Can. J. Math., Vol. XXV, No. 1, 1973, pp. 194–203

INTEGRAL LIMIT LAWS FOR ADDITIVE FUNCTIONS

J. GALAMBOS

1. Introduction. In the present paper a general form of integral limit laws for additive functions is obtained. Our limit law contains Kubilius' results [**5**] on his class H. In the proof we make use of characteristic functions (Fourier transforms), which reduces our problem to finding asymptotic formulas for sums of multiplicative functions. This requires an extension of previous results in order to enable us to take into consideration the parameter of the characteristic function in question. We call this extension a parametric mean value theorem for multiplicative functions and its proof is analytic on the line of [**4**]. The present investigation was induced by the work of Levin and Fainleib [**6**], who obtained integral limit laws for a class of additive functions, which class overlaps (but does not contain) the class H of Kubilius. Their investigation is more specific than ours also in the choices of the normalizing constants A_N and B_N for obtaining limit laws for $(f(n) - A_N)/B_N$ with $f(n)$ additive. As a matter of fact, they restrict themselves to the case when both A_N and $B_N{}^2$ are constant multiples of loglog N. In our case, however, the constants A_N and $B_N{}^2$ are not restricted in magnitude. It is also to be emphasized that they are not the usual values $\sum f(p)/p$ and $\sum f^2(p)/p$, respectively, where the summations are over all primes p not exceeding N.

Acknowledgement. I wish to express my thanks to Professor Walter Philipp for his comments on an earlier version of this paper.

2. A parametric mean value theorem for multiplicative functions. Let $f(n)$ be a real valued strongly additive arithmetical function, i.e., for any coprime u and v,

$$f(uv) = f(u) + f(v), \tag{1}$$

and for any integer $a \geqq 1$ and any prime number p,

$$f(p^a) = f(p). \tag{2}$$

Throughout this paper, p will denote prime numbers.

Let $N\nu_N(n : \ldots)$ denote the number of positive integers $n \leqq N$, for which the property given in the dotted place holds. Given two sequences A_N and $B_N > 0$ of real numbers, the arithmetical function $(f(n) - A_N)/B_N$ is said to have a limit law if there is a distribution function $F(x)$ such that, for all

Received December 7, 1971 and in revised form, July 31, 1972.

continuity points of $F(x)$, as $N \to +\infty$,

$$\lim \nu_N(n : f(n) - A_N < xB_N) = F(x). \tag{3}$$

Put

$$g(n, t) = \exp(itf(n)), \tag{4}$$

where t is a real variable. Note that (1) and (2) imply that for any coprime u and v,

$$g(uv, t) = g(u, t)g(v, t), \tag{5}$$

and that

$$g(p^a, t) = g(p, t), \quad a \geqq 1. \tag{6}$$

In other words, (5) and (6) yield that $g(n, t)$ is a strongly multiplicative function of n. The well-known continuity theorem [**7**, p. 191] of characteristic functions or Fourier transforms, stated as Lemma 1 below, thus reduces the validity of (3) to an asymptotic relation for the arithmetical mean of the strongly multiplicative function $g(n, t)$.

LEMMA 1. *The limit relation* (3) *holds if, and only if,*

$$\lim_{N=+\infty} \frac{\exp(-itA_N/B_N)}{N} \sum_{n=1}^{N} g(n, t/B_N) = \varphi(t) \tag{7}$$

exists and is continuous at $t = 0$. *In this case*

$$\varphi(t) = \int_{-\infty}^{+\infty} e^{itx} dF(x). \tag{8}$$

In view of Lemma 1, our problem is reduced to giving an asymptotic formula expressed in (7). There are several results on the arithmetical mean of multiplicative functions with modulus bounded by one. All of them are, however, in terms of N and thus the parameter t of $g(n, t)$ can not be taken into account. Since the sequences A_N and B_N enter our expression in (7) through the parameter t, we need a parametric mean value theorem for multiplicative functions. Such a result will be obtained by applying Satz 1′ and the method of proof of Satz 2 of [**4**]. In spite of this close relation of our proof to the work [**4**], our conclusion is essentially new by having the parameter t in the final form. It is worthwhile pointing out that a step on p. 380 of [**4**] (the equivalence of the last two formulas) makes Satz 2 of [**4**] unapplicable to our problem.

Before turning to the mean value theorems, we need some notations.

Let us put

$$\sigma_0 = 1 + 1/\log N, \tag{9}$$

and

$$f_c(n) = f(n) - c \log n, \tag{10}$$

where c is a real number. For an arithmetical function $g(n)$ we introduce the Dirichlet series

$$G(s) = \sum_{n=1}^{+\infty} \frac{g(n)}{n^s}, \qquad s = \sigma + iu.$$

We need the following result of [**4**].

LEMMA 2. *Let $g(n)$ be multiplicative and let $|g(n)| \leqq 1$. Assume that for $G(s)$ the asymptotic equality*

$$G(s) = \frac{CL((\sigma - 1)^{-1})}{s - (1 + ia)} + o\left(\frac{1}{\sigma - 1}\right), \qquad \sigma \to 1, \tag{11}$$

holds uniformly on $|u| \leqq K$ for any finite K. Here the values C and a are constants with a real and the function $L(y)$ is assumed to satisfy the following conditions. For any y, $|L(y)| = 1$ and for $y = O(\log N)$, as $N \to +\infty$,

$$L(y_1)/L(y) \to 1 \text{ uniformly for } y \leqq y_1 \leqq 3y. \tag{12}$$

We then have

$$\sum_{n=1}^{N} g(n) = \frac{CL(\log N)}{1 + ia} N^{1+ia} + o(N). \tag{13}$$

We are now in the position to give our parametric mean value theorem.

LEMMA 3. *Let $f(n)$ be a strongly additive function and let $g(n, t) = \exp(itf(n))$. Let further B_N be a sequence of positive real numbers, tending to $+\infty$ with N. Let us assume that there is a real number c such that for all real numbers σ with $\frac{1}{2}(\sigma_0 - 1) \leqq \sigma - 1 \leqq \frac{3}{2}(\sigma_0 - 1)$,*

$$\lim_{N=+\infty} \sum_p p^{-\sigma}\{1 - \operatorname{Re}[\exp(itf_c(p)/B_N)]\} = \alpha(f, t) \tag{14}$$

exists and is independent of σ. Then

$$\frac{1}{N}\sum_{n=1}^{N} g\left(n, \frac{t}{B_N}\right) = N^{ict/B_N} H(N, t) \times \exp\left\{\sum_p \sum_{k=1}^{+\infty} \frac{1}{kp^{k\sigma_0}} [\exp(itkf_c(p)/B_N) - 1]\right\} + o(1), \tag{15}$$

where

$$H(N, t) = \frac{1}{1 + ict/B_N} \prod_p \left[1 - \frac{\exp(itf_c(p)/B_N)[\exp(itf_c(p)/B_N) - 1]}{p^{2(1+ict/B_N)} - p^{1+ict/B_N}}\right]. \tag{16}$$

Proof. Let $g^*(n, t)$ be completely multiplicative and let it coincide with $g(n, t)$ when n is a prime number, i.e., $g^*(p^k, t) = g^k(p, t)$ for all primes p and for $k = 1, 2, \ldots$. Let $G^*(s, t, N)$ denote the Dirichlet series of $g^*(n, t/B_N)$. We first show that Lemma 2 is applicable to $g^*(n, t/B_N)$ and then by an elementary argument, well-known in analytic number theory, we shall deduce the conclusion of Lemma 3. In order to show that (11) holds, we specify the values C, a and $L(y)$. Let $a = ct/B_N$, occuring in the condition (14) and let

$$C = \exp[-\alpha(f, t)] \tag{17}$$

and

$$L\left(\frac{1}{\sigma - 1}\right) = \exp\left\{i \sum_p \sum_{k=1}^{+\infty} \frac{1}{kp^{k\sigma}} \operatorname{Im}[g^*(n, t/B_N)n^{-ict/B_N}]\right\}. \tag{18}$$

In order to simplify the double sum in (18) and in the statement itself we introduce

$$\lambda(n) = \begin{cases} k^{-1}, & \text{if } n = p^k \\ 0, & \text{otherwise.} \end{cases}$$

Let us first note that for $N \to +\infty$

$$\exp\left\{\sum_{n=1}^{+\infty} \frac{\lambda(n)}{n^{\sigma_0}} [\operatorname{Re}(g^*(n, t/B_N)n^{-ict/B_N}) - 1]\right\} = C + o(1). \tag{19}$$

Indeed, for any fixed m, and for $\sigma \geqq 1$,

$$\sum_{k=2}^{+\infty} \sum_p \frac{1}{kp^{k\sigma}} \{1 - \operatorname{Re}[\exp(itf_c(p)k/B_N)]\} \tag{20}$$

$$\leqq \sum_{k=2}^{+\infty} \frac{1}{k}\left[\sum_{p \leqq m} \frac{|tf_c(p)|k}{B_N p^{k\sigma}} + \sum_{p>m} \frac{2}{p^{k\sigma}}\right] \leqq \frac{m|t|}{2B_N} \max_{p \leqq m} |f_c(p)| + \sum_{p>m} \sum_{k=2}^{+\infty} \frac{2}{kp^{k\sigma}}.$$

Since $B_N \to +\infty$ by assumption, letting $N \to +\infty$ and then $m \to +\infty$, (20) now yields (19). In order to show the validity of (11), we compare $G^*(s, t, N)$ with $CL((\sigma - 1)^{-1})\zeta(s)$, where $\zeta(s)$ is the Riemann zeta function. Since $g^*(n, t)$ is completely multiplicative,

$$G^*(s, t, N) = \exp\left[\sum_{n=1}^{+\infty} \frac{\lambda(n)g^*(n, t/B_N)}{n^s}\right],$$

and thus by the definition of the values involved and in view of (19),

$$\frac{G^*(s + ict/B_N, t, N)}{\zeta(s)CL((\sigma - 1)^{-1})} \sim \exp\left\{\sum_{n=1}^{+\infty} \lambda(n)(n^{-\sigma} - n^{-s})[1 - w(n)]\right\}, \tag{21}$$

where

$$w(n) = g^*(n, t/B_N)n^{ict/B_N}.$$

We therefore have to show that the exponent on the right hand side of (21) tends to zero, uniformly on any finite interval of Im s, as $\sigma \to 1$. As a matter of fact, when showing this limit relation, the well-known property of the zeta function,

$$\zeta(s) = \frac{1}{s - 1} + o\left(\frac{1}{\sigma - 1}\right), \sigma \to 1,$$

yields

$$G^*(s, t, N) = \frac{CL((\sigma - 1)^{-1})}{s - (1 + ict/B_N)} + o\left(\frac{1}{\sigma - 1}\right),$$

as $\sigma \to 1$, the asymptotic equality being uniform on any finite interval of $\operatorname{Im} s$. We then have to show that the conditions on $L(y)$ are satisfied. Since this latter calculation will be of the same kind as estimating the exponent in (21), let us first turn to $L(y)$. The condition $|L(y)| = 1$ is evident from the definition (18). Also from (18),

$$\frac{L((\sigma - 1)^{-1})}{L((\tau - 1)^{-1})} = \exp\left\{i \sum_{n=1}^{+\infty} \lambda(n)(n^{-\sigma} - n^{-\tau}) \operatorname{Im}[w(n)]\right\}.$$

Hence all conditions of Lemma 2 will be satisfied if we show that for $\frac{1}{2}(\sigma - 1) \leqq \operatorname{Re} s - 1 \leqq \frac{3}{2}(\sigma - 1)$ and for $|\operatorname{Im} s| \leqq K_0(\sigma - 1)$, with $\sigma - 1 = O(\sigma_0 - 1)$,

$$\sum_{n=1}^{+\infty} \lambda(n)|n^{-\sigma} - n^{-s}|\,|1 - w(n)| \to 0, \text{ uniformly in } s.$$

This requires only a slight modification of the argument on p. 382 of [**4**]. We first note that

$$|n^{-\sigma} - n^{-s}| \leqq 3(1 + K_0)(\sigma - 1)n^{-\sigma} \log n.$$

Since $|1 - w(n)|^2 = 2(1 - \operatorname{Re} w(n))$, in view of (19),

$$\sum_{n=1}^{+\infty} \lambda(n)n^{-\sigma}|1 - w(n)|^2$$

uniformly converges (uniform in N). We can therefore choose a value T, independently of N, such that

$$\sum_{n=T+1}^{+\infty} \lambda(n)n^{-\sigma}|1 - w(n)|^2 < \epsilon.$$

We thus have

$$\begin{aligned}\sum_{n=1}^{+\infty} \lambda(n)\left|\frac{1}{n^\sigma} - \frac{1}{n^s}\right| |1 - w(n)| &\leqq 3(K_0 + 1)(\sigma - 1)\sum_{n=1}^{+\infty} \frac{\lambda(n) \log n}{n^\sigma} |1 - w(n)| \\ &\leqq 3(K_0 + 1)(\sigma - 1)\sum_{n=1}^{T} \frac{\lambda(n) \log n}{n^\sigma} |1 - w(n)| \\ &\quad + 3(K_0 + 1)(\sigma - 1)\sum_{n=T+1}^{+\infty} \frac{\lambda(n) \log n}{n^\sigma} |1 - w(n)|.\end{aligned}$$

The first term tends to 0 as $\sigma \to 1$ (T is fixed). By an appeal to the Cauchy inequality in the second sum, we get

$$\left[\sum_{n=T+1}^{+\infty} \frac{\lambda(n) \log n}{n^\sigma} |1 - w(n)|\right]^2 \leqq \sum_{n=1}^{+\infty} \frac{\lambda(n) \log^2 n}{n^\sigma} \sum_{n=T+1}^{+\infty} \frac{\lambda(n)}{n^\sigma} |1 - w(n)|^2.$$

The second factor is smaller than ϵ by the choice of T. On the other hand, the first one, as it is well-known, is $O((\sigma - 1)^{-2})$, which remains bounded when multiplied by $\sigma - 1$ (note that we estimated the square of the sum in question). (11) is thus established. We can now apply Lemma 2 to $g^*(n, t/B_N)$ and then the argument on p. 370 of [**4**]. This completes the proof of Lemma 3.

The following elementary lemma will simplify some of our expressions.

LEMMA 4. *Let a_n and b_N be two sequences of real numbers and let $b_N \to 0$ as $N \to +\infty$. Assume that*

$$\sum_{n=1}^{+\infty} a_n < +\infty, \qquad a_n > 0. \tag{22}$$

Then, for any sequence c_n of real numbers,

$$\lim_{N=+\infty} \sum_{n=1}^{+\infty} a_n |\exp(ic_n b_N) - 1| = 0.$$

Proof. Let $\epsilon > 0$ be any prescribed real value. In view of (22), there is an integer M such that

$$\sum_{n=M+1}^{+\infty} a_n < \epsilon.$$

Thus

$$\sum_{n=1}^{+\infty} a_n |\exp(ic_n b_N) - 1| \leqq b_N \sum_{n=1}^{M} a_n |c_n| + 2\epsilon.$$

Since M does not depend on N, letting $N \to +\infty$ and then $\epsilon \to 0$ establishes the lemma.

3. The limit laws for additive functions. The main result of the present paper is given in the following

THEOREM. *Let $f(n)$ be a real valued strongly additive arithmetical function. Let B_N be a sequence of positive real numbers, tending to $+\infty$. Assume that the condition, concerning* (14), *in Lemma* 3 *is satisfied. We further assume that there is a function $\beta(f, t)$, not depending on N, such that*

$$A_N^* = (B_N/t)\{\textstyle\sum p^{-\sigma_0} \operatorname{Im}[\exp(itf_c(p)/B_N)] - \beta(f, t)\} + o(B_N) \tag{23}$$

is independent of t, the summation being over primes p. Then, if both $\alpha(f, t)$ and $\beta(f, t)$ are continuous at $t = 0$,

$$(f(n) - c \log N - A_N^*)/B_N$$

has a limit law, the characteristic function $\varphi(t)$ of which is given by

$$\log \varphi(t) = -\alpha(f, t) + i\beta(f, t). \tag{24}$$

Proof. Because of the preparation on the preceding pages, there remain only a few steps to establish our theorem. In view of Lemma 1, we have to show that the assumptions of our Theorem imply (7) with $A_N = c \log N + A_N^*$ and with $\varphi(t)$ determined in (24). By an appeal to Lemma 3, this goal is accomplished if we show that under our conditions the limit relations

$$\lim H(N, t) = 1 \quad (N \to +\infty) \tag{25}$$

and

$$\lim_{N=+\infty} \sum_p \sum_{k=2}^{+\infty} \frac{1}{kp^{k\sigma_0}} [\exp(itkf_c(p)/B_N) - 1] = 0 \tag{26}$$

hold. Applying the inequality

$$|\log(1 + x)| \leqq 2|x|, \quad \text{for } |x| \leqq \tfrac{1}{2}$$

in the definition (16) of $H(N, t)$, we get the estimate

$$|\log H(N, t)| \leqq 2|c|\,|t|/B_N + 2\sum [p(p - 1)]^{-1}|\exp(itf_c(p)/B_N) - 1|,$$

the summation being over all primes p. By Lemma 4 we therefore have (25). A direct application of Lemma 4 gives (26) as well and the proof of the theorem is thus complete.

As a corollary, we shall show that the sufficiency part of the integral limit law of Kubilius [**5**] for his class H is contained in our Theorem. For this we first need a definition.

Definition. A strongly additive function $f(n)$ is said to belong to the class H of Kubilius if

$$D_N^2(f) = D_N^2 = \sum_{p \leqq N} \frac{f^2(p)}{p} \to +\infty \tag{27}$$

and if there is an integer valued function $r(N) \to +\infty$ with N and such that $\log r(N)/\log N \to 0$ and $D_{r(N)}/D_N \to 1$ as $N \to +\infty$.

Let us put

$$E_N(f) = E_N = \sum_{p \leqq N} \frac{f(p)}{p} \tag{28}$$

and

$$K_N(u) = D_N^{-2} \sum f^2(p)/p \tag{29}$$

where the summation is over all primes $p \leqq N$ for which $f(p) < uD_N$. We now deduce from our Theorem the following result of Kubilius [**5**, p. 58].

Corollary. *Let $f(n) \in H$ and assume that there is a distribution function $K(u)$ such that*

$$K_N(u) \to K(u) \quad (N \to +\infty) \tag{30}$$

for all continuity points of $K(u)$. Then $(f(n) - E_N)/D_N$ has a limit law of which *the characteristic function $\varphi(t)$ is determined by the Kolmogorov formula*

$$\log \varphi(t) = \int_{-\infty}^{+\infty} (e^{itu} - 1 - itu)u^{-2}dK(u). \tag{31}$$

Proof. We have to verify that the conditions of our Theorem are satisfied for $f(n) \in H$ whenever (30) holds and when $c = 0$, $B_N = D_N$ and $A_N = E_N$. To accomplish this goal, let the summations $\sum$ with respect to all prime numbers p in (14) and (23) be split into four sums

$$\sum = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 + \sum\nolimits_4,$$

where

$$\begin{array}{ll} \text{in} \sum\nolimits_1 : p \leqq r(N); & \text{in} \sum\nolimits_2 : r(N) < p \leqq N; \\ \text{in} \sum\nolimits_3 : N < p \leqq N^M; & \text{in} \sum\nolimits_4 : \quad p > N^M, \end{array} \tag{32}$$

with the function $r(N)$ specified in the definition of the class H and where M is a sufficiently large real number, not depending on N. We first show that $\sum_4$ is smaller than any prescribed value $\epsilon > 0$ in both sums in question. Namely, for any real $\sigma \geqq \frac{1}{2}(\sigma_0 + 1)$, in both sums, $\sum_4$ is majorized by (we put $N_j = N^{jM}$, $j = 1, 2, \ldots$)

$$\sum\nolimits_4 p^{-\sigma} = \sum_{j=1} \sum_{N_j < p \leqq N_{j+1}} p^{-\sigma} < C \sum_{j=1}^{+\infty} e^{-\frac{1}{2}jM} < \epsilon$$

for M sufficiently large. Indeed, from the elementary relation [8, p. 58]

$$\sum_{p \leqq x} \frac{1}{p} = \log \log x + o(1) \tag{33}$$

and by the choice of σ, we get

$$\sum_{N_j < p \leqq N_{j+1}} p^{-\sigma} \leqq \sum_{N_j < p \leqq N_{j+1}} p^{-(1+1/2\log N)} \leqq N^{-jM/2\log N} \sum_{N_j < p \leqq N_{j+1}} \frac{1}{p}$$
$$= e^{-\frac{1}{2}jM}(\log \log N_{j+1} - \log \log N_j + O(1)) < Ce^{-\frac{1}{2}jM}$$

with a suitable constant C. Turning to $\sum_3$ we first note that for any real number z, $|\mathrm{Im} \exp(iz)| \leqq |z|$ and $|1 - \mathrm{Re} \exp(iz)| \leqq |z|$. Thus, for any real number $\sigma \geqq 1$, both in (14) and in (23),

$$\sum\nolimits_3 \leqq \frac{|t|}{D_N} \sum\nolimits_3 \frac{|f(p)|}{p} \leqq \frac{|t|}{D_N} \left(\sum\nolimits_3 \frac{f^2(p)}{p} \right)^{\frac{1}{2}} \left(\sum\nolimits_3 \frac{1}{p} \right)^{\frac{1}{2}},$$

where in the last step we made use of the Cauchy inequality. (33) yields that the last factor is bounded as $N \to +\infty$. The estimate above therefore gives that $\sum_3$ is also smaller than any real $\epsilon > 0$ for any $f(n) \in H$. As a matter of fact, since $\log r(N^M)/\log N^M \leqq M^{-1}$ for N sufficiently large, therefore $r(N^M) \leqq N$ and thus $D_{N_1} \sim D_N$, where, as before, $N_1 = N^M$. For evaluating the limits for $\sum_1$ and $\sum_2$ we need the condition in (30). By the definition of $K_N(u)$,

$$(34)\qquad \sum_{p \leqq N} \frac{1}{p}\,[\exp(itf(p)/D_N) - 1 - itf(p)/D_N] = \int_{-\infty}^{+\infty} (e^{itu} - 1 - itu)u^{-2}dK_N(u).$$

Since the integrand on the right hand side is continuous and bounded in u, by the Helly-Bray theorem [**7**, p. 182] and by (30) we have that the limit as $N \to +\infty$ exists in (34). Putting

$$-\alpha(f, t) = \lim_{N=+\infty} \sum_{p \leqq N} p^{-1}\{\mathrm{Re}[\exp(itf(p)/D_N) - 1]\}$$

and

$$\beta(f, t) = \lim_{N=+\infty} \sum_{p \leqq N} p^{-1}\{\mathrm{Im}[\exp(itf(p)/D_N)] - tf(p)/D_N\},$$

another appeal to the Helly-Bray theorem gives

$$(35)\qquad -\alpha(f, t) + i\beta(f, t) = \int_{-\infty}^{+\infty} (e^{itu} - 1 - itu)u^{-2}dK(u).$$

It is evident from (35) that both $\alpha(f, t)$ and $\beta(f, t)$ are continuous in t. The proof of the corollary is thus completed if we show that for all σ occuring in (14)

$$(36)\qquad \lim_{N=+\infty} \sum_{p \leqq N} \left(\frac{1}{p} - \frac{1}{p^\sigma}\right) \{1 - \mathrm{Re}[\exp(itf(p)/D_N)]\} = 0$$

and that, as $N \to +\infty$,

$$(37)\qquad (D_N/t) \sum_{p \leqq N} \frac{1}{p^{\sigma_0}}\,\mathrm{Im}[\exp(itf(p)/D_N)] = \sum_{p \leqq N} \frac{f(p)}{p} + o(D_N).$$

Both (36) and (37) easily follow from the fact that $f(n) \in H$ and from the elementary relations [**8**, p. 56]

$$(38)\qquad \sum_{p \leqq x} \frac{\log p}{p} \sim \log x$$

and

$$(39)\qquad \frac{1}{p} - \frac{1}{p^\sigma} \leqq \frac{1}{p}\,(1 - \exp[-(\sigma - 1)\log p]) \leqq \frac{(\sigma - 1)\log p}{p} \leqq \frac{2 \log p}{p \log N},$$

being valid for all σ with $1 < \sigma \leqq 1 + 2/\log N$. As a matter of fact, by (38) and (39)

$$\sum\nolimits_1 \leqq \sum\nolimits_1 \left(\frac{1}{p} - \frac{1}{p^\sigma}\right) \leqq \frac{2}{\log N} \sum\nolimits_1 \frac{\log p}{p} < \frac{4 \log r(N)}{\log N} < \epsilon$$

https://doi.org/10.4153/CJM-1973-017-3 Published online by Cambridge University Press

for N large. Also by (38) and (39) and by the Cauchy inequality

$$\left|\sum\nolimits_2 \left(\frac{1}{p} - \frac{1}{p^\sigma}\right) \operatorname{Im}[\exp(itf(p)/D_N)]\right| \leqq \frac{2|t|}{D_N \log N} \sum\nolimits_2 \frac{|f(p)|\log p}{p}$$
$$\leqq \frac{2|t|}{D_N \log N}\left(\sum\nolimits_2 \frac{f^2(p)}{p}\right)^{\frac{1}{2}}\left(\sum\nolimits_2 \frac{\log^2 p}{p}\right)^{\frac{1}{2}} < \frac{2|t|}{D_N}(D_N{}^2 - D_{r(N)}{}^2)^{\frac{1}{2}} \to 0$$

as $N \to +\infty$. Finally, by making use of the inequality $1 - \cos x \leqq x^2$, we have

$$\sum\nolimits_2 \left(\frac{1}{p} - \frac{1}{p^\sigma}\right)\{1 - \operatorname{Re}[\exp(itf(p)/D_N)]\} \leqq \frac{2t^2}{D_N} \sum\nolimits_2 \frac{f^2(p)}{p} < \epsilon$$

for N sufficiently large, and the proof is thus complete.

We conclude the paper with a few remarks. First of all, we wish to re-emphasize the fact that in our Theorem we made no assumption on the relation of $D_N(f)$ to B_N and neither were $E_N(f)$ and A_N related. A deep analysis of the present work seems to justify to conjecture that the assumptions of the Theorem are not only sufficient but also necessary for $(f(n) - A_N)/B_N$ to have a limit law. In this regard, see the comment on p. 137 of [**3**]. We point out that a thorough theory of additive functions exists only for the class H (for recent results see [**2**]). In the past decade, much attention was paid to the behaviour of arithmetical functions $f(n)$ when the argument n goes through a given sequence of positive integers (not necessarily the successive ones). In this direction, no results are known for functions not belonging to class H. The method of the present paper could lead to new results in this regard if Lemma 3 were extended in such a way that the argument were not restricted to run through the consecutive integers. For such an extension, a probabilistic argument similar to the one applied in [**1**], may lead to the most general result.

References

1. J. Galambos, *A probabilistic approach to mean values of multiplicative functions*, J. London Math. Soc. *2* (1970), 405–419.

2. ——— *Distribution of arithmetical functions. A survey*, Ann. Inst. H. Poincaré, Sect. B. *6* (1970), 281–305.

3. ——— *Distribution of additive and multiplicative functions*, The theory of Arithmetic Functions, Lecture Notes Series (Springer Verlag, Vol. 251, 1972, pp. 127–139).

4. G. Halàsz, *Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen*, Acta Math. Acad. Sci. Hungar. *19* (1968), 365–403.

5. J. Kubilius, *Probabilistic methods in the theory of numbers*, Transl. Math. Monographs, Amer. Math. Soc. *11*, 1964.

6. B. V. Levin and A. S. Fainleib, *Applications of some integral equations to problems of number theory*, Russian Math. Surveys *22* (1967), 119–204.

7. M. Loéve, *Probability theory*, 3rd ed. (Van Nostrand, Princeton, N.J., 1963).

8. E. C. Titchmarsh, *The theory of the Riemann zeta-function* (Claredon Press, Oxford, 1951).

Temple University,
Philadelphia, Pennsylvania