THIRD-ORDER SOLUTION OF AN ARTIFICIAL-SATELLITE THEORY

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ABSTRACT. A third-order solution is developed for the motions of artificial satellites moving in the gravitational field of the Earth, whose potential includes the second-, third-, and fourth-order zonal harmonics. Third-order periodic perturbations with fourth-order secular perturbations are derived by Hori's perturbations method. All quantities are expanded into power series of the eccentricity, but the solution is obtained so as to be closed with respect to the inclination. A comparison with the results of numerical integration of the equations of motion indicates that the solution can predict the position of a close-earth satellite with a small eccentricity with an accuracy of better than 1 cm over 1 month.

1. INTRODUCTION

Second-order theories of artificial satellites have been established by many authors during the past 15 years; an excellent review is given by Hori and Kozai (1975). A third-order solution was derived by Deprit and Rom (1970), but their solution does not include J_3 and J_4. According to Aksnes's numerical experiments comparing his second-order solution (Aksnes, 1970) with numerical integration, residuals of a few decimeters remain in position, most of which come from the third-order interaction among J_2, J_3, and J_4. On the other hand, the accuracy of a recently launched geodetic satellite equipped with retroreflectors for laser ranging will reach 3 to 5 cm. Therefore, we can expect to obtain more accurate information on satellite motions by taking into account the third-order periodic perturbations. Here we consider only the second-, third-, and fourth-order zonal-harmonics perturbations. We chose a Keplerian motion as an intermediate orbit and adopted Hori's (1966) perturbation method. We assumed that the eccentricity of a geodetic satellite requiring highly accurate solutions is usually low; therefore, all quantities in the present solution are expanded into power series of the eccentricity, but the solution is obtained in closed form with respect to the inclination. Delaunay variables were selected as the canonical elements used to construct the new Hamiltonian

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and the determining functions eliminating periodic terms, which do not depend on the chosen canonical variables as long as Keplerian motion is adopted as an intermediate orbit. However, the final expressions of the periodic perturbations are given in $\ell + g$, h, $e \cos g$, $e \sin g$, and L, which are not singular at zero eccentricity. All literal calculations were carried out by means of the computer algebra program Smithsonian Package for Algebra and Symbolic Mathematics (SPASM) (Hall and Cherniack, 1969). Final results were checked analytically in various ways and were compared with the results of numerical integration.

2. OUTLINE OF THE METHOD OF SOLUTION

Let us consider an artificial satellite orbiting in an axially symmetric gravitational field of the earth, whose force function is of the form

$$U = \frac{\mu}{r} \left[1 - \sum_{n=2}^{4} \left(\frac{a_e}{r}\right)^n J_n P_n(\sin \beta)\right], \tag{1}$$

where a_e is the equatorial radius of the Earth, r the radius vector of the satellite, and β the declination. In this paper, the coefficient J_2 is assumed to be a small quantity of the first order and J_3 and J_4 are of the second order.

To solve for the motion of the satellite under the force function (1), we adopted Hori's (1966) perturbation method, which utilizes Lie transformation; all formulas are canonically invariant, and the perturbations of any quantity are given by simple formulas and in explicit form. Because of the generality of Hori's method, we can choose any set of canonical variables. In the present paper, we use Delaunay variables as a canonical set for their simplicity, where

$$L = \sqrt{\mu a}, \qquad \ell = \text{mean anomaly},$$

$$G = L\sqrt{1 - e^2}, \qquad g = \text{argument of perigee},$$

$$H = G \cos i, \qquad h = \text{longitude of ascending node}.$$

The equations of motions are

$$\frac{d}{dt}(L,G,H) = \frac{\partial F}{\partial(\ell,g,h)}, \quad \frac{d}{dt}(\ell,g,h) = -\frac{\partial F}{\partial(L,G,H)}, \tag{2}$$

where

$$F = F_0 + F_1 + F_2,$$

$$F_0 = \frac{\mu^2}{2L^2},$$

$$F_1 = -\frac{\mu a_e^2}{r^3} J_2 P_2(\sin\beta), \tag{3}$$

$$F_2 = -\frac{\mu a_e^3}{r^4} J_3 P_3(\sin\beta) - \frac{\mu a_e^4}{r^5} J_4 P_4(\sin\beta).$$

Under the assumption that the eccentricity is small, we expanded the disturbing functions F_1 and F_2 into a power series of the eccentricity:

$$F_1 = \frac{\mu^4 J2}{L^6} \sum_{p=0}^{1} \sum_{k=-\infty}^{\infty} X_k^{2,2p}(e)\, B_{2,2p}(i) \cos(k\ell + 2pg),$$

$$F_2 = \frac{\mu^5 J_3}{L^8} \sum_{p=0}^{1} \sum_{k=-\infty}^{\infty} X_k^{3,2p+1}(e)\, B_{3,2p+1}(i) \sin[k\ell + (2p+1)g]$$

$$+ \frac{\mu^6 J_4}{L^{10}} \sum_{p=0}^{2} \sum_{k=-\infty}^{\infty} X_k^{4,2p}(e)\, B_{4,2p}(i) \cos(k\ell + 2pg), \tag{4}$$

where $X_k^{n,m}(e)$ is a Hansen coefficient,

$$B_{20} = -\frac{1}{4}(-2 + 3\sin^2 i), \quad B_{22} = \frac{3}{4}\sin^2 i,$$

$$B_{31} = -\frac{3}{8}\sin i\,(-4 + 5\sin^2 i), \quad B_{33} = \frac{5}{8}\sin^3 i,$$

$$B_{40} = -\frac{3}{8}(8 - 40\sin^2 i + 35\sin^4 i),$$

$$B_{42} = \frac{5}{16}\sin^2 i\,(-6 + 7\sin^2 i), \quad B_{44} = -\frac{35}{64}\sin^4 i.$$

When the lowest powers of e and sin i of the coefficient of $\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(k\ell + qg)$ in the trigonometric series (4) are α and δ, we have the following relations:

$$\alpha = |k - q| \quad (\text{mod. } 2),$$

$$\delta = |q| \quad (\text{mod. } 2),$$

which are called the d'Alembert characteristics.

The algorithm for deriving the new Hamiltonian F* (Hori, 1966) and the determining function S eliminating short-period terms is

zeroth order:

$$F_0^* = F_0, \tag{5a}$$

first order:

$$F_1^* = F_{1s},$$

$$S_1 = \frac{L'^3}{\mu^3} \int F_{1p}\, d\ell',$$

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second order:

$$F_2^* = F_{2s} + \frac{1}{2}\{F_1 + F_1^*, S_1\}_s,$$
$$S_2 = \frac{L'^3}{\mu^3}\int (F_{2p} + \frac{1}{2}\{F_1 + F_1^*, S_1\}_p)d\ell', \qquad (5b)$$

third order:

$$F_3^* = \frac{1}{12}\{\{F_{1p}, S_1\}, S_1\}_s + \frac{1}{2}\{F_2 + F_2^*, S_1\}_s + \frac{1}{2}\{F_1 + F_1^*, S_2\}_s,$$
$$S_3 = \frac{L'^3}{\mu^2}\int (\frac{1}{12}\{\{F_{1p}, S_1\}, S_1\}_p + \frac{1}{2}\{F_2 + F_2^*, S_1\}_p \qquad (5d)$$
$$+ \frac{1}{2}\{F_1 + F_1^*, S_2\}_p)\, d\ell',$$

fourth order:

$$F_4^* = \frac{1}{2}\{F_1 + F_1^*, S_3\}_s + \frac{1}{2}\{F_2 + F_2^*, S_2\}_s + \frac{1}{2}\{F_3 + F_3^*, S_1\}_s$$
$$+ \frac{1}{12}\{\{F_{1p}, S_1\}, S_2\}_s + \frac{1}{12}\{\{F_{1p}, S_2\}, S_1\}_s \qquad (5e)$$
$$+ \frac{1}{12}\{\{F_{2p}, S_1\}, S_1\}_s,$$

where the braces represent Poisson brackets and the subscripts s and p indicate the constant and periodic parts in ℓ'. It should be noted that we can add any function of L', G', H', and g' to S.

The algorithm for calculating the new Hamiltonian F** and the determining function S* eliminating long-period terms is given by the following equations:

first order:

$$F_1^{**} = F_1^*, \qquad (6a)$$

second order:

$$F_2^{**} = F_{2s}^*,$$
$$S_1^* = -\left(\frac{\partial F_1}{\partial G''}\right)^{-1}\int F_{2p}^*\, dg'', \qquad (6b)$$

third order:

$$F_3^{**} = \frac{1}{2}\{F_2^* + F_2^{**}, S_1^*\}_s,$$
$$S_2^* = -\frac{1}{2}\left(\frac{\partial F_1^*}{G''}\right)^{-1}\int \{F_2^* + F_2^{**}, S_1^*\}_p\, dg'', \qquad (6c)$$

fourth order:

$$F_4^{**} = \frac{1}{12}\{\{F_{2p}^*, S_1^*\}, S_1^*\}_s + \frac{1}{2}\{F_3^* + F_3^{**}, S_1^*\}_s$$

$$+ \frac{1}{2} \{F_2^* + F_2^{**}, S_2^*\}_s ,$$

$$S_3^* = - (\frac{\partial F_1^*}{\partial G''})^{-1} \int (\frac{1}{12} \{\{F_{2p}^*, S_1^*\}, S_1^*\}_p + \frac{1}{2} \{F_3^* + F_3^{**}, S_1^*\}_p$$

$$+ \frac{1}{2} \{F_2^* + F_2^{**}, S_2^*\}_p) \, dg'', \qquad (6d)$$

where the subscripts s and p indicate the constant and periodic parts in g",

These algorithms are actually very simple, but calculating them by hand is laborious. Therefore, all computations were carried out by the computer program SPASM (Hall and Cherniack, 1969). SPASM handles the operations in (5) and (6) easily, keeping rational fractions for coefficients.

A key operation in (5) and (6) is evaluating the Poisson bracket {A,B}:

$$\{A,B\} = \frac{\partial A}{\partial L}\frac{\partial B}{\partial \ell} - \frac{\partial A}{\partial \ell}\frac{\partial B}{\partial L} + \frac{\partial A}{\partial G}\frac{\partial B}{\partial g} - \frac{\partial A}{\partial g}\frac{\partial B}{\partial G} + \frac{\partial A}{\partial H}\frac{\partial B}{\partial h} + \frac{\partial A}{\partial h}\frac{\partial B}{\partial H} . \qquad (7)$$

In the present theory, as we check to obtain a solution that is closed with respect to the inclination, the atomic variables in the computer algebra are L, e, s = sin i, c = cos i, and $\gamma = 1 - 5 \cos^2 i$ (γ appears in the denominator of S*). The derivatives with respect to L, G, and H are

$$\frac{\partial}{\partial L} = (\frac{\partial}{\partial L}) + \frac{1 - e^2}{eL}\frac{\partial}{\partial e} ,$$

$$\frac{\partial}{\partial G} = \frac{1}{L\sqrt{1-e^2}} (- \frac{1 - e^2}{e}\frac{\partial}{\partial e} + \frac{1 - \sin^2 i}{\sin i}\frac{\partial}{\partial s} - \cos i \frac{\partial}{\partial c}$$

$$+ 10(1 - \sin^2 i) \frac{\partial}{\partial \gamma}) ,$$

$$\frac{\partial}{\partial H} = - \frac{1}{L\sqrt{1 - e^2}} (\frac{\cos i}{\sin i}\frac{\partial}{\partial s} + \frac{\partial}{\partial c} + 10 \cos i \frac{\partial}{\partial \gamma}) ,$$

where the factor $(1 - e^2)^{-1/2}$ will be replaced by a Taylor expansion in powers of e. If both A and B satisfy the d'Alembert characteristics, then even if these derivatives include the terms 1/e and 1/s in i, the Poisson bracket {A,B} keeps the d'Alembert characteristics and does not have 1/e and 1/sin i in the expression. This serves as a good check on literal manipulation by a computer. In Deprit and Rom's (1970) theory, $\cos i = H/L(1 - e^2)^{1/2}$ is developed in power series of e. Therefore, their determining function W apparently loses the d'Alembert characteristics with respect to the inclination. If A and B are expanded into power series of e and truncated at e^n, the derivatives with respect to

L and G are correct up to e^{n-2}; however {A,B} is correct up to e^{n-1}. In other words, with one operation of the Poisson bracket, only one degree in e (not two) is lost. Our program, which takes this fact into consideration, saves a lot of computer time.

Complete to J_2^3, the analytical solution must take into account all the following terms:

first order:

J_2, J_3/J_2, J_4/J_2,

second order:

J_3, J_3J_4/J_2^2, J_2^2, J_4,

$(J_3/J_2)^2$, $(J_4/J_2)^2$,

third order:

J_2J_3, $J_3J_4^2/J_2^2$, $(J_3/J_2)^3$, J_3J_4/J_2,

J_2^3, J_2J_4, $(J_4/J_2)^3$, $J_3^2J_4/J_2^3$,

J_3^2/J_2, J_4^2/J_2.

Most of these terms arise from the interaction among J_2, J_3, and J_4. Tables I and II list the numbers of terms involved in S_2 and S_3, respectively. The total number of terms with the factor J_2^3 in S_3^* is 106. On the other hand, Deprit and Rom's W_3 contains 192 terms up to e4, partly because they chose η = H/L as one of the arguments. Figures 1 and 2 show parts of S_3 and S_3^*.

Table I
Numbers of terms in the determining function S_3

	J_2^3	J_2J_3	J_2J_4	
e^0	6	6	6	18
e^1	16	8	16	40
e^2	19	18	18	55
e^3	32	19	32	38
e^4	33	30	31	94
Total	106	81	103	290

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Table II
Numbers of terms in the determining function S_3^*

	$J_3J_4^2J_2^3$	$(J_3/J_2)^3$	J_3J_4/J_2	J_2J_3	$(J_4/J_2)^3$	$(J_3^2J_4/J_2^3)$	J_4^2/J_2	J_3^2/J_2	J_2J_4	J_2^3	
e	5	4	8	9	-	-	-	-	-	-	26
e^2	-	-	-	-	6	7	15	7	15	15	65
e^3	19	14	26	26	-	-	-	-	-	-	86
Total	24	18	34	35	6	7	15	7	15	15	176

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$$E*SI^{6}*(235/8192-19531/65536*E^{2}) \qquad SIN\ (6*SG+7*M)$$

$$+E^{3}*SI^{4}*(2870799/114688-6818593/229376*SI^{2}) \qquad SIN\ (4*SG+7*M)$$

$$+SI^{6}*(15/4096-63/1024*E^{2}+14463/65536*E^{4}) \qquad SIN\ (6*SG+6*M)$$

$$+E^{2}*SI^{4}*(18517/2048-44301/4096*SI^{2}-282515/6144*E^{2}+105793/2048*E^{2}*SI^{2}) \qquad SIN\ (4*SG+6*M)$$

$$+E^{4}*SI^{2}*(397793/4096-5639543/24576*SI^{2}+26702183/196608*SI^{4}) \qquad SIN\ (2*SG+6*M)$$

$$+E*SI^{6}*(-61/8192+2731/65536*E^{2}) \qquad SIN\ (6*SG+5*M)$$

$$+E*SI^{4}*(25907/10240-12537/4096*SI^{2}-1572423/81920*E^{2}+705243/32768*E^{2}*SI^{2}) \qquad SIN\ (4*SG+5*M)$$

Fig. 1. Determining function S_3 with factor J_2^3.

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COEFFICIENT OF COS(SG)

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1/A2^3*A3*A4^2*E/L^9*SI*(-50/ING^2+3325/8/ING^2*SI^2-616575/512/ING^2*SI^4+723275/512/ING^2*SI^6-3543925/6144/ING^2*SI^8+1125/32*E^2/ING^3
*SI^2-120375/512*E^2/ING^3*SI^4+16625/32*E^2/ING^3*SI^6-2925125/6144*E^2/ING^3*SI^8+961625/6144*E^2/ING^3*SI^10-2925/8*E^2/ING^2+392475
/128*E^2/ING^2*SI^2-71025/8*E^2/ING^2*SI^4+7975975/768*E^2/ING^2*SI^6-12986225/3072*E^2/ING^2*SI^8)
+ 1/A2^3*A3^3*E/L^5*SI*(35/32/ING*SI^2-35/32/ING*SI^4+1/6-35/48*SI^2-25/64*E^2/ING^2*SI^2+25/32*E^2/ING^2*SI^4-25/64*E^2/ING^2*SI^6-35/32*E^2
/ING+1055/128*E^2/ING*SI^2-915/128*E^2/ING*SI^4+47/48*E^2-575/192*E^2*SI^2)
+ 1/A2*A3*A4*E/L^9*SI*(150/ING^2-15825/16/ING^2*SI^2+574175/256/ING^2*SI^4-1640825/768/ING^2*SI^6+750575/1024/ING^2*SI^8+285/4/ING
-21705/64/ING*SI^2+125555/256/ING*SI^4-115605/512/ING*SI^6+75/32*E^2/ING^3*SI^2-13225/256*E^2/ING^3*SI^4+240175/1536*E^2/ING^3*SI^6
-258925/1536*E^2/ING^3*SI^8+7875/128*E^2/ING^3*SI^10+13825/16*E^2/ING^2-707445/128*E^2/ING^2*SI^2+12423585/1024*E^2/ING^2*SI^4-68666125
/6144*E^2/ING^2*SI^6+15171975/4096*E^2/ING^2*SI^8+25725/64*E^2/ING-934995/512*E^2/ING*SI^2+2583775/1024*E^2/ING*SI^4-569695/512*E^2/ING
*SI^6)
+ A2*A3*E/L^9*SI*(-108/ING^2+8619/16/ING^2*SI^2-1519933/1536/ING^2*SI^4+407615/512/ING^2*SI^6-30375/128/ING^2*SI^8-99/4/ING+4329/64/ING
*SI^2-7535/128/ING*SI^4+7875/512/ING*SI^6-195/8*E^2/ING^3*SI^2+201475/1536*E^2/ING^3*SI^4-386335/1536*E^2/ING^3*SI^6+105975/512*E^2/ING^3
*SI^8-31875/512*E^2/ING^3*SI^10-7323/16*E^2/ING^2+51769/24*E^2/ING^2*SI^2-11443601/3072*E^2/ING^2*SI^4+5738015/2048*E^2/ING^2*SI^6-3178525
/4096*E^2/ING^2*SI^8-6593/64*E^2/ING+129189/512*E^2/ING*SI^2-366959/2048*E^2/ING*SI^4+108245/4096*E^2/ING*SI^6)
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Fig. 2. Determining function S_3^*.

When both short- and long-period terms have been eliminated, the new Hamiltonian F** contains no angular variables. The action variables L'', G'', and H'' are constant, and the angular variables ℓ'', g'', and h'' are expressed as follows:

$$\ell'' = -\frac{\partial}{\partial L''} F^{**} + \ell_0'',$$

$$g'' = -\frac{\partial}{\partial G''} F^{**} + g_0'', \qquad (9)$$

$$h'' = -\frac{\partial}{\partial H''} F^{**} + h_0''.$$

The generating functions S and S* determine a completely canonical transformation from the osculating elements (L, G, H, ℓ, g, and h) to the mean elements (L'', G'', H'', ℓ'', and h''). Because ℓ, g, and e do not satisfy the d'Alembert characteristics with respect to the eccentricity, we selected ℓ + g, h, e cos i, e sin i, L, and H as the set of elements for calculating the ephemeris of a satellite, as Deprit and Rom (1970) did. If ε stands for one of these elements, the osculating element ε is obtained from the following formulas through the third order of J_2^3:

$$\varepsilon = \varepsilon' + \{\varepsilon', S\} + \frac{1}{2}\{\{\varepsilon', S\}, S\} - \frac{1}{6}\{\{\{\varepsilon', S\}, S\}, S\} + O(J_2^4),$$

$$\varepsilon' = \varepsilon'' + \{\varepsilon'', S^*\} + \frac{1}{2}\{\{\varepsilon'', S^*\}, S^*\} + \frac{1}{6}\{\{\{\varepsilon'', S^*\}, S^*\}, S^*\} + O(J_2^4). \qquad (10)$$

These expressions do not contain negative powers of the eccentricity. Even though neither ℓ + g nor h satisfies the d'Alembert characteristics with respect to the inclination, the sum of ℓ + g + h does, thus providing another good check on the lengthy calculations. Figures 3 and 4 show parts of the third-order short- and long-period perturbations of ℓ + g. The method of calculating position and velocity from these elements is given in Deprit and Rom (1970).

3. ANALYTICAL AND NUMERICAL CHECK

Hori (1970a) and Yuasa (1971) showed that both Hori's and von Zeipel's perturbation theories give the same canonical transformation, together with the same new Hamiltonian, up to the third order. This allows us to compare the present solution, based on Hori's theory, with Kozai's (1962) solution, based on von Zeipel's theory. Our F_3^{**} completely agress with Kozai's F_3^{**}. Although Hori (1966) gave a relation between the determining functions of his and von Zeipel's theories, we have not compared these functions, because of the complexity of Kozai's ex-

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$$E/SI*(-69/16+1737/64*SI^{2}-5909/128*SI^{4}+1029/32*SI^{6}+87/64*E^{2}-35307/512*E^{2}*SI^{2}+155723/512*E^{2}*SI^{4}-299091/1024*E^{2}*SI^{6})\quad COS\ (3*SG+4*M)$$

$$+\ E^{3}/SI*(2115/64-110099/512*SI^{2}+178439/768*SI^{4}-81685/2048*SI^{6})\quad COS\ (SG+4*M)$$

$$+\ E^{2}*SI*(-225/256+135/512*SI^{2}+5865/4096*SI^{4})\quad COS\ (5*SG+3*M)$$

$$+\ 1/SI*(-9/8+189/32*SI^{2}-15/2*SI^{4}+555/128*SI^{6}+9/8*E^{2}-3735/128*E^{2}*SI^{2}+62009/512*E^{2}*SI^{4}-57333/512*E^{2}*SI^{6})\quad COS\ (3*SG+3*M)$$

$$+\ E^{2}/SI*(543/32-5583/64*SI^{2}+45049/1024*SI^{4}+143131/4096*SI^{6})\quad COS\ (SG+3*M)$$

$$+\ E^{3}*SI*(25/128-55/768*SI^{2}-1565/6144*SI^{4})\quad COS\ (5*SG+2*M)$$

$$+\ E/SI*(9/8-195/16*SI^{2}+665/16*SI^{4}-2265/64*SI^{6}+219/32*E^{2}-14745/256*E^{2}*SI^{2}+47773/256*E^{2}*SI^{4}-299649/2048*E^{2}*SI^{6})\quad COS\ (3*SG+2*M)$$

Fig. 3. Third-order short-period perturbation of $\ell + g$ with the factor J_2J_3.

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COEFFICIENT OF COS(SG)

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1/A2^3*A3*A4^2*E/L^10*(-1000/ING^3*SI+37625/4/ING^3*SI^3-527375/16/ING^3*SI^5+1712375/32/ING^3*SI^7-1306375/32/ING^3*SI^9+189875/16/ING^3
*SI^11+50/ING^2/SI-7125/4/ING^2*SI+2897125/256/ING^2*SI^3-14174475/512/ING^2*SI^5+29614375/1024/ING^2*SI^7-22217825/2048/ING^2*SI^9)
+ 1/A2^3*A3^3*E/L^6*(25/2/ING^2*SI^3-25/ING^2*SI^5+25/2/ING^2*SI^7-55/16/ING*SI+505/32/ING*SI^3-395/32/ING*SI^5-1/8/SI+53/16*SI-413/64
*SI^3)
+ 1/A2*A3*A4*E/L^10*(3000/ING^3*SI-91475/4/ING^3*SI^3+519325/8/ING^3*SI^5-5623675/64/ING^3*SI^7+7356475/128/ING^3*SI^9-1875125/128/ING^3
*SI^11-150/ING^2/SI+42085/8/ING^2*SI-3546125/128/ING^2*SI^3+14268245/256/ING^2*SI^5-25027215/512/ING^2*SI^7+16119775/1024/ING^2*SI^9
-285/4/ING/SI+56355/32/ING*SI-847595/128/ING*SI^3+546425/64/ING*SI^5-1861195/512/ING*SI^7)
+ A2*A3*E/L^10*(-2160/ING^3*SI+25575/2/ING^3*SI^3-478465/16/ING^3*SI^5+2214225/64/ING^3*SI^7-1269275/64/ING^3*SI^9+144375/32/ING^3*SI^11
+108/ING^2/SI-23883/8/ING^2*SI+3200581/256/ING^2*SI^3-10716335/512/ING^2*SI^5+8069435/512/ING^2*SI^7-2581125/512/ING^2*SI^9+99/4/ING
/SI-14905/32/ING*SI+151543/128/ING*SI^3-538593/512/ING*SI^5+311625/1024/ING*SI^7)
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Fig. 4. Third-order long-period perturbation of $\ell + g$.

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pression of S_2 and the tediousness of the calculations. Instead, we compared our S_2 with that derived later by Hori (1970b). In determining S, we have an ambiguity and may add an arbitrary function of L, G, H, and g to S, giving us $\bar{S}$. In the present theory, the disturbing function is expanded into a Fourier series with arguments of ℓ and g; therefore, it is natural to determine S in such a manner that its mean vlue with respect to the mean anomaly is zero. On the other hand, the mean value of Hori's S is not zero. The relation between S and $\bar{S}$, both of which are determining functions eliminating short-period terms in the Hamiltonian, is

$$\bar{S}_1 - S_1 + f_1,$$
$$\bar{S}_2 = S_2 + \frac{1}{2}\{S_1, f_1\} + f_2, \tag{11}$$

where f_1 and f_2 are arbitrary functions of L, G, H, and g. Then, the relation between S^* and $\bar{S}^*$, which are determining functions eliminating long-period terms, is

$$\bar{S}^*_1 = S^*_1 - f_1,$$
$$\bar{S}^*_2 = S^*_2 + \frac{1}{2}\{S^*_1, f_1\} - f_2. \tag{12}$$

Even if the functional forms of these determining functions are different, the composite canonical transformation of (S, S*) is identical to that of ($\bar{S}$, $\bar{S}$*). The second-order determining function S_2 fo the present theory and that derived by Hori are found to satisfy relation (11).

It is of interest to compare the present solution with that due to Deprit and Rom (1970), in which J_3 and J_4 are zeros. Their solution was obtained by their own perturbation method, which, like Hori's, is based on Lie transformation. Campbell and Jefferys (1970) showed that the perturbation theories of Hori and Deprit are equivalent and derived explicit relations between the determining functions for the two:

$$W_1 = -S_1,$$
$$W_2 = -2S_2, \tag{13}$$
$$W_3 = -6S_3 - \{S_1, S_2\},$$

where W_n are the determining functions in Deprit's theory. Using these relations, we compared our solution with Deprit and Rom's, which also serves as an independent check. We found only a few disagreements with their terms, which all seemed to be typographical errors. Kutuzov (1975) also discovered some discrepancies between his solution, obtained by computer algebra, and theirs.

As an internal consistency check, we wanted to make sure that a small divisor (1-5 $\cos^2$ i) disappeared for the even-order harmonics when the geopotential was equal to that in Vinti (1959) [$J_{2n} = (-1)^{n+1} J_2^n$, $n \geq 2$]. In checking the third order, we had to add long-period perturba-

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tions arising from J_6 and J_8.

Finally, the present solution was compared with the results obtained from numerical integration. A Taylor-type integrator was adopted, in which the positions and velocities are expanded into a power series of time and the coefficients of the series are determined by recurrent formulas (Rabe, 1961; Deprit and Zahar, 1966). The order of the power series and the step size of time were chosen so as to maintain about 12 significant figures in the integral of energy; the integration step is roughly one-fifth of the convergence radius τ of the two-body problem, $\tau = (1/n)\ \{\ln\ [(1/e) + \sqrt{(1/e^2) - 1}] - \sqrt{1 - e^2}\}$, when the degree of the Taylor series is 16. The numerical calculations were carried out by a CDC 6400 computer in double precision in order to avoid roundoff errors. It takes about 0.7 s to evaluate the series of the present theory and about 0.3 s to integrate the equation of motion for one step.

The initial conditions were computed from the present theory from a set of mean elements a" , e" , i", ℓ" , g" , and h" . It should be noted that the mean motion of the mean anomaly of the integrated orbit is expected to be different from the computed mean motion by an order of J_2^4 because the accuracy of the periodic terms is of the order of J_2^3. This discrepancy can be avoided by adjusting the semimajor axis so as to remove the secular term in the residuals of the mean anomaly.

Such comparisons were done for the artificial satellites Geos 3, Starlette, and Lageos. The results for Starlette are plotted in Figures 5 and 6. Figures 5a, 5b, and 5c show in-track, along-track, and across-track errors over two revolutions. The deviations are less than 2×10^{-4} m and are totally negligible. Figures 6a, 6b, and 6c show the errors over about 600 revolutions. The deviations along and across the track are totally insignificant, but the in-track error has a secular trend that seems to be proportional to the square of time. The error first increases, then decreases, and finally vanishes at about t = 22 days, at which time the semimajor axis was adjusted. This secular trend can be explained by the accumulation of truncation errors, which are caused by replacing infinitesimal operations with finite operations. The round-off errors do not accumulate significantly, because our calculations were carried out in double precision, which amounts to an accuracy of about 30 significant figures in decimal notation. The accumulation error due to discretization (Kinoshita, 1968) is

$$\Delta r_{\text{in-track}} \propto a\ell^2 h^p, \tag{14}$$

where a is the semimajor axis, ℓ the mean anomaly, h the step size, and p the degree of Taylor expansion. About 1 cm of the in-track error at t = 40 days (1 = 3600 radians) is obtained from using Equation (14) with $a = 7.335 \times 10^6$ m and h = 600 s = 1/10 radians, the order of the error agreeing with that of the numerical experiment. We can avoid discretization errors by employing a much more accurate integrator, but it seems to be an unnecessary use of computer time. The comparisons for Geos 3 and Lageos gave roughly the same results as for Starlette.

We are now confident that the present solution can predict the position of a geodetic satellite with a small eccentricity with an accuracy of better than 1 cm over 1 month.

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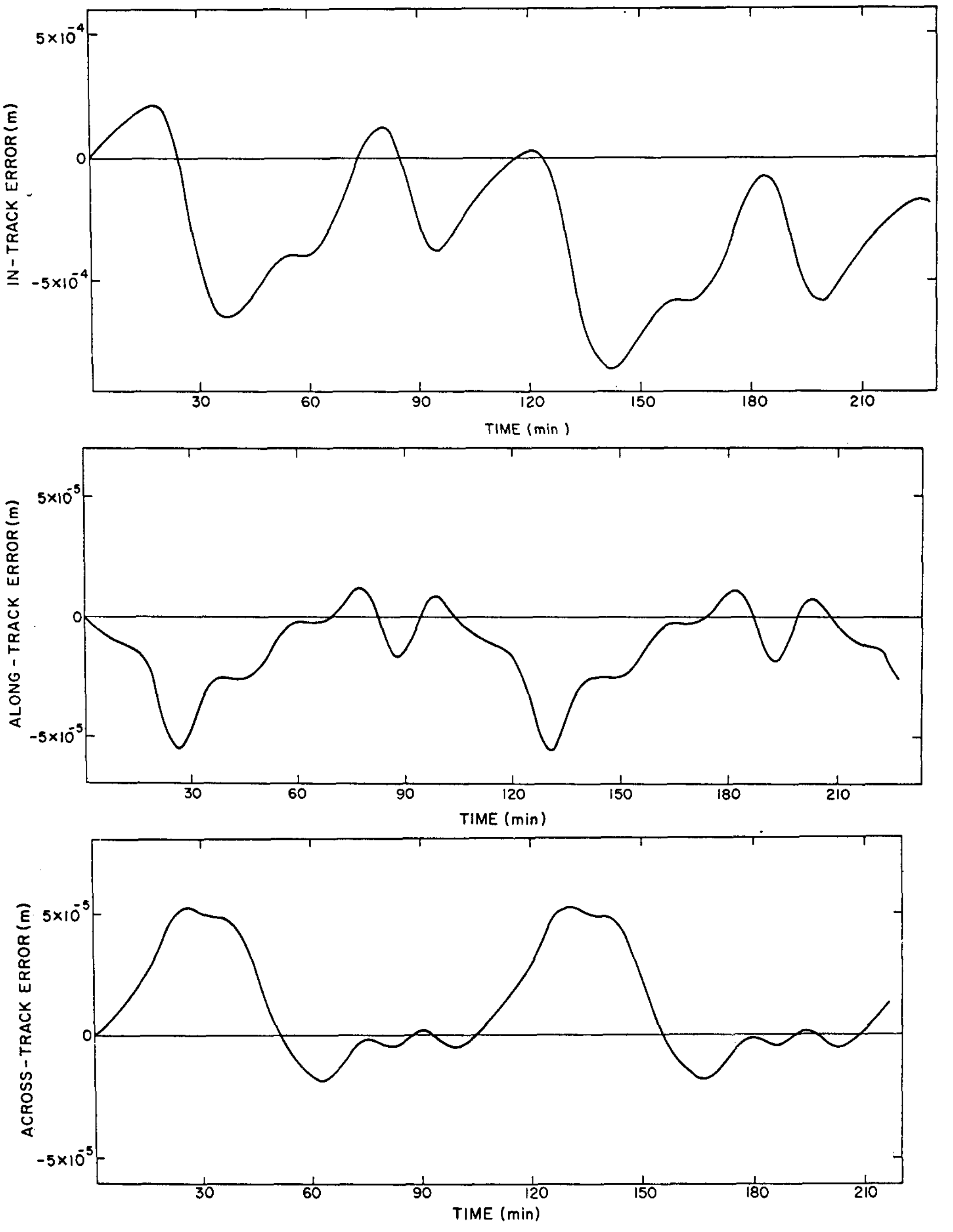

Fig. 5. Prediction error of the third-order solution for Starlette plotted over two revolutions: $J_2 = 1.082 \times 10^{-3}$, $J_3 = -2.54 \times 10^{-6}$, $J_4 = -1.619 \times 10^{-6}$, $a'' = 7.335 \times 10^{6}$ m, $e'' = 0.020636$, $i'' = 49.8223$, $\ell_0'' = 350.23968$, $g_0'' = 82.7702$, $h_0'' = 125.0266$, $P = 104.2$ min.

(a) in-track error.

(b) along-track error.

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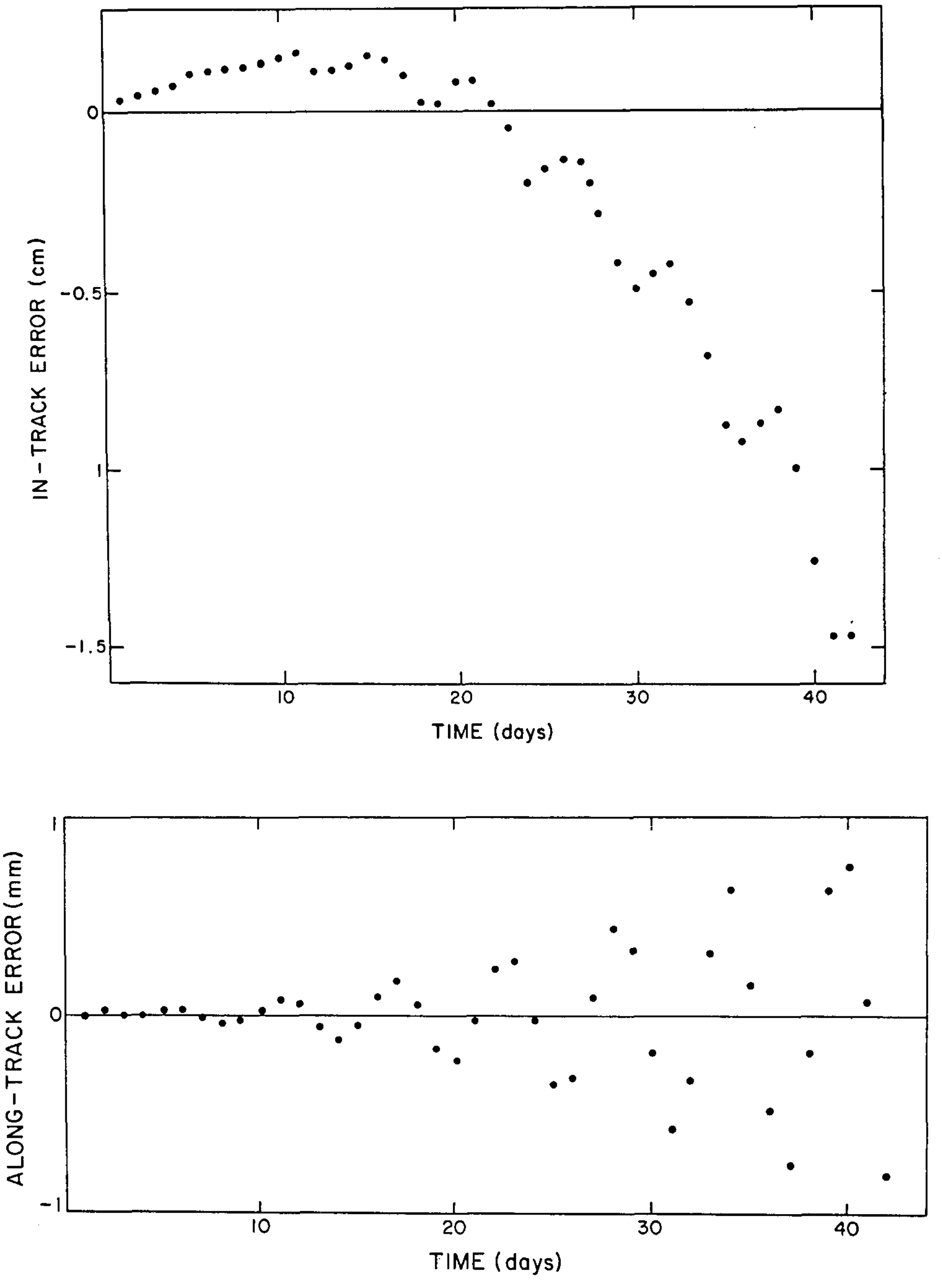

Fig. 6. The same as Figure 5 plotted over about 600 revolutions.

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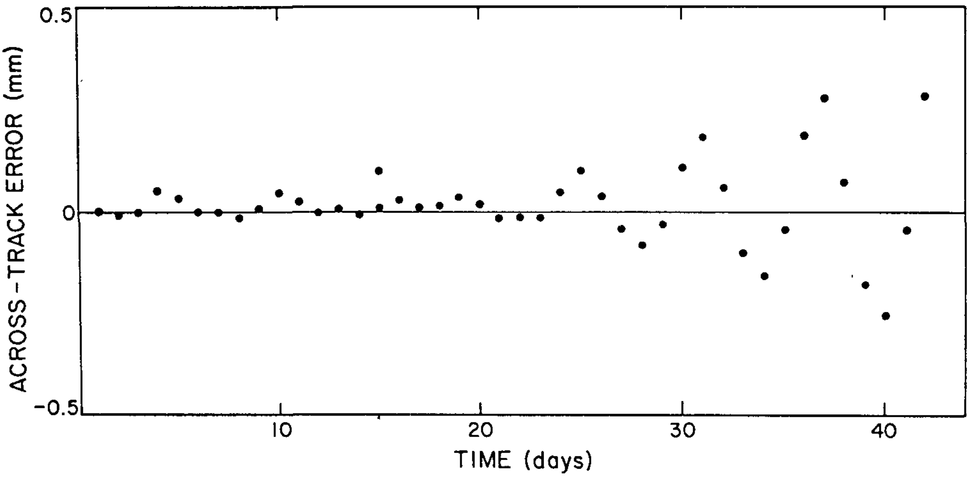

(c) across-track error.

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