LONG WAVES ON A ROTATING EARTH IN THE PRESENCE OF A SEMI-INFINITE BARRIER

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(Received 24th September 1963)
(Received in Revised Form 6th December 1963)

Summary

The problem considered is that of long gravity waves approaching, from an arbitrary direction, a semi-infinite barrier, the whole system being in rotation. It is shown that the rotation gives rise to a wave in the shadow region whose amplitude depends upon the angle of incidence, but whose form is independent of it and which travels along the barrier without attenuation in that direction. The work is an extension and simplification of previous work by Crease, involving the use of methods previously developed by the author.

1. Introduction

The problem of long gravity waves approaching a semi-infinite barrier, the whole system being in rotation, arises for certain aspects of an investigation into the origin of storm surges. A full discussion of the oceanographic problems involved has been given in a recent paper (Crease, 1958). In this a solution has been given using Wiener-Hopf techniques. It will be shown in this paper that these long and complicated techniques are not in fact necessary for the solution of this problem, and that the solution for an arbitrary angle of incidence (it having been assumed in the previous treatment that the barrier was parallel to the wave crests) can be obtained quite simply using a technique for the solution of the Sommerfeld problem developed by the author (**1**) in 1954.

The equations associated with long-wave theory are well known (e.g. see (**3**)) but will for convenience be reproduced here. The horizontal equation of motion is

$$\frac{\partial \boldsymbol{q}}{\partial t} + \Omega \boldsymbol{k}_\times \boldsymbol{q} = -g\nabla\zeta$$

where $\boldsymbol{q}$ is the horizontal fluid velocity, ζ is the elevation of the free surface above its mean level, ∇ the two dimensional gradient operator, $\boldsymbol{k}$ unit vector vertically upwards and Ω the Coriolis parameter ($= 2\omega \sin\alpha$ where ω is the angular velocity of the earth and α the north latitude).

The continuity equation is

$$\nabla . \boldsymbol{q} = -\frac{1}{h}\frac{\partial \zeta}{\partial t}.$$

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It may be seen that a solution is given by

$$q = \frac{1}{\Omega}\frac{\partial}{\partial t}\nabla A - k_{\times}\nabla A$$

$$\zeta = -\frac{h}{\Omega}\left(\frac{1}{gh}\frac{\partial^2 A}{\partial t^2} + \frac{\Omega^2}{gh}A\right)$$

where $\nabla^2 A - \dfrac{\Omega^2}{gh}A - \dfrac{1}{gh}\dfrac{\partial^2 A}{\partial t^2} = 0.$

It can fairly easily be seen that, if a time variation $\exp\{i\sigma t\}$ be imposed, the elevation ζ obeys a differential equation

$$\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2} + k^2\zeta = 0 \quad \text{.................................(1)}$$

$$\left(k^2 = \frac{\sigma^2}{gh} - \frac{\Omega^2}{gh} > 0\right)$$

and the boundary condition of zero fluid velocity across a boundary with normal n is

$$ip\frac{\partial\zeta}{\partial s} + \frac{\partial\zeta}{\partial n} = 0,$$

where $p = \Omega/\sigma < 1$.

If the semi-infinite barrier is $x > 0$, $y = 0$ (i.e. the positive x-axis), the boundary condition is

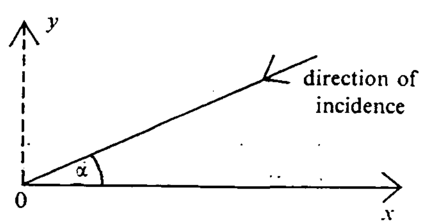
$$\frac{\partial\zeta}{\partial y} + ip\frac{\partial\zeta}{\partial x} = 0. \quad \text{......................................(2)}$$

Equations (1) and (2) together with the incident field define the problem.

The incident wave is of the form

$$\exp\{ik(x\cos\alpha + y\sin\alpha)\}$$

which represents a wave travelling in the direction $\alpha+\pi$ (see figure), and the



solution required is that which obeys the required boundary condition (2) and which reduces to the incident field at great distances from the origin in the region $\pi - \alpha < \phi < \pi + \alpha$, where ρ, ϕ are polar coordinates.

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2. Analysis

Consider the diffraction function

$$D(\rho,\ \phi,\ \chi) = \frac{1}{\sqrt{\pi}} \exp\left\{i\left[k\rho \cos(\phi+\chi) - \frac{\pi}{4}\right]\right\} \int_{-\infty}^{m} \exp\{-i\gamma^2\} d\gamma, \quad \text{.........(3)}$$

where $x = \rho \cos \phi$, $y = \rho \sin \phi$, $m = \sqrt{2k\rho} \cos \frac{1}{2}(\phi+x)$. Then it follows (1) that

$$\frac{1}{ik}\frac{\partial D}{\partial x} = D \cos \chi + F_0 \cos \tfrac{1}{2}(\phi-\chi) \quad \text{.......................(4a)}$$

and

$$\frac{1}{ik}\frac{\partial D}{\partial y} = -D \sin \chi + F_0 \sin \tfrac{1}{2}(\phi-\chi), \quad \text{.....................(4b)}$$

where

$$F_0 = \sqrt{\frac{1}{2\pi k\rho}} \exp\{-i(k\rho + \tfrac{1}{4}\pi)\}. \quad \text{..........................(5)}$$

Bearing in mind these differential coefficients, we look for a ζ of the form

$$\zeta = D(\rho,\ \phi,\ -\alpha) + KD(\rho,\ \phi,\ \alpha) + LD(\rho,\ \phi,\ -\alpha^*) \quad \text{..............(6)}$$

such that

$$\frac{\partial \zeta}{\partial y} + ip\frac{\partial \zeta}{\partial x} \text{ vanishes on } \phi = 0 \textit{ and } \phi = 2\pi.$$

$D(\rho,\ \phi,\ -\alpha)$ and $D(\rho,\ \phi,\ \alpha)$ are the terms of the type usually associated with the Sommerfeld Diffraction Problem. $D(\rho,\ \phi,\ -\alpha^*)$ is a term which is inserted in order to cancel out the terms involving F_0 in the boundary condition. We note that

$$D(\rho,\ 0,\ -\alpha) = D(\rho,\ 0,\ \alpha) = D_0, \text{ say,}$$

and

$$D(\rho,\ 2\pi,\ -\alpha) = D(\rho,\ 2\pi,\ \alpha) = D_0', \text{ say.}$$

On $\phi = 0$, the boundary equation becomes

$$\begin{aligned} &D_0\{[\sin \alpha + ip \cos \alpha] + [-\sin \alpha + ip \cos \alpha]K\} \\ &\quad + F_0\{[(\sin \tfrac{1}{2}\alpha + ip \cos \tfrac{1}{2}\alpha) + K(-\sin \tfrac{1}{2}\alpha + ip \cos \tfrac{1}{2}\alpha)\} \\ &\quad + LD(\rho,\ 0,\ -\alpha^*)[\sin a^* + ip \cos \alpha^*] + LF_0\left\{\sin \frac{\alpha^*}{2} + ip \cos \frac{\alpha^*}{2}\right\} = 0. \quad \text{......(7)} \end{aligned}$$

The condition on $\phi = 2\pi$ is similar, D_0' replacing D_0 and $-F_0$ replacing F_0.

Both of these conditions can be satisfied it

$$\sin \alpha + ip \cos \alpha + K(ip \cos \alpha - \sin \alpha) = 0, \quad \text{....................(8)}$$

$$\sin \tfrac{1}{2}\alpha + ip \cos \frac{\alpha}{2} + K\left(-\sin \frac{\alpha}{2} + ip \cos \frac{\alpha}{2}\right) + L\left(\sin \frac{\alpha^*}{2} + ip \cos \frac{\alpha^*}{2}\right) = 0, \quad \text{......(9)}$$

and

$$\sin \alpha^* + ip \cos \alpha^* = 0. \quad \text{..............................(10)}$$

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It follows, from equation (8), that

$$K = \frac{\sin\alpha + ip\cos\alpha}{\sin\alpha - ip\cos\alpha} = \exp\{2i\tan^{-1}(p\cot\alpha)\} = \exp(i\psi), \quad \text{.........(11)}$$

and, from equation (10), that

$$\tan\alpha^* = -ip. \quad \text{.................................(12)}$$

It further follows that

$$\sin\alpha^* = \frac{ip}{(1-p^2)^{\frac{1}{2}}}, \quad \cos\alpha^* = -\frac{1}{(1-p^2)^{\frac{1}{2}}}$$

and, writing $s = (1-p^2)^{\frac{1}{2}}$, we deduce that

$$\cos\frac{\alpha^*}{2} = \left(\frac{s-1}{2s}\right)^{\frac{1}{2}} \text{ and } \sin\frac{\alpha^*}{2} = -\left(\frac{s+1}{2s}\right)^{\frac{1}{2}}.$$

That these are the appropriate solutions will be shown in the appendix.

Equation (9) becomes, on using equations (11) and (12),

$$L = \frac{2\sin\frac{\alpha}{2}}{\sin\alpha - ip\cos\alpha} \cdot \frac{\tan\alpha^*}{\sin\frac{\alpha^*}{2} + ip\cos\frac{\alpha^*}{2}}$$

$$= \frac{-2\sqrt{2}\sin\frac{\alpha}{2}}{\sin\alpha - ip\cos\alpha}\left(\frac{s-1}{s}\right)^{\frac{1}{2}}. \quad \text{..............................(13)}$$

Substituting the expressions for K and L given by (11) and (13) into equation (6), it follows that

$$\zeta = D(\rho, \phi, -\alpha) + \exp\{i\psi\} D(\rho, \phi, \alpha)$$

$$+ \frac{2\sqrt{2i}\sin\frac{\alpha}{2}}{\sin\alpha - ip\cos\alpha}\left(\frac{1-s}{s}\right)^{\frac{1}{2}} D(\rho, \phi, -\alpha^*), \quad \text{.................(14)}$$

$$= \zeta_1 + \zeta_2 + \zeta_3.$$

It may be verified that if $\alpha = 3\pi/2$ this is equivalent to the previously obtained expression (2).

3. Discussion

Ignoring for the present the third term in equation (14), the first two terms represent Sommerfeld Diffraction type fields of the usual type.

If ρ is large, i.e. if points far from the diffracting edge are considered, then $D(\rho, \phi, -\alpha)$ behaves like $\exp\{ik\rho\cos(\phi-\alpha)\}$ for $0<\phi<\pi+\alpha$, and vanishes for $\pi+\alpha<\phi<2\pi$. Similarly $D(\rho, \phi, \alpha)$ behaves like $\exp\{ik\rho\cos(\phi+\alpha)\}$ in the region $0<\phi<\pi-\alpha$ and is zero in $\pi-\alpha<\phi<2\pi$. Thus there are three regions:

(a) $\pi+\alpha<\phi<2\pi$. This is the geometrical shadow region and at large

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distances from the origin the field is effectively zero, the region being screened by the half axis.

(b) $\pi-\alpha<\phi<\pi+\alpha$. At large distances from the origin the field is effectively the incident field.

(c) $0<\phi<\pi-\alpha$. At large distances from the origin the field is effectively $\exp\{ik\rho\cos(\phi-\alpha)\}+\exp\{i\psi\}\exp\{ik\rho\cos(\phi+\alpha)\}$ which is the same as that for reflection by a complete axis.

It will be observed that when $\alpha = 3\pi/2$ (the case previously treated) $\cot\alpha$ and hence ψ are zero, and the following statement (**2**) is true: "Neither ζ_2 nor ζ_1 contains the rotational parameters at all; these functions may be expected to represent the usual diffraction and reflection effects of acoustics when the boundary condition is that the normal gradient of the dependent variable is zero on the barrier." However if α is not $\pi/2$ or $3\pi/2$, there is a phase factor involving p in ζ_2, and the statement ceases to hold.

The third term ζ_3 is the product of two terms, one of which is a function of α and the other of which is a function of the field variables ρ and ϕ, but is independent of α. In fact, one may write (**2**)

$$\zeta_3 = f(\alpha)\zeta_3', \qquad\qquad (15)$$

where $f(\alpha) = -\dfrac{\sqrt{2}\sin\dfrac{\alpha}{2}}{\sin\alpha - ip\cos\alpha}$, and so $f(3\pi/2) = 1$. The quantity that is of importance is

$$|f(\alpha)| = \frac{\sin\dfrac{\alpha}{2}}{\sqrt{2}(\sin^2\alpha+p^2\cos^2\alpha)^{\frac{1}{2}}}.$$

It is fairly easy to to see that this is a smooth function whose behaviour is given by the following table:

α :	0	$\pi/2$	π	$3\pi/2$	2π
$\lvert f(\alpha)\rvert$:	0	1	$\dfrac{1}{\sqrt{2}p}$	1	0.

The intervening points can be filled in without difficulty.

As the spatial behaviour has already been discussed (**2**), it will not be discussed here. The only difference that the alteration in the angle of incidence makes is that the magnitude of the effect will be different.

Appendix

The equation $\tan\alpha^* = -ip$ will have four possible solutions relevant to this problem since $D(\rho, \phi, -\alpha^*)$ has period 4π in the variable $(\phi-\alpha^*)$. The correct one is the one which will lead to evanescence of the disturbances at great distances from the barrier. That the solution taken is the appropriate

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one may be seen from a consideration of the asymptotic expansion of the Fresnel Integral.

Consider first Stokes' Asymptotic formula (4):

$$\int_0^x e^{x^2-t^2}dt = \pm\tfrac{1}{2}e^{x^2}\sqrt{\pi}+O\left(\frac{1}{|x|}\right) \quad \text{(A1)}$$

for $|x|$ large where the positive sign is taken if $-\tfrac{1}{2}\pi<\text{phase } x<\tfrac{1}{2}\pi$ and the negative sign is taken if $\tfrac{1}{2}\pi<\text{phase } x<\frac{3\pi}{2}$. It follows that

$$e^{i\pi/4}e^{im^2}\int_0^m e^{-i\gamma^2}d\gamma = \pm\frac{\sqrt{\pi}}{2}e^{im^2}+O\left(\frac{1}{|m|}\right), \quad \text{(A2)}$$

where $\pm$ arises according as $-3\pi/4<\text{phase } m<\pi/4$ or $\pi/4<\text{phase } m<5\pi/4$, and that

$$\frac{1}{\sqrt{\pi}}e^{i\pi/4}e^{im^2}\int_{-\infty}^m e^{-i\gamma^2}d\gamma = Pe^{im^2}+O\left(\frac{1}{|m|}\right), \quad \text{(A3)}$$

where P is 1 if $-3\pi/4<\text{phase } m<\pi/4$ and is 0 if $\pi/4<\text{phase } m<5\pi/4$. Now

$$D(\rho,\phi,-\alpha^*) = \frac{e^{-i\pi/4}}{\sqrt{\pi}}\exp\{ik\rho\cos(\phi-\alpha^*)\}.$$

$$\int_{-\infty}^{\sqrt{2k\rho}\cos\left(\frac{\phi-\alpha^*}{2}\right)}\exp(-i\gamma^2)d\gamma,$$

$$= \frac{e^{-i\pi/4}}{\sqrt{\pi}}e^{ik\rho}e^{im^2}\int_{-\infty}^m e^{-i\gamma^2}d\gamma, \quad \text{(A4)}$$

where $m=\sqrt{2k\rho}\cos\left(\frac{\phi-\alpha^*}{2}\right)$.

Thus $D(\rho,\phi,-\alpha^*)$ is, apart from a constant factor, of the form of the expressions in (A3).

Now $\sin\alpha^*/2$ is of the form $-\mu$ and $\cos\alpha^*/2$ is of the form $-i\nu$, where μ and ν are positive. This follows from the fact that $0<p<1$. For $0<\phi<\pi$, we have $0<\sin\theta/2<1$, $0<\cos\theta/2<1$, and

$$\cos\left(\frac{\phi-\alpha^*}{2}\right) = -i\nu\cos\frac{\phi}{2}-\mu\sin\frac{\phi}{2}. \quad \text{(A5)}$$

For $\pi<\phi<2\pi$, we have $0<\sin\phi/2<1$, $-1<\cos\theta/2<0$, and

$$\cos\left(\frac{\phi-\alpha^*}{2}\right) = i\nu\left[-\cos\frac{\phi}{2}\right]-\mu\sin\frac{\phi}{2}. \quad \text{(A6)}$$

Clearly phase m is the same as the phase of the right hand sides of (A5), (A6).

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It follows therefore from (A5) that, for $0<\phi<\pi$,

$$\begin{aligned}\exp\{im^2\} &= \exp i\left[2k\rho\left(\mu\sin\frac{\phi}{2}+i\nu\cos\frac{\phi}{2}\right)^2\right]\\ &= \exp i\left[2k\rho\left(\mu^2\sin^2\frac{\phi}{2}-\nu^2\cos^2\frac{\phi}{2}+i\mu\nu\sin\frac{\phi}{2}\cos\frac{\phi}{2}\right)\right]\\ &= \exp i\left[2k\rho\left(\mu^2\sin^2\frac{\phi}{2}-\nu^2\cos^2\frac{\phi}{2}\right)\right].\exp\left[-k\rho\mu\nu\sin\phi\right].\end{aligned}$$

This tends to zero as ρ tends to infinity for $0<\phi<\pi$, and so does Pe^{im^2}; thus the value of P, and hence the phase of m, is immaterial. Hence, for $0<\phi<\pi$, $D(\rho, \phi, -\alpha^*)\to 0$ as $\rho\to\infty$. Looking at (A6), it can be seen that, for $\pi<\phi<2\pi$, we have $\frac{1}{2}\pi\leqq$ phase $m\leqq\pi$ and so $D(\rho, \phi, -\alpha^*)=O\left(\frac{1}{\rho^{\frac{1}{2}}}\right)$ as $\rho\to\infty$.

It will be observed that D does not tend to zero for $\phi = 0$, at large distances from the origin. This is in order. As pointed out (2) by Crease, the waves represented by ζ_3 can be greater on the barrier than the incident waves. What matters is that they die out away from the barrier. Thus the particular set α^*, $\frac{1}{2}\alpha^*$ chosen satisfy the appropriate radiation conditions, and so the solution obtained is the required one.

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