfield-cotorsion. By Lemma 4.4, in its injective resolution $0 \to M \to E \to D \to 0$ both $E$ and $D$ are injective. From Theorem 5.2 we deduce that $K \otimes_R M$ is isomorphic to the torsion submodule of $D$, so it is injective.

(ii) Suppose $D$ is a weak-injective torsion module, and let $0 \to M \to G \xrightarrow{\alpha} D \to 0$ be a flat cover sequence of $D$ (cf. [5]). Then $E = Q \otimes_R G$ is flat, and so w.d.$E/G \leq 1$, and $\alpha$ extends to $\beta : E \to D$. By the cover property, there is a $\gamma : E \to G$ such that $\alpha\gamma = \beta$. Clearly, $\gamma$ must be the identity on $G$, so $G$ is a summand of $E$. This means that $G$ is divisible, whence we conclude that in the given exact sequence $M$ and $D$ correspond to each other in the Matlis category equivalence. As the kernel of a flat cover, $M$ is Enochs-cotorsion. (The hypothesis on $Q$ is not needed in this part of the proof.)

For the converse, assume that in the Matlis exact sequence $0 \to M \to Q \otimes_R M \to K \otimes_R M = D \to 0$ the $h$-reduced torsion-free module $M$ is Enochs-cotorsion. Then for any $A$ of w.d. $\leq 1$, in the induced exact sequence

$$\mathrm{Ext}^1_R(A, Q \otimes_R M) \to \mathrm{Ext}^1_R(A, D) \to \mathrm{Ext}^2_R(A, M) \tag{5.3}$$

the last term is 0 by Theorem 4.5. By virtue of (1.1), we have the isomorphism $\mathrm{Ext}^1_R(A, Q \otimes_R M) \cong \mathrm{Ext}^1_R(A \otimes_R Q, Q \otimes_R M)$. As w.d.$A \otimes_R Q \leq 1$ both as an $R$- and as a $Q$-module, the hypothesis on $Q$ implies that $A \otimes_R Q$ is a projective $Q$-module. Therefore, the first Ext in (5.3) is also 0, and we conclude that the middle Ext vanishes, i.e. $D$ is weak-injective. □

The next result exhibits a special case where the Matlis exact sequence coincides with the injective resolution sequence.

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Corollary 5.4. *If $M$ is an $h$-reduced torsion-free Warfield-cotorsion $R$-module, then $Q \otimes_R M$ is its injective envelope.*

Proof. By Proposition 5.3, from the Matlis exact sequence we deduce

$$0 = \mathrm{Ext}^1_R(F, M) \to \mathrm{Ext}^1_R(F, Q \otimes_R M) \to \mathrm{Ext}^1_R(F, K \otimes_R M) = 0$$

for any $M$ as stated and for any torsion-free $F$. Hence $Q \otimes_R M$ is Warfield-cotorsion. Manifestly, $M$ is essential in $Q \otimes_R M$.

It remains to verify that a torsion-free divisible Warfield-cotorsion module $N$ is injective. As $\mathrm{Tor}^R_1(Q, C) = 0$ for any module $C$ and $\mathrm{Ext}^1_R(Q, N) = 0$, we can make use of (1.1) to obtain the isomorphism

$$\mathrm{Ext}^1_R(C \otimes_R Q, N) \cong \mathrm{Ext}^1_R(C, \mathrm{Hom}_R(Q, N)).$$

The first Ext is 0 ($C \otimes_R Q$ is always torsion-free), and since $N$ is torsion-free divisible, the right-hand side Ext is $\cong \mathrm{Ext}^1_R(C, N)$. Thus $\mathrm{Ext}^1_R(C, N) = 0$ for every $C$, i.e. $N$ is injective. □

## 6. Matlis rings

Recall that a Matlis domain is an integral domain such that p.d.$Q = 1$; see [10]. Accordingly, a ring will be called a *Matlis ring* if p.d.$Q = 1$. Then p.d.$K = 1$ as well.

The next theorem is well known for modules over domains; interestingly, it holds over all commutative rings. Part (a) is vacuous if $K$ does not have any summand of p.d. $\leq 1$, so the interesting case is (b), generalizing Lee [14]. In the proof, we use the category equivalence verified above.

Theorem 6.1.

(a) *An $h$-divisible torsion module of* p.d. $\leq 1$ *is a summand of a direct sum of copies of $K$.*

(b) *If $R$ is a Matlis ring, then the converse is also true.*

Proof. (a) Let $T$ be $h$-divisible torsion of p.d. $\leq 1$, and $M \cong \mathrm{Hom}_R(K, T)$ the $h$-reduced torsion-free Matlis-cotorsion module corresponding to $T$ in the category equivalence of Theorem 5.1, i.e. we have the exact sequence $0 \to M \to Q \otimes_R M \to T \to 0$. If $C$ is any $h$-reduced torsion-free Matlis-cotorsion module, then there is an exact sequence

$$0 = \mathrm{Ext}^1_R(Q \otimes_R M, C) \to \mathrm{Ext}^1_R(M, C) \to \mathrm{Ext}^2_R(T, C) = 0$$

where the first Ext vanishes because of Lemma 4.1. Hence $\mathrm{Ext}^1_R(M, C) = 0$.

Let $F$ be a free module and $\phi : F \to M$ an epimorphism. As $M$ is $h$-reduced Matlis-cotorsion, $\phi$ extends to a homomorphism $\overline{\phi} : \overline{F} \to M$ that is evidently surjective. (Observe that $\overline{F}$ is $h$-reduced.) From the exact sequence $0 \to C = \mathrm{Ker}\,\overline{\phi} \to \overline{F} \to M \to 0$ we obtain the exactness of $0 \to \mathrm{Ext}^1_R(Q, C) \to \mathrm{Ext}^1_R(Q, \overline{F}) = 0$, showing that $\mathrm{Ext}^1_R(Q, C) = 0$, i.e. $C$ is also Matlis-cotorsion (and evidently $h$-reduced). By the preceding paragraph, $\overline{F} \cong C \oplus M$, and therefore, $T \cong K \otimes_R M$ is isomorphic to a summand of $K \otimes_R \overline{F} \cong K \otimes_R F \cong \oplus K$.

(b) For Matlis rings, p.d.$K \leq 1$ makes this claim evident. □

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A submodule $N$ of $M$ is called *tight* if $\mathrm{p.d.}M/N \leq \mathrm{p.d.}M$ (then also $\mathrm{p.d.}N \leq \mathrm{p.d.}M$); see, for example, [10, page 214]. Let $\mathrm{p.d.}M = m \geq 1$. A *tight system* for $M$ is a family $\mathcal{T} = \{M_i \mid i \in I\}$ of tight submodules of $M$ such that:

(i) $0, M \in \mathcal{T}$;
(ii) $\mathcal{T}$ is closed under unions of chains;
(iii) if $M_i < M_j$ in $\mathcal{T}$, then $\mathrm{p.d.}M_j/M_i \leq m$;
(iv) given $M_i \in \mathcal{T}$ and a subset $X \subset M$ of cardinality less than or equal to $\aleph_{m-1}$, there is an $M_j \in \mathcal{T}$ satisfying: $M_i \leq M_j$, $X \subseteq M_j$, and $M_j/M_i$ is $\leq \aleph_{m-1}$-generated.

Tight systems exist over integral domains for all modules of p.d. at most 1. For rings with divisors of zero we have the following weaker result.

Lemma 6.2. *Over a commutative ring $R$, every torsion $R$-module of* p.d. $\leq 1$ *admits a tight system.*

Proof. If $T$ is a torsion module, then $\mathrm{Ext}^1_R(T, D) = 0$ holds for all torsion-free divisible modules $D$ (Lemma 3.8(iii)). If, in addition, $\mathrm{p.d.}T \leq 1$, then it satisfies the hypotheses of Bazzoni *et al.* [3, Lemma 3.1] on every uncountable regular cardinal $\kappa \geq \aleph_1$. '$\aleph_1$-tight system' in their sense is the same as 'tight system' in the sense above. Consequently, tight systems exist for all torsion modules of p.d. $\leq 1$. □

The proof of the following lemma is a modified version of the proof by Hamsher [12, Theorem 2.1]. The latter theorem deals with certain localizations $R_S$ of $R$ at submonoids $S \subseteq R^\times$.

Lemma 6.3. *Suppose $R$ is a Matlis ring, and $S$ is a multiplicative monoid of elements in $R^\times$. If* $\mathrm{p.d.}Q/R_S \leq 1$, *then $R_S/R$ is a direct summand of $K$.*

Proof. In the paper [12], the proofs of Proposition 1.2, Corollaries 1.3 and 1.4, and Lemma 2.2 (ii) do not use the domain property of the commutative ring $R$, only the hypothesis that $\mathrm{p.d.}Q \leq 1$. Hence they establish our claim as well. □

In the following proof, by a *$G(\aleph_0)$-family in $M$* is meant a collection $\mathcal{G}$ of submodules of $M$ such that: (i) $0, M \in \mathcal{G}$; (ii) $\mathcal{G}$ is closed under unions of chains; (iii) if $A \in \mathcal{G}$ and $X$ is a countable subset of $M$, then there is a $B \in \mathcal{G}$ such that $A \leq B, X \subseteq B$ and $B/A$ is countably generated. We can now verify our main theorem on Matlis rings, generalizing theorems by Matlis [17] and Lee [14].

Theorem 6.4. *For a commutative ring $R$, the following conditions are equivalent:*

(i) *$R$ is a Matlis ring;*
(ii) *$K$ is a direct sum of countably presented submodules;*
(iii) *divisible $R$-modules are h-divisible.*

Proof. (i) ⇒ (ii) We begin by noting that by Lemma 6.2, $K$ admits a tight system $\mathcal{T}$. Furthermore, the collection $\mathcal{G}$ of submodules $R_S/R$ with $\mathrm{p.d.}Q/R_S \leq 1$ is a $G(\aleph_0)$-family of summands in $K$ where the $S$ are submonoids of $R^\times$ (see [10, page 141]).

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Evidently, the intersection of two $G(\aleph_0)$-families,

$$\mathcal{T}^* = \mathcal{T} \cap \mathcal{G} = \{T \in \mathcal{T} \mid T \text{ is a summand of } K\},$$

is again a $G(\aleph_0)$-family; it is likewise a tight system in $K$. With the aid of this $\mathcal{T}^*$, we use a transfinite process to construct a continuous well-ordered ascending chain

$$0 = K_0 < K_1 < \cdots < K_\alpha < \cdots < K_\tau = K \quad (\alpha < \tau)$$

of submodules $K_\alpha \in \mathcal{T}^*$ such that for each $\alpha < \tau$, $K_{\alpha+1}/K_\alpha$ is a countably generated module of p.d. $\leq 1$. It is obvious that each submodule in this chain is a direct summand of the next one, so that we can write $K_{\alpha+1} = K_\alpha \oplus A_\alpha$ where $A_\alpha$ is a countably generated divisible module. It is even countably presented, since from the proof it is clear that it is isomorphic to $R_S/R$ for some countable monoid $S$. Hence $K = \bigoplus_\alpha A_\alpha$, and (ii) follows.

(ii) $\Rightarrow$ (iii) The proof of [10, Theorem 2.8, page 253] applies to conclude that every element of a torsion divisible module $D$ is contained in an $h$-divisible submodule. Thus every such $D$ is $h$-divisible.

(iii) $\Rightarrow$ (i) Condition (iii) implies that the module $\partial$ (in Section 3) is $h$-divisible. Let $\phi : R \to \partial$ be the injection such that $\phi(1) = (\emptyset)_R$. By $h$-divisibility, $\phi$ extends to a homomorphism $\psi : Q \to \partial$ which must also be a monic map. It is straightforward to see that $\partial = \psi(Q) \oplus t(\partial)$. Hence p.d.$Q \leq 1$ is immediate. □

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