Adv. Appl. Prob. **46,** 1148–1171 (2014)
Printed in Northern Ireland

CHARACTERIZATION OF THE GENERALIZED PÓLYA PROCESS AND ITS APPLICATIONS

JI HWAN CHA,* *Ewha Womans University*

Abstract

In this paper some important properties of the generalized Pólya process are derived and their applications are discussed. The generalized Pólya process is defined based on the stochastic intensity. By interpreting the defined stochastic intensity of the generalized Pólya process, the restarting property of the process is discussed. Based on the restarting property of the process, the joint distribution of the number of events is derived and the conditional joint distribution of the arrival times is also obtained. In addition, some properties of the compound process defined for the generalized Pólya process are derived. Furthermore, a new type of repair is defined based on the process and its application to the area of reliability is discussed. Several examples illustrating the applications of the obtained properties to various areas are suggested.

Keywords: Generalized Pólya process; restarting property; distribution of the number of events; conditional arrival time distribution; repair type; reliability application

2010 Mathematics Subject Classification: Primary 60K10
Secondary 62P30

1. Introduction

A stochastic process $\{N(t),\ t \geq 0\}$ is said to be a *counting process* if $N(t)$ represents the total number of 'events' that occur by time t. Counting processes (point processes) are useful tools for modelling random recurrent events. Recently, there has been a rapidly increasing literature concerning modelling and analysis of recurrent events, with a wide range of applications in, e.g. reliability analysis of repairable items, queueing analysis, insurance risk analysis, biology, and telecommunications.

The range of applications for counting processes is very wide. A few outline examples illustrate the breadth of potential applications. Emissions from a radioactive source occur in an irregular sequence in time in the area of physics. As another example, in the area of electronic engineering, the occurrences of peak signals of electrical energy define a sequence of points in time. In road traffic studies, one may consider the sequence of time points at which vehicles pass a reference point. Notably, almost all stochastic problems of operational research involve a point process. See also Cox and Isham (2000) for more examples of applications in different areas.

Traditionally, the most commonly used counting processes for modelling random recurrent events are renewal processes and nonhomogeneous Poisson processes (NHPPs), including the homogeneous Poisson process (HPP) as a special case in both models. The simplest but one of the most important counting processes is the HPP. The HPP can be characterized by the independent and identically distributed interarrival times with an exponential distribution. It is

Received 12 April 2013; revision received 2 December 2013.
* Postal address: Department of Statistics, Ewha Womans University, Seoul, 120-750, Korea.
Email address: jhcha@ewha.ac.kr

well known that the HPP possesses both the independent increments and stationary increments properties. A renewal process is a counting process where the interarrival times are independent and identically distributed with arbitrary distribution and nonnegative support (see Cox (1962)). The NHPP differs from the HPP in that the rate of occurrence varies with time rather than being constant, dropping the stationary increments property of the HPP. However, as the NHPP still possesses the independent increments property, it leads to nice closed-form results in many applications (see Cha and Finkelstein (2009), (2011a)).

To date, much effort has been invested into generalizing the basic counting processes stated above to more generalized counting processes. The generalization of the HPP or the NHPP may include the compound, filtered, two-dimensional, and marked Poisson processes (see Kao (1997)). Recently, semi-Markov processes were intensively studied and employed as one of the generalized counting processes in many applications (see Limnios and Oprişan (2001) and Barbu and Limnios (2008)). While the semi-Markov processes are a natural generalization of Markov processes, the renewal processes can also be regarded as a particular case of the semi-Markov processes.

In this paper we will characterize a new counting process called the 'generalized Pólya process' (GPP) suggested in Konno (2010). The GPP can be viewed as a further generalization of the NHPP which possesses neither the independent increments nor the stationary increments property. While only the marginal distribution of the number of events in $(0, t]$ was obtained in Konno (2010) based on differential equations, further detailed characterization of the GPP will be performed in this paper by deriving various properties which can usefully be used in many applications. Even though the GPP has neither the independent increments nor the stationary increments property, it will be shown that the GPP possesses very 'nice' properties which may yield closed-form results in various applications. Furthermore, one of the most important contributions to the area of reliability will be that the GPP allows us to define a 'new repair type' and a 'new failure process'. This will eventually contribute to the development of a variety of new maintenance models and related topics in the area of reliability.

The organization of this paper is as follows. In Section 2, some fundamental properties of the GPP are discussed. The 'restarting property' of the GPP is primarily discussed based on the stochastic intensity of the counting process. Based on the restarting property of the GPP, the joint distribution of the number of events in arbitrary, nonoverlapping intervals is derived. In Section 3, the conditional joint distributions of the arrival times in an arbitrary interval are derived. An example which illustrates the utility of the obtained results is provided. In Section 4, the compound process for the GPP is defined and useful results are derived. In Section 5, based on the GPP, a new repair type is defined and the effect of the defined repair is discussed. We show that, depending on the parameters of the GPP, the 'degree of the repair' can be modelled continuously. A replacement model is considered and the optimal replacement problem is studied as an illustration of application. Finally, in Section 6, some concluding remarks are given and potential areas to which the GPP can be applied are discussed.

2. Fundamental property

Let $\{N(t),\ t \geq 0\}$ be an orderly point process, and let $\mathcal{H}_{t-} \equiv \{N(u),\ 0 \leq u < t\}$ be the history (internal filtration) of the process in $[0, t)$, i.e. the set of all point events in $[0, t)$. Observe that $\mathcal{H}_{t-}$ can equivalently be defined in terms of $N(t-)$ and the sequential arrival points of the events $0 \leq T_1 \leq T_2 \leq \cdots \leq T_{N(t-)} < t$ in $[0, t)$, where T_i is the time from 0 until the arrival of the ith event in $[0, t)$. A convenient mathematical description of point processes follows from using the concept of the stochastic intensity $\lambda_t,\ t \geq 0$ (the intensity process) (see

Aven and Jensen (1999), (2000)). As discussed in Cha and Finkelstein (2011b), the stochastic intensity λ_t of an orderly point process $\{N(t),\ t \geq 0\}$ is defined as the limit

$$\lambda_t \equiv \lim_{\Delta t \to 0} \frac{\mathbb{P}(N(t, t+\Delta t) = 1 \mid \mathcal{H}_{t-})}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbb{E}[N(t, t+\Delta t) \mid \mathcal{H}_{t-}]}{\Delta t}, \tag{1}$$

where $N(t_1, t_2),\ t_1 < t_2$, represents the number of events in $[t_1, t_2)$. Then the above stochastic intensity has the heuristic interpretation

$$\lambda_t \, \mathrm{d}t = \mathbb{E}[\mathrm{d}N(t) \mid \mathcal{H}_{t-}], \tag{2}$$

which is very similar to the ordinary failure rate or hazard rate of a random variable (see Aven and Jensen (1999)). A clear understanding of the definition of the stochastic intensity given in (1) and the heuristic interpretation in (2) is crucial to deriving the fundamental properties of the GPP in this and the subsequent sections. A classical example of λ_t is the intensity process generated by the renewal process

$$\lambda_t = \sum_{n=0}^{\infty} \lambda(t - T_n)\, \mathbf{1}(T_n < t \leq T_{n+1}), \qquad T_0 \equiv 0,$$

where $\lambda(t)$ is the failure rate function of the distribution of interarrival times in the renewal process. Another standard example is the 'deterministic stochastic intensity' $\lambda_t = \lambda(t),\ t \geq 0$, which defines the NHPP with intensity function $\lambda(t)$. It is then clear that, in the NHPP, the future process (i.e. the process after time t) does not depend on the process history $\mathcal{H}_{t-}$, but depends only on the process 'age' t. Note that the renewal process and the NHPP can be interpreted as the 'perfect repair process' and the 'minimal repair process' in reliability applications (see Aven and Jensen (2000), and Finkelstein and Cha (2013)), and this type of repair-based interpretation of the point process is important in reliability applications (see Section 5).

Now the GPP is formally defined in terms of the stochastic intensity.

Definition 1. (*Generalized Pólya process.*) A counting process $\{N(t),\ t \geq 0\}$ is called the generalized Pólya process (GPP) with parameter set $(\lambda(t), \alpha, \beta)$, $\alpha \geq 0$ and $\beta > 0$, if

(i) $N(0) = 0$;

(ii) $\lambda_t = (\alpha N(t-) + \beta)\lambda(t)$.

Note that the GPP with $(\lambda(t),\ \alpha = 0,\ \beta = 1)$ reduces to the NHPP with intensity function $\lambda(t)$ and, accordingly, the GPP can be understood as a generalized version of the NHPP. Obviously, the GPP with $\alpha > 0$ does not possess the independent increments property. In the following discussions, we will implicitly assume that $\alpha > 0$, unless otherwise specified.

It is clear that, from Definition 1, the GPP possesses the Markovian property. In many applications, in addition to the Markovian property, the following restarting property makes the stochastic analysis much simpler.

Definition 2. (*Restarting property.*) Let $t > 0$ be an 'arbitrary' time point. If the conditional future stochastic process from t, given the history until time t, follows the same type of stochastic process with a possibly different set of process parameters, then the process is said to possess the *restarting property*. A stochastic process with the restarting property is called the *restarting process*.

Note that the Markovian property does not imply the restarting property. A counter example is the Yule process, which has the Markovian property but does not possess the restarting property. Observe that the ordinary renewal process does not possess the restarting property as it restarts only at each renewal point. However, let us consider the delayed renewal process $\{N(t),\ t \geq 0\}$ with the first interarrival time distribution $(F(v+t) - F(v))/(1 - F(v))$ and the common remaining interarrival times distribution $F(t)$, i.e. the 'initial age' of the first interarrival time distribution in this case is v. Then this delayed renewal process is characterized by the set of parameters $(v, F(t))$ and, at an arbitrary time $u > 0$, given $T_{N(u-)} = x^*$, the conditional future process $\{N_u(t),\ t \geq 0\}$, where $N_u(t) \equiv N(u+t) - N(u)$, is also the delayed renewal process with parameter set $(u - x^*, F(t))$. The simplest restarting process is obviously the HPP. For the NHPP with intensity function (process parameter) $\lambda(t)$, at an arbitrary time $u > 0$, the conditional future process $\{N_u(t),\ t \geq 0\}$ is the NHPP with process parameter $\lambda(u+t),\ t \geq 0$. Note that, for the HPP and NHPP, the restarting parameters do not depend on the given history. From Definition 1, it is now clear that the GPP has the restarting property, which we state in detail in the following proposition.

Proposition 1. *Let $\{N(t),\ t \geq 0\}$ be the GPP with parameter set $(\lambda(t), \alpha, \beta)$. At an arbitrary time $u > 0$, given $\{N(u-) = n,\ T_1 = t_1,\ T_2 = t_2, \dots, T_n = t_n\}$, the conditional future process $\{N_u(t),\ t \geq 0\}$, where $N_u(t) \equiv N(u+t) - N(u)$, is also the GPP with parameter set $(\lambda(u+t), \alpha, \beta + n\alpha),\ t \geq 0$.*

Throughout the rest of the paper, the restarting property stated in Proposition 1 will be critical in deriving the properties of the GPP.

We will now discuss the distribution of the number of events. The following first 'general result', along with some important marginal and conditional distributions of the number of events, gives the joint distribution of the number of events in an arbitrary number of consecutive, nonoverlapping time intervals. From this result, it will be shown that all other joint and conditional distributions for the number of events in different time intervals can also be obtained. In the following, we define $\Lambda(t) \equiv \int_0^t \lambda(u)\,\mathrm{d}u$.

Theorem 1. *Let $t > 0$ and $0 \equiv u_0 < u_1 < u_2 < \cdots < u_m$. Then*

$$\text{(i)}\quad \mathbb{P}(N(t) = n) = \frac{\Gamma(\beta/\alpha + n)}{\Gamma(\beta/\alpha)n!}(1 - \exp\{-\alpha\Lambda(t)\})^n(\exp\{-\alpha\Lambda(t)\})^{\beta/\alpha};$$

$$\begin{aligned}\text{(ii)}\quad &\mathbb{P}(N(u_2) - N(u_1) = n)\\ &= \frac{\Gamma(\beta/\alpha + n)}{\Gamma(\beta/\alpha)n!}\left(\frac{1 - \exp\{-\alpha[\Lambda(u_2) - \Lambda(u_1)]\}}{1 + \exp\{-\alpha\Lambda(u_2)\} - \exp\{-\alpha[\Lambda(u_2) - \Lambda(u_1)]\}}\right)^n\\ &\quad\times\left(\frac{\exp\{-\alpha\Lambda(u_2)\}}{1 + \exp\{-\alpha\Lambda(u_2)\} - \exp\{-\alpha[\Lambda(u_2) - \Lambda(u_1)]\}}\right)^{\beta/\alpha};\end{aligned}$$

$$\begin{aligned}\text{(iii)}\quad &\mathbb{P}(N(u_i) - N(u_{i-1}) = n_i,\ i = 1, 2, \dots, m)\\ &= \prod_{i=1}^{m}\Bigg[\frac{\Gamma(\beta/\alpha + \sum_{k=1}^{i} n_k)}{\Gamma(\beta/\alpha + \sum_{k=1}^{i-1} n_k)n_i!}(1 - \exp\{-\alpha[\Lambda(u_i) - \Lambda(u_{i-1})]\})^{n_i}\\ &\qquad\times(\exp\{-\alpha[\Lambda(u_i) - \Lambda(u_{i-1})]\})^{\sum_{k=1}^{i-1} n_k + \beta/\alpha}\Bigg],\end{aligned}$$

where $\sum_{k=1}^{i-1} n_k \equiv 0$ when $i = 1$;

(iv) $\mathbb{P}(N(u_2) - N(u_1) = n_2 \mid N(u_1) = n_1)$

$$= \frac{\Gamma(\beta/\alpha + n_1 + n_2)}{\Gamma(\beta/\alpha + n_1)n_2!}(1 - \exp\{-\alpha[\Lambda(u_2) - \Lambda(u_1)]\})^{n_2} \times (\exp\{-\alpha[\Lambda(u_2) - \Lambda(u_1)]\})^{n_1+\beta/\alpha}.$$

Proof. See Appendix A.

Remark 1. (i) The marginal and conditional distributions in Theorem 1 follow negative binomial distributions.

(ii) The joint distribution of the number of events in separate time intervals and related conditional distributions could be obtained using Theorem 1(iii) and following the procedures described in the proof of Theorem 1 in Appendix A. For example, the joint distribution of $(N(u_4) - N(u_3), N(u_2) - N(u_1))$ can be obtained from the joint distribution of $(N(u_i) - N(u_{i-1}),\ i = 1, 2, 3, 4)$.

(iii) The restarting property of the GPP can be usefully used to characterize the conditional future process of the Yule process with parameter λ and no individual at time 0 (i.e. $N(0) = 0$). Specifically, given $N(u-) = n$, the future process follows the GPP with parameter set $(\lambda, 1, n)$.

It was stated in Proposition 1 that, given $N(u-)$, the conditional future process $\{N_u(t),\ t \geq 0\}$ in the GPP is also the GPP. It was also mentioned that the HPP and the NHPP possess the restarting property. However, it is important to understand that the future processes in the HPP and the NHPP are 'unconditionally' the HPP and the NHPP, respectively. That is, without any information on the history of the processes, the future processes can be perfectly described in the same manner in the cases of the HPP and the NHPP. This type of stronger property makes the relevant analysis much simpler in many applications.

How about the GPP? If the future process from an arbitrary time point u is 'unconditionally' the GPP as in the cases of the HPP and the NHPP, then this property could be usefully used in many applications. For example, one may start to observe the GPP from time u without any information on the history of the process before u (see also Section 3). Now let us see whether or not the future process from an arbitrary time point u in the GPP is 'unconditionally' the GPP.

Let us fix $u > 0$, and, as before, define $N_u(t) \equiv N(u + t) - N(u)$. Then $\{N_u(t),\ t \geq 0\}$ represents the future process from the time point u. Let T_{ui} be the time from 0 until the arrival of the ith event in (u, ∞), $u \leq T_{u1} \leq T_{u2} \leq \cdots$. In order to characterize the process $\{N_u(t),\ t \geq 0\}$, it is sufficient to specify the stochastic intensity of the future process λ_t^u, which is defined by

$$\lambda_t^u \equiv \lim_{\Delta t \to 0} \frac{\mathbb{P}(N_u(t, t + \Delta t) = 1 \mid \mathcal{H}_{[u,u+t)})}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbb{E}[N_u(t, t + \Delta t) \mid \mathcal{H}_{[u,u+t)}]}{\Delta t},$$

where $N_u(t_1, t_2)$, $t_1 < t_2$, represents the number of events in $[u + t_1, u + t_2)$ and $\mathcal{H}_{[u,u+t)}$ is the history of the process in $[u, u + t)$. Note that $\mathcal{H}_{[u,u+t)}$ can be completely defined by the number of events and the sequential arrival times in the interval $[u, u + t)$. The following theorem implies that, 'unconditionally', the future process of the GPP is also the GPP.

Theorem 2. *The stochastic intensity λ_t^u is given by*

$$\lambda_t^u = (\alpha[N((u + t)-) - N(u-)] + \beta)\psi(t, u),$$

https://doi.org/10.1239/aap/1418396247 Published online by Cambridge University Press

where

$$\psi(t,u) \equiv \frac{\lambda(u+t)\exp\{\alpha\Lambda(u+t)\}}{1+\exp\{\alpha\Lambda(u+t)\}-\exp\{\alpha\Lambda(u)\}},$$

and, thus, the future process of the GPP, $\{N_u(t),\ t \geq 0\}$, is 'unconditionally' the GPP with parameter set $(\psi(t,u), \alpha, \beta)$.

Proof. See Appendix B.

Remark 2. Recall that, for the GPP $\{N(t),\ t \geq 0\}$ with parameter set $(\lambda(t), \alpha, \beta)$, $\mathbb{P}(N(t) = n)$ was obtained as (see Theorem 1)

$$\mathbb{P}(N(t)=n) = \frac{\Gamma(\beta/\alpha+n)}{\Gamma(\beta/\alpha)n!}(1-\exp\{-\alpha\Lambda(t)\})^n(\exp\{-\alpha\Lambda(t)\})^{\beta/\alpha}. \tag{3}$$

Now, owing to the property described in Theorem 2, the distribution of the number of events in an arbitrary time interval $(u, u+t]$, $\mathbb{P}(N(t+u)-N(u)=n)$, can be obtained as follows. Note that the parameter function $\psi(t,u)$ in $\{N_u(t),\ t \geq 0\}$ corresponds to $\lambda(t)$ in $\{N(t),\ t \geq 0\}$. Thus, in order to use (3), we obtain

$$\int_0^t \psi(w,u)\,\mathrm{d}w = \frac{1}{\alpha}\ln(1+\exp\{\alpha\Lambda(u+t)\}-\exp\{\alpha\Lambda(u)\}),$$

which corresponds to $\Lambda(t)$ in (3). Therefore, using (3), we have

$$\begin{aligned}\mathbb{P}(N(u+t)-N(u)=n) &= \frac{\Gamma(\beta/\alpha+n)}{\Gamma(\beta/\alpha)n!}\bigg(\frac{\exp\{\alpha\Lambda(u+t)\}-\exp\{\alpha\Lambda(u)\}}{1+\exp\{\alpha\Lambda(u+t)\}-\exp\{\alpha\Lambda(u)\}}\bigg)^n \\ &\quad\times\bigg(\frac{1}{1+\exp\{\alpha\Lambda(u+t)\}-\exp\{\alpha\Lambda(u)\}}\bigg)^{\beta/\alpha},\end{aligned} \tag{4}$$

which is Theorem 1(ii).

3. Conditional distribution of the arrival times

In this section we derive the conditional distribution of the arrival times in an 'arbitrary time interval' $(u, v]$, given $N(v)-N(u)$, $v > u$, in the GPP. If the process $\{N(t),\ t \geq 0\}$ is the NHPP with intensity function $\lambda(t)$ then it is well known that the conditional arrival time distribution of $T_1, T_2, \ldots, T_{N(t)}$ in $(0, t]$, given that $N(t) = n$, is given by (see Ross (1996))

$$n!\prod_{i=1}^{n}\bigg(\frac{\lambda(t_i)}{\Lambda(t)}\bigg), \qquad 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq t. \tag{5}$$

Now we consider the conditional arrival time distribution in an arbitrary time interval $(u, v]$, $v > u$. As before, let T_{ui} be the time from 0 until the arrival of the ith event in (u, ∞), $u \leq T_{u1} \leq T_{u2} \leq \cdots$. As the future process of the NHPP from an arbitrary time point u is 'unconditionally' (i.e. without any information on the history) the NHPP with process parameter $\lambda(u+t)$, $t \geq 0$, the conditional arrival time distribution of $T_{u1}, T_{u2}, \ldots, T_{u(N(v)-N(u))}$ in $(u, v]$, $v > u$, given that $N(v)-N(u) = n$, is, using (5),

$$n!\prod_{i=1}^{n}\bigg(\frac{\lambda(u+(t_{ui}-u))}{\int_0^{v-u}\lambda(u+w)\,\mathrm{d}w}\bigg) = n!\prod_{i=1}^{n}\bigg(\frac{\lambda(t_{ui})}{\Lambda(v)-\Lambda(u)}\bigg), \qquad u \leq t_{u1} \leq t_{u2} \leq \cdots \leq t_{un} \leq v. \tag{6}$$

In the following, we will apply a similar procedure to derive the conditional distribution of the arrival times in the GPP.

Theorem 3. *The conditional joint distribution of the arrival times*

$$(T_{u1}, T_{u2}, \dots, T_{u(N(v)-N(u))})$$

in $(u, v]$, $v > u$, *given that* $N(v) - N(u) = n$, *is*

$$f_{T_{u1},T_{u2},\dots,T_{u(N(v)-N(u))} \mid N(v)-N(u)}(t_{u1}, t_{u2}, \dots, t_{un} \mid n) \\ = n! \prod_{i=1}^{n} \bigg(\frac{\alpha\lambda(t_{ui}) \exp\{\alpha\Lambda(t_{ui})\}}{\exp\{\alpha\Lambda(v)\} - \exp\{\alpha\Lambda(u)\}} \bigg), \tag{7}$$

where $u < t_{u1} \le t_{u2} \le \cdots \le t_{un} \le v$.

Proof. We first obtain the conditional arrival time distribution of $T_1, T_2, \dots, T_{N(t)}$ in $(0, t]$, given that $N(t) = n$. Observe that the joint distribution of $(T_1, T_2, \dots, T_{N(t)}, N(t))$ is given by

$$f_{T_1,T_2,\dots,T_{N(t)},N(t)}(t_1, t_2, \dots, t_n, n) \\ = \frac{\Gamma(\beta/\alpha + n)}{\Gamma(\beta/\alpha)} \bigg(\prod_{i=1}^{n} \alpha\lambda(t_i) \exp\{\alpha\Lambda(t_i)\} \bigg) \exp\{-(\beta + n\alpha)\Lambda(t)\},$$

whereas

$$\mathbb{P}(N(t) = n) = \frac{\Gamma(\beta/\alpha + n)}{\Gamma(\beta/\alpha)n!} (1 - \exp\{-\alpha\Lambda(t)\})^n (\exp\{-\alpha\Lambda(t)\})^{\beta/\alpha}.$$

Thus, the conditional arrival time distribution of $T_1, T_2, \dots, T_{N(t)}$ in $(0, t]$, given that $N(t) = n$, is

$$f_{T_1,T_2,\dots,T_{N(t)} \mid N(t)}(t_1, t_2, \dots, t_n \mid n) \\ = n! \prod_{i=1}^{n} \bigg(\frac{\alpha\lambda(t_i) \exp\{\alpha\Lambda(t_i)\}}{\exp\{\alpha\Lambda(t)\} - 1} \bigg), \qquad 0 < t_1 \le t_2 \le \cdots \le t_n \le t. \tag{8}$$

Now we consider the conditional arrival time distribution in an arbitrary time interval $(u, u + t]$, $t > 0$, based on the above result. From Theorem 2, the unconditional process $\{N_u(t),\ t \ge 0\}$ is the GPP with parameter set $(\psi(t, u), \alpha, \beta)$, where

$$\psi(t, u) \equiv \frac{\lambda(u + t) \exp\{\alpha\Lambda(u + t)\}}{1 + \exp\{\alpha\Lambda(u + t)\} - \exp\{\alpha\Lambda(u)\}}.$$

Therefore, applying (8), the conditional arrival time distribution of

$$(T_{u1}, T_{u2}, \dots, T_{u(N(u+t)-N(u))})$$

in $(u, u + t]$, given that $N(u + t) - N(u) = n$, is

$$f_{T_{u1},T_{u2},\dots,T_{u(N(u+t)-N(u))} \mid N(u+t)-N(u)}(t_{u1}, t_{u2}, \dots, t_{un} \mid n) \\ = n! \prod_{i=1}^{n} \bigg(\frac{\alpha\psi(t_{ui} - u, u) \exp\{\alpha \int_0^{t_{ui}-u} \psi(w, u)\, \mathrm{d}w\}}{\exp\{\alpha \int_0^{t} \psi(w, u)\, \mathrm{d}w\} - 1} \bigg) \\ = n! \prod_{i=1}^{n} \bigg(\frac{\alpha\lambda(t_{ui}) \exp\{\alpha\Lambda(t_{ui})\}}{\exp\{\alpha\Lambda(u + t)\} - \exp\{\alpha\Lambda(u)\}} \bigg)$$

for $u < t_{u1} \le t_{u2} \le \cdots \le t_{un} \le u + t$. Setting $u + t \equiv v$ yields the desired result.

Remark 3. It can be seen from Theorem 3 that, given that $N(v) - N(u) = n$, the n random variables $T_{u1}, T_{u2}, \ldots, T_{un}$ have the same distribution as the order statistics corresponding to the n independent random variables, identically distributed according to

$$\left(\frac{\alpha\lambda(x)\exp\{\alpha\Lambda(x)\}}{\exp\{\alpha\Lambda(v)\} - \exp\{\alpha\Lambda(u)\}}\right), \qquad u < x \leq v.$$

Having derived the conditional joint distribution of the arrival times in the interval $(u, v]$, given $N(v) - N(u)$, in Theorem 3, it is sometimes of interest to derive this distribution given the event history in the previous interval $(0, u]$, $\{N(u), T_1, T_2, \ldots, T_{N(u)}\}$, in addition to $N(v) - N(u)$. That is, the conditional joint distribution of

$$(T_{u1}, T_{u2}, \ldots, T_{u(N(v)-N(u))} \mid T_1, T_2, \ldots, T_{N(u)}, N(u), N(v) - N(u))$$

can be important in some cases (see Example 1 below). Suppose that $\{N(t),\ t \geq 0\}$ is the NHPP with intensity function $\lambda(t)$. Then, owing to the independent increments property of the NHPP, it is clear that

$$\begin{aligned}(T_{u1}, T_{u2}, \ldots, T_{u(N(v)-N(u))} &\mid T_1, T_2, \ldots, T_{N(u)}, N(u), N(v) - N(u)) \\ &\stackrel{\mathrm{D}}{=} (T_{u1}, T_{u2}, \ldots, T_{u(N(v)-N(u))} \mid N(v) - N(u)),\end{aligned}$$

where '$\stackrel{\mathrm{D}}{=}$' stands for equality in distribution, and this conditional distribution is also given by (6). However, as the GPP does not possess the independent increments property, the conditional joint distribution of

$$(T_{u1}, T_{u2}, \ldots, T_{u(N(v)-N(u))} \mid T_1, T_2, \ldots, T_{N(u)}, N(u), N(v) - N(u))$$

would depend on the event history in the previous interval $(0, u]$, $\{N(u), T_1, T_2, \ldots, T_{N(u)}\}$, in some way. Interpreting the definition of the GPP in Section 1 and considering the Markov property, it should depend only on $N(u)$ among the elements of the event history in the previous interval, i.e.

$$\begin{aligned}(T_{u1}, T_{u2}, \ldots, T_{u(N(v)-N(u))} &\mid T_1, T_2, \ldots, T_{N(u)}, N(u), N(v) - N(u)) \\ &\stackrel{\mathrm{D}}{=} (T_{u1}, T_{u2}, \ldots, T_{u(N(v)-N(u))} \mid N(u), N(v) - N(u)).\end{aligned}$$

However, the following result shows that, given $N(v) - N(u)$, the conditional joint distribution of the arrival times in the interval $(u, v]$ does not depend on the event history in the previous interval.

Theorem 4. *For the conditional joint distribution of the arrival times in the interval $(u, v]$,*

$$\begin{aligned}(T_{u1}, T_{u2}, \ldots, T_{u(N(v)-N(u))} &\mid T_1, T_2, \ldots, T_{N(u)}, N(u), N(v) - N(u)) \\ &\stackrel{\mathrm{D}}{=} (T_{u1}, T_{u2}, \ldots, T_{u(N(v)-N(u))} \mid N(v) - N(u)),\end{aligned}$$

and, thus, the conditional joint distribution of

$$(T_{u1}, T_{u2}, \ldots, T_{u(N(v)-N(u))} \mid T_1, T_2, \ldots, T_{N(u)}, N(u), N(v) - N(u))$$

is also given by (7).

Proof. From the proof of Theorem 2, the joint distribution of

$$(T_1, T_2, \ldots, T_{N(u)}, T_{u1}, T_{u2}, \ldots, T_{u(N(v)-N(u))}, N(u), N(v) - N(u))$$

is given by

$$\begin{aligned} &f_{T_i,\,1\le i\le N(u),T_{uj},\,1\le j\le N(v)-N(u),N(u),N(v)-N(u)}(t_1, t_2, \ldots, t_{n_1}, t_{u1}, t_{u2}, \ldots, t_{un_2}, n_1, n_2) \\ &\quad = \Bigg[\frac{\Gamma(\beta/\alpha + n_1)}{\Gamma(\beta/\alpha)}\bigg(\prod_{i=1}^{n_1} \alpha\lambda(t_i)\exp\{\alpha\Lambda(t_i)\}\bigg)\frac{\Gamma(\beta/\alpha + n_1 + n_2)}{\Gamma(\beta/\alpha + n_1)} \\ &\qquad\times \bigg(\prod_{j=1}^{n_2} \alpha\lambda(t_{uj})\exp\{\alpha\Lambda(t_{uj})\}\bigg)\exp\{-(\beta + n_1\alpha + n_2\alpha)\Lambda(v)\}\Bigg], \end{aligned} \tag{9}$$

whereas the joint distribution of

$$(T_1, T_2, \ldots, T_{N(u)}, N(u), N(v) - N(u))$$

is given by

$$\begin{aligned} &f_{T_i,\,1\le i\le N(u),N(u),N(v)-N(u)}(t_1, t_2, \ldots, t_{n_1}, n_1, n_2) \\ &\quad = \Bigg[\frac{\Gamma(\beta/\alpha + n_1)}{\Gamma(\beta/\alpha)}\bigg(\prod_{i=1}^{n_1} \alpha\lambda(t_i)\exp\{\alpha\Lambda(t_i)\}\bigg)\frac{\Gamma(\beta/\alpha + n_1 + n_2)}{\Gamma(\beta/\alpha + n_1)} \\ &\qquad\times \int_u^v \cdots \int_u^{t_{u3}} \int_u^{t_{u2}} \bigg(\prod_{j=1}^{n_2} \alpha\lambda(t_{uj})\exp\{\alpha\Lambda(t_{uj})\}\bigg)\,\mathrm{d}t_{u1}\,\mathrm{d}t_{u2}\cdots\mathrm{d}t_{un_2} \\ &\qquad\times \exp\{-(\beta + n_1\alpha + n_2\alpha)\Lambda(v)\}\Bigg] \\ &\quad = \Bigg[\frac{\Gamma(\beta/\alpha + n_1)}{\Gamma(\beta/\alpha)}\bigg(\prod_{i=1}^{n_1} \alpha\lambda(t_i)\exp\{\alpha\Lambda(t_i)\}\bigg)\frac{\Gamma(\beta/\alpha + n_1 + n_2)}{\Gamma(\beta/\alpha + n_1)n_2!} \\ &\qquad\times (\exp\{\alpha\Lambda(v)\} - \exp\{\alpha\Lambda(u)\})^{n_2}\exp\{-(\beta + n_1\alpha + n_2\alpha)\Lambda(v)\}\Bigg]. \end{aligned} \tag{10}$$

Then the result follows from (9) and (10).

Remark 4. From Theorem 4, it can be seen that, given $N(v) - N(u)$, $\{T_{u1}, T_{u2}, \ldots, T_{u(N(v)-N(u))}\}$ and $\{T_1, T_2, \ldots, T_{N(u)}, N(u)\}$ are conditionally independent.

Example 1. Suppose that each event from the GPP with parameter set $(\lambda(t), \alpha, \beta)$ is classified as being either a type-1 or type-2 event, and suppose that the probability of an event being classified as type 1 depends on the time at which it occurs. More specifically, suppose that if an event occurs at time t then, independently of all else, it is classified as being a type-1 event with probability $p(t)$ and a type-2 event with probability $1 - p(t)$. This type of classification model has many useful applications in reliability and queueing analysis, e.g. two types of shocks causing the system failure or two types of customers arriving at the server, etc. (see also Cha and Finkelstein (2009), (2011a)).

Let $N_i(t)$, $i = 1, 2$, represent the number of type-i events that occur by time t. Suppose that the event history in the previous interval $(0, u]$, $\{N(u), T_1, T_2, \ldots, T_{N(u)}\}$, was observed and that we are now interested in obtaining the distribution of $N_i(v) - N_i(u),\ v > u, i = 1, 2$. Consider the conditional distribution of $N_1(v) - N_1(u)$, given that $\{N(u) = m,\ T_1 = t_1,\ T_2 = t_2, \dots, T_m = t_m\}$. Observe that

$$\begin{aligned}
&\mathbb{P}(N_1(v) - N_1(u) = n \mid T_1 = t_1,\ T_2 = t_2, \dots, T_m = t_m,\ N(u) = m) \\
&\quad = \mathbb{E}_{(N(v)-N(u)\,|\,T_1=t_1,\,T_2=t_2,\dots,T_m=t_m,\,N(u)=m)}[\mathbb{P}(N_1(v) - N_1(u) = n \mid T_1 = t_1, \\
&\qquad\qquad T_2 = t_2, \dots, T_m = t_m, \\
&\qquad\qquad N(u) = m,\ N(v) - N(u))], \qquad (11)
\end{aligned}$$

where '$\mathbb{E}_{(N(v)-N(u)\,|\,T_1=t_1,\,T_2=t_2,\dots,T_m=t_m,\,N(u)=m)}$' stands for the expectation with respect to the conditional distribution of

$$(N(v) - N(u) \mid T_1 = t_1,\ T_2 = t_2, \dots, T_m = t_m,\ N(u) = m).$$

It can be shown that

$$(N(v) - N(u) \mid T_1 = t_1,\ T_2 = t_2, \dots, T_m = t_m,\ N(u) = m) \overset{\mathrm{D}}{=} (N(v) - N(u) \mid N(u) = m),$$

and the expectation in (11) can be written as

$$\begin{aligned}
&\mathbb{E}_{(N(v)-N(u)\,|\,N(u)=m)}[\mathbb{P}(N_1(v) - N_1(u) = n \mid T_1 = t_1, T_2 = t_2, \dots, T_m = t_m, \\
&\qquad N(u) = m,\ N(v) - N(u))]. \qquad (12)
\end{aligned}$$

Furthermore, from the assumption on the classification (i.e. the classification depends only on the occurrence time) and the property that (see Theorem 4)

$$\begin{aligned}
&(T_{u1}, T_{u2}, \dots, T_{u(N(v)-N(u))} \mid T_1, T_2, \dots, T_{N(u)}, N(u), N(v) - N(u)) \\
&\quad \overset{\mathrm{D}}{=} (T_{u1}, T_{u2}, \dots, T_{u(N(v)-N(u))} \mid N(v) - N(u)),
\end{aligned}$$

it holds that

$$\begin{aligned}
&\mathbb{P}(N_1(v) - N_1(u) = n \mid T_1 = t_1,\ T_2 = t_2, \dots, T_m = t_m,\ N(u) = m,\ N(v) - N(u) = k) \\
&\quad = \mathbb{P}(N_1(v) - N_1(u) = n \mid N(v) - N(u) = k).
\end{aligned}$$

Now let us consider an arbitrary event that occurred in the interval $(u, v]$. If it had occurred at time $x \in (u, v]$ then the probability that it would be a type-1 event would be $p(x)$. Hence, by Theorem 3 (see also Remark 3), it follows that the probability that it will be a type-1 event is

$$\phi(u, v) \equiv \left(\frac{\alpha \int_u^v p(x)\lambda(x) \exp\{\alpha\Lambda(x)\}\,\mathrm{d}x}{\exp\{\alpha\Lambda(v)\} - \exp\{\alpha\Lambda(u)\}}\right),$$

independently of the other events. Hence,

$$\mathbb{P}(N_1(v) - N_1(u) = n \mid N(v) - N(u) = k) = \binom{k}{n}(\phi(u, v))^n(1 - \phi(u, v))^{k-n}.$$

On the other hand, from Theorem 1,

$$\begin{aligned}
&\mathbb{P}(N(v) - N(u) = k \mid N(u) = m) \\
&\quad = \frac{\Gamma(\beta/\alpha + m + k)}{\Gamma(\beta/\alpha + m)k!}(1 - \exp\{-\alpha[\Lambda(v) - \Lambda(u)]\})^k(\exp\{-\alpha[\Lambda(v) - \Lambda(u)]\})^{m+\beta/\alpha}.
\end{aligned}$$

https://doi.org/10.1239/aap/1418396247 Published online by Cambridge University Press

Finally, from (12),

$$\begin{aligned}
&\mathbb{P}(N_1(v) - N_1(u) = n \mid T_1 = t_1,\ T_2 = t_2, \dots, T_m = t_m,\ N(u) = m) \\
&\quad = \sum_{k=n}^{\infty} \binom{k}{n} (\phi(u, v))^n (1 - \phi(u, v))^{k-n} \frac{\Gamma(\beta/\alpha + m + k)}{\Gamma(\beta/\alpha + m) k!} \\
&\qquad\qquad \times (1 - \exp\{-\alpha[\Lambda(v) - \Lambda(u)]\})^k (\exp\{-\alpha[\Lambda(v) - \Lambda(u)]\})^{m+\beta/\alpha} \\
&\quad = \sum_{l=0}^{\infty} \frac{1}{l!\, n!} (\phi(u, v))^n (1 - \phi(u, v))^l \frac{\Gamma(\beta/\alpha + m + n + l)}{\Gamma(\beta/\alpha + m)} \\
&\qquad\qquad \times (1 - \exp\{-\alpha[\Lambda(v) - \Lambda(u)]\})^{n+l} (\exp\{-\alpha[\Lambda(v) - \Lambda(u)]\})^{m+\beta/\alpha} \\
&\quad = \left[\sum_{l=0}^{\infty} \frac{\Gamma(\beta/\alpha + m + n + l)}{\Gamma(\beta/\alpha + m + n) l!} ((1 - \phi(u, v))(1 - \exp\{-\alpha[\Lambda(v) - \Lambda(u)]\}))^l\right] \\
&\qquad \times \frac{\Gamma(\beta/\alpha + m + n)}{\Gamma(\beta/\alpha + m) n!} ((\phi(u, v))(1 - \exp\{-\alpha[\Lambda(v) - \Lambda(u)]\}))^n \\
&\qquad \times (\exp\{-\alpha[\Lambda(v) - \Lambda(u)]\})^{m+\beta/\alpha} \\
&\quad = \frac{\Gamma(\beta/\alpha + m + n)}{\Gamma(\beta/\alpha + m) n!} \\
&\qquad \times \left(\frac{\phi(u, v) - \phi(u, v) \exp\{-\alpha[\Lambda(v) - \Lambda(u)]\}}{\phi(u, v) + \exp\{-\alpha[\Lambda(v) - \Lambda(u)]\} - \phi(u, v) \exp\{-\alpha[\Lambda(v) - \Lambda(u)]\}}\right)^n \\
&\qquad \times \left(\frac{\exp\{-\alpha[\Lambda(v) - \Lambda(u)]\}}{\phi(u, v) + \exp\{-\alpha[\Lambda(v) - \Lambda(u)]\} - \phi(u, v) \exp\{-\alpha[\Lambda(v) - \Lambda(u)]\}}\right)^{m+\beta/\alpha}.
\end{aligned}$$

Note that the unconditional distribution of $\mathbb{P}(N_1(v) - N_1(u) = n)$ can also be obtained in a similar, but much simpler, way.

4. Compound GPP

A stochastic process $\{W(t),\ t \geq 0\}$ is said to be a *compound GPP* if it can be represented as

$$W(t) = \sum_{i=1}^{N(t)} X_i, \qquad t \geq 0, \tag{13}$$

where $\{N(t),\ t \geq 0\}$ is the GPP, and $\{X_i,\ i \geq 1\}$ is a family of independent and identically distributed random variables that is independent of $\{N(t),\ t \geq 0\}$. Several practical applications of the compound process defined in (13) can be found in Ross (2003).

In the following discussions, instead of the basic compound process defined in (13), we will consider a more general case. Let us consider a compound process in an arbitrary time interval $(u, u + t]$, $t \geq 0$:

$$W_u(t) = \sum_{i=1}^{N_u(t)} X_i, \qquad t \geq 0.$$

Here, as before, $N_u(t) \equiv N(u + t) - N(u)$. First we will obtain some unconditional properties of $W_u(t)$ and then we will consider some of its conditional properties.

Let $M_X(s) \equiv \mathbb{E}[\mathrm{e}^{sX_i}]$. The following result gives the moment generating function, the mean, and the variance of $W_u(t)$.

https://doi.org/10.1239/aap/1418396247 Published online by Cambridge University Press

Theorem 5. *The moment generating function of $W_u(t)$, denoted by $M_{W_u(t)}(s)$, is given by*

$$M_{W_u(t)}(s) = (1 - [\exp\{\alpha\Lambda(u+t)\} - \exp\{\alpha\Lambda(u)\}](M_X(s) - 1))^{-\beta/\alpha},$$

and the mean and variance of $W_u(t)$ are given by

$$\mathbb{E}[W_u(t)] = \frac{\beta}{\alpha}(\exp\{\alpha\Lambda(u+t)\} - \exp\{\alpha\Lambda(u)\})\mathbb{E}[X]$$

and

$$\begin{aligned}\operatorname{var}[W_u(t)] &= \frac{\beta}{\alpha}(\exp\{\alpha\Lambda(u+t)\} - \exp\{\alpha\Lambda(u)\})\mathbb{E}[X^2] \\ &\quad + \frac{\beta}{\alpha}([\exp\{\alpha\Lambda(u+t)\} - \exp\{\alpha\Lambda(u)\}]\mathbb{E}[X])^2.\end{aligned}$$

Proof. From Theorem 2, the process $\{N_u(t),\ t \geq 0\}$ is the GPP with parameter set $(\psi(t, u), \alpha, \beta)$, where

$$\psi(t, u) \equiv \frac{\lambda(u+t)\exp\{\alpha\Lambda(u+t)\}}{1 + \exp\{\alpha\Lambda(u+t)\} - \exp\{\alpha\Lambda(u)\}}.$$

By conditioning on $N_u(t)$,

$$\begin{aligned}M_{W_u(t)}(s) &= \sum_{n=0}^{\infty}\mathbb{E}[\exp\{sW_u(t)\} \mid N_u(t) = n]\mathbb{P}(N_u(t) = n) \\ &= \sum_{n=0}^{\infty}\mathbb{E}[\exp\{s(X_1 + X_2 + \cdots + X_n)\} \mid N_u(t) = n]\mathbb{P}(N_u(t) = n) \\ &= \sum_{n=0}^{\infty}\mathbb{E}[\exp\{s(X_1 + X_2 + \cdots + X_n)\}]\mathbb{P}(N_u(t) = n) \\ &= \sum_{n=0}^{\infty}(M_X(s))^n\frac{\Gamma(\beta/\alpha + n)}{\Gamma(\beta/\alpha)n!}\bigg(\frac{\exp\{\alpha\Lambda(u+t)\} - \exp\{\alpha\Lambda(u)\}}{1 + \exp\{\alpha\Lambda(u+t)\} - \exp\{\alpha\Lambda(u)\}}\bigg)^n \\ &\quad\times\bigg(\frac{1}{1 + \exp\{\alpha\Lambda(u+t)\} - \exp\{\alpha\Lambda(u)\}}\bigg)^{\beta/\alpha} \\ &= (1 - [\exp\{\alpha\Lambda(u+t)\} - \exp\{\alpha\Lambda(u)\}](M_X(s) - 1))^{-\beta/\alpha}.\end{aligned} \tag{14}$$

Differentiating (14) yields

$$\begin{aligned}M_{W_u(t)}(s)' &= \frac{\beta}{\alpha}[\exp\{\alpha\Lambda(u+t)\} - \exp\{\alpha\Lambda(u)\}]M_X(s)' \\ &\quad\times(1 - [\exp\{\alpha\Lambda(u+t)\} - \exp\{\alpha\Lambda(u)\}](M_X(s) - 1))^{-\beta/\alpha - 1}\end{aligned}$$

and

$$\begin{aligned}M_{W_u(t)}(s)'' &= \frac{\beta}{\alpha}[\exp\{\alpha\Lambda(u+t)\} - \exp\{\alpha\Lambda(u)\}]M_X(s)'' \\ &\quad\times(1 - [\exp\{\alpha\Lambda(u+t)\} - \exp\{\alpha\Lambda(u)\}](M_X(s) - 1))^{-\beta/\alpha - 1} \\ &\quad+ \frac{\beta}{\alpha}\bigg(\frac{\beta}{\alpha} + 1\bigg)([\exp\{\alpha\Lambda(u+t)\} - \exp\{\alpha\Lambda(u)\}]M_X(s)')^2 \\ &\quad\times(1 - [\exp\{\alpha\Lambda(u+t)\} - \exp\{\alpha\Lambda(u)\}](M_X(s) - 1))^{-\beta/\alpha - 2}.\end{aligned}$$

Finally, $\mathbb{E}[W_u(t)]$ and $\operatorname{var}[W_u(t)]$ can be obtained from $M_{W_u(t)}(s)'|_{s=0}$ and $M_{W_u(t)}(s)''|_{s=0}$.

In the above discussion we considered the 'unconditional' compound process $\{W_u(t), t \geq 0\}$. Now we will consider the 'conditional' compound process given the process history in the previous interval $(0, u]$, $\{N(u), T_1, T_2, \ldots, T_{N(u)}\}$. The conditional property of $\{W_u(t), t \geq 0\}$ is more important in some situations. For example, $W_u(t)$ can be understood as the accumulated claims during the interval $(u, u+t]$ in insurance applications and one may have observed the occurrences of these events during the previous interval $(0, u]$. Note that if the counting process $\{N(t), t \geq 0\}$ is the NHPP then the conditional compound process has the same stochastic property as the unconditional compound process since the NHPP possesses the independent increments property. For the GPP, we have the following property for the conditional compound process.

Theorem 6. *The conditional moment generating function of $W_u(t)$, given that $\{N(u) = n, T_1 = t_1, T_2 = t_2, \ldots, T_n = t_n\}$, denoted by $M_{W_u(t)\,|\,T_1=t_1,\,T_2=t_2,\ldots,T_n=t_n,\,N(u)=n}(s)$, is given by*

$$\begin{aligned} &M_{W_u(t)\,|\,T_1=t_1,\,T_2=t_2,\ldots,T_n=t_n,\,N(u)=n}(s) \\ &\quad = (\exp\{\alpha[\Lambda(u+t) - \Lambda(u)]\} - M_X(s)(\exp\{\alpha[\Lambda(u+t) - \Lambda(u)]\} - 1))^{-(\beta/\alpha+n)}, \end{aligned}$$

and the conditional mean and variance of $W_u(t)$ are given by

$$\begin{aligned} &\mathbb{E}[W_u(t) \mid T_1 = t_1,\, T_2 = t_2, \ldots, T_n = t_n,\, N(u) = n] \\ &\quad = \left(\frac{\beta}{\alpha} + n\right)(\exp\{\alpha[\Lambda(u+t) - \Lambda(u)]\} - 1)\mathbb{E}[X] \end{aligned}$$

and

$$\begin{aligned} &\operatorname{var}[W_u(t) \mid T_1 = t_1,\, T_2 = t_2, \ldots, T_n = t_n,\, N(u) = n] \\ &\quad = \left(\frac{\beta}{\alpha} + n\right)(\exp\{\alpha[\Lambda(u+t) - \Lambda(u)]\} - 1)\mathbb{E}[X^2] \\ &\quad\quad + \left(\frac{\beta}{\alpha} + n\right)([\exp\{\alpha[\Lambda(u+t) - \Lambda(u)]\} - 1]\mathbb{E}[X])^2. \end{aligned}$$

Proof. From Proposition 1, the conditional future process $\{N_u(t), t \geq 0\}$ is the GPP with parameter set $(\lambda(u+t), \alpha, \beta + n\alpha)$, $t \geq 0$. Then, similar to the proof of Theorem 5,

$$\begin{aligned} &M_{W_u(t)\,|\,T_1=t_1,\,T_2=t_2,\ldots,T_n=t_n,N(u)=n}(s) \\ &\quad = \mathbb{E}[\exp\{sW_u(t)\} \mid T_1 = t_1,\, T_2 = t_2, \ldots, T_n = t_n,\, N(u) = n] \\ &\quad = \sum_{m=0}^{\infty} \mathbb{E}[\exp\{sW_u(t)\} \mid T_1 = t_1,\, T_2 = t_2, \ldots, T_n = t_n,\, N(u) = n,\, N_u(t) = m] \\ &\quad\quad\quad \times \mathbb{P}(N_u(t) = m \mid T_1 = t_1,\, T_2 = t_2, \ldots, T_n = t_n,\, N(u) = n) \\ &\quad = \sum_{m=0}^{\infty} \mathbb{E}[\exp\{s(X_1 + X_2 + \cdots + X_m)\}] \\ &\quad\quad\quad \times \mathbb{P}(N_u(t) = m \mid T_1 = t_1,\, T_2 = t_2, \ldots, T_n = t_n,\, N(u) = n) \\ &\quad = \sum_{m=0}^{\infty} (M_X(s))^m \frac{\Gamma(\beta/\alpha + n + m)}{\Gamma(\beta/\alpha + n)m!}(1 - \exp\{-\alpha[\Lambda(u+t) - \Lambda(u)]\})^m \\ &\quad\quad\quad \times (\exp\{-\alpha[\Lambda(u+t) - \Lambda(u)]\})^{n+\beta/\alpha} \\ &\quad = (\exp\{\alpha[\Lambda(u+t) - \Lambda(u)]\} - M_X(s)(\exp\{\alpha[\Lambda(u+t) - \Lambda(u)]\} - 1))^{-(\beta/\alpha+n)}. \end{aligned}$$

https://doi.org/10.1239/aap/1418396247 Published online by Cambridge University Press

Then, following the same procedures as those described in the proof of Theorem 5, the desired result is obtained.

5. Reliability application

In the reliability area many different types of repair have been suggested and applied. When the failure rate of the item is increasing, the performance of the item deteriorates with time. This eventually results in a low efficiency of the item and, at the same time, a high operational cost. In order to maximize the item efficiency or minimize the operational cost, various repair and replacement policies have been studied and thoroughly discussed in the literature. Surveys on various maintenance models from a practical point of view can be found in, e.g. Sherif and Smith (1981), Valdez-Flores and Feldman (1989), Wang (2002), and Tadj *et al.* (2011). An overview on the maintenance theory from a theoretical point of view can be found in Nakagawa (2005). More theoretical and sophisticated models have also been developed (see Mi (1994), Ebrahimi (1997), Aven (1996), Aven and Jensen (2000), Cha (2001), (2003), and Badía *et al.* (2001)).

It is important to understand that *one counting process corresponds to one repair type and vice versa* in reliability applications. The most basic, but important, types of repair are 'perfect' and 'minimal' repairs. In a perfect repair, the system is returned to a state that is *as good as new*. This implies that the interfailure times in this case are independent and identically distributed, and, accordingly, the failure process of the repairable system with perfect repair is described by the renewal process. On the other hand, in a 'minimal repair', the state of the item after the repair is restored to the *as-bad-as-old* condition. More precisely, if the system with survival function $\bar{F}(t)$ has failed at time x then this type of repair implies that the survival function of the repaired system is given by

$$\bar{F}_x(t) \equiv \frac{\bar{F}(x+t)}{\bar{F}(x)} = \exp\left\{-\int_0^t r(x+u)\,\mathrm{d}u\right\},$$

where $r(t)$ is the failure rate function of the system. Thus, this type of repair restores our system to the state it had prior to the failure. It is well known that the failure process of the repairable system with minimal repair is the NHPP with intensity function

$$\lambda(t) = r(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\ln(\bar{F}(t)).$$

We will now define a new repair type based on the GPP. From Definition 1, recall that the stochastic intensity of the GPP is given by

$$\lambda_t \equiv \lim_{\Delta t \to 0} \frac{\mathbb{P}(N(t, t+\Delta t) = 1 \mid \mathcal{H}_{t-})}{\Delta t} = (\alpha N(t-) + \beta)\lambda(t). \tag{15}$$

To define a new repair type from the GPP, we interpret the 'event' as the 'failure' and define $N(t)$ as the number of 'failures' in the time interval $(0, t]$. Suppose that the failure process of the system is the GPP with parameter set $(\lambda(t), \alpha, \beta)$. Then the survival function of the time to the first failure is given by

$$\bar{F}(t) = \exp\left\{-\int_0^t \beta\lambda(u)\,\mathrm{d}u\right\}, \qquad t \ge 0. \tag{16}$$

Therefore, from (16), the failure rate of our system is given by $r(t) = \beta\lambda(t)$. Now suppose that this system has failed at time x_1 (the first failure) and has instantly been repaired according to

https://doi.org/10.1239/aap/1418396247 Published online by Cambridge University Press

the repair type which corresponds to the stochastic intensity in (15). Then the survival function of the repaired system is given by

$$\begin{aligned}\bar{F}_{x_1}^{[1]}(t) &\equiv \exp\bigg\{-\int_0^t (\beta\lambda(x_1+u)+\alpha\lambda(x_1+u))\,\mathrm{d}u\bigg\}\\ &= \exp\bigg\{-\int_0^t (r(x_1+u)+\alpha\lambda(x_1+u))\,\mathrm{d}u\bigg\},\end{aligned}$$

where the notation $\bar{F}_s^{[n]}(t)$ stands for the (residual) survival function of the system which has been repaired at time s for the nth time, $n = 1, 2, \ldots$. Suppose now that the above repaired system has failed at time x_2, $x_2 > x_1$, for the second time and has been repaired according to the repair type which corresponds to the stochastic intensity in (15). Then the survival function of the repaired system is given by

$$\begin{aligned}\bar{F}_{x_2}^{[2]}(t) &\equiv \exp\bigg\{-\int_0^t (\beta\lambda(x_2+u)+2\alpha\lambda(x_2+u))\,\mathrm{d}u\bigg\}\\ &= \exp\bigg\{-\int_0^t (r(x_2+u)+2\alpha\lambda(x_2+u))\,\mathrm{d}u\bigg\}.\end{aligned}$$

In a similar fashion, we can generally define $\bar{F}_s^{[n]}(t)$:

$$\begin{aligned}\bar{F}_{s}^{[n]}(t) &\equiv \exp\bigg\{-\int_0^t (\beta\lambda(s+u)+n\alpha\lambda(s+u))\,\mathrm{d}u\bigg\}\\ &= \exp\bigg\{-\int_0^t (r(s+u)+n\alpha\lambda(s+u))\,\mathrm{d}u\bigg\}.\end{aligned}$$

In the following discussions, for convenience, we will call this type of repair a 'GPP repair'. Furthermore, we define (α, β) as the set of parameters of the GPP repair defined above. Then the failure process $\{N(t),\ t \geq 0\}$ for the system with failure rate $r(t)$ under the GPP repair with parameter set (α, β) is the GPP with parameter set $(r(t)/\beta, \alpha, \beta)$. Clearly, the GPP repair is a *worse than minimal repair* as the survival function of the system on each repair is smaller than that of the system on which the minimal repair is applied. Note that the parameters α and β determine the degree of the repair. Specifically, the case in which $\alpha = 0$ corresponds to the minimal repair, whereas $\alpha > 0$ implies that the repair is a worse than minimal repair. Furthermore, as α increases and β decreases to 0, the repair becomes worse and worse.

Now we will consider a simple replacement policy employing the GPP repair defined above. A system with failure rate $r(t)$ starts its operation at time 0 and is GPP repaired on each failure. We assume that the distribution of the lifetime of the system is proper, i.e. $\int_0^\infty r(t)\,\mathrm{d}t = \infty$, and without loss of generality that $\lim_{t\to\infty} r(t) > 0$. The system is replaced by an identical new system when its age reaches T, and the repair and replacement process is repeated again and again. The cost for a GPP repair is $c_{\mathrm{GPP}} > 0$ and that for a replacement is c_{r}. Then, in this case, the expected number of GPP repairs in one renewal cycle is, from Theorem 1,

$$\mathbb{E}[N(T)] = \frac{\beta}{\alpha}(\exp\{\alpha\Lambda(T)\} - 1),$$

where $\lambda(t) = r(t)/\beta$ and, thus, $\Lambda(t) = \int_0^t r(u)\,\mathrm{d}u/\beta$. From the renewal reward theorem (see, e.g. Ross (1996)), the long-run average cost rate function $C(T)$, as the function of T, is given by

$$C(T) = \frac{\beta(\exp\{\alpha\Lambda(T)\}-1)c_{\mathrm{GPP}}/\alpha + c_{\mathrm{r}}}{T} = \frac{\beta(\exp\{\alpha R(T)/\beta\}-1)c_{\mathrm{GPP}}/\alpha + c_{\mathrm{r}}}{T}, \qquad (17)$$

where $R(t) \equiv \int_0^t r(u)\,\mathrm{d}u$. The problem is to find the optimal T^* which minimizes $C(T)$ in (17). The property of the optimal T^* is given in the following theorem.

Theorem 7. *If $r(t)$ satisfies*

$$r'(t) + \frac{\alpha}{\beta} r^2(t) > 0 \quad \textit{for all } t > 0, \tag{18}$$

then there exists a unique optimal $T^* \in (0, \infty)$, which is the solution of the equation

$$Tr(T)\exp\left\{\frac{\alpha R(T)}{\beta}\right\} - \frac{\beta}{\alpha}\exp\left\{\frac{\alpha R(T)}{\beta}\right\} - \left(\frac{c_{\mathrm{r}}}{c_{\mathrm{GPP}}} - \frac{\beta}{\alpha}\right) = 0.$$

Proof. Differentiating $C(T)$ we obtain

$$C'(T) = \frac{c_{\mathrm{GPP}}}{T^2}\left[Tr(T)\exp\left\{\frac{\alpha R(T)}{\beta}\right\} - \frac{\beta}{\alpha}\exp\left\{\frac{\alpha R(T)}{\beta}\right\} - \left(\frac{c_{\mathrm{r}}}{c_{\mathrm{GPP}}} - \frac{\beta}{\alpha}\right)\right].$$

Let

$$\Phi(T) \equiv Tr(T)\exp\left\{\frac{\alpha R(T)}{\beta}\right\} - \frac{\beta}{\alpha}\exp\left\{\frac{\alpha R(T)}{\beta}\right\} - \left(\frac{c_{\mathrm{r}}}{c_{\mathrm{GPP}}} - \frac{\beta}{\alpha}\right).$$

Then we have $\Phi(0) = -c_{\mathrm{r}}/c_{\mathrm{GPP}} < 0$ and $\lim_{T\to\infty}\Phi(T) = \infty$. Furthermore, if condition (18) is satisfied then

$$\begin{aligned}\Phi'(T) &= Tr'(T)\exp\left\{\frac{\alpha R(T)}{\beta}\right\} + \frac{\alpha}{\beta}Tr^2(T)\exp\left\{\frac{\alpha R(T)}{\beta}\right\} \\ &= T\exp\left\{\frac{\alpha R(T)}{\beta}\right\}\left[r'(T) + \frac{\alpha}{\beta}r^2(T)\right] \\ &> 0 \quad \text{for all } T > 0.\end{aligned}$$

Thus, $\Phi(T)$ is strictly increasing with $\lim_{T\to\infty}\Phi(T) = \infty$. Therefore, there is a unique solution $T^* \in (0, \infty)$ which satisfies $\Phi(T^*) = 0$. It is now clear that this T^* satisfies $C'(T^*) = 0$ with $C'(T) < 0$ for $T < T^*$ and $C'(T) > 0$ for $T > T^*$. Therefore, this T^* is the optimal replacement time for the replacement policy.

Remark 5. Suppose that $r(t)$ is increasing. Then it is obvious that condition (18) is satisfied and, thus, there exists a unique optimal T^*. However, condition (18) does not necessarily require that $r(t)$ should be increasing. Even decreasing $r(t)$ can satisfy condition (18).

Suppose now that the failure rate function $r(t)$ is decreasing. If the repair type performed on each failure of the system is the minimal repair then the long-run average cost rate function $C(T)$ is simply given by

$$C(T) = \frac{c_{\mathrm{m}}R(T) + c_{\mathrm{r}}}{T},$$

where c_{m} is the cost for a minimal repair. When the failure rate function $r(t)$ is decreasing, it can be shown that the cost function $C(T)$ is strictly decreasing and that the optimal replacement time in this case is $T^* = \infty$. However, when the repair type performed on each failure of the system is the GPP repair, the situation dramatically changes as illustrated in the following example.

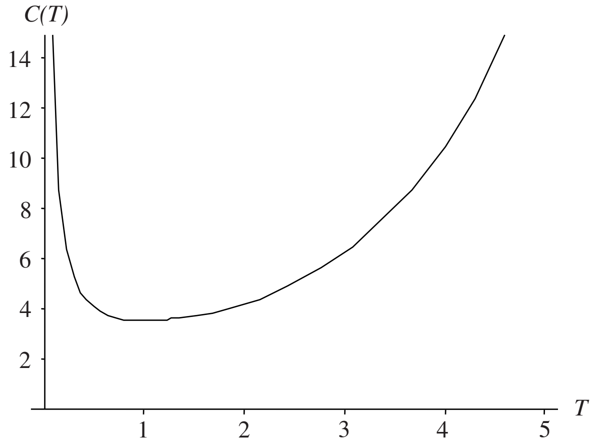

FIGURE 1: The long-run average cost rate $C(T)$.

Example 2. Suppose that the failure rate function of the system is given by

$$r(t) = \begin{cases} \frac{1}{4}(t-2)^2 + 1 & \text{if } 0 \leq t < 2, \\ 1 & \text{if } t \geq 2. \end{cases}$$

Therefore, the failure rate function is decreasing. Suppose further that the repair performed on the system is the GPP repair with parameter set ($\alpha = 0.8$, $\beta = 1$). The parameters for the costs are given by $c_{\mathrm{GPP}} = 0.8$ and $c_{\mathrm{r}} = 1.0$.

Then it is clear that inequality (18) is satisfied for $t \geq 2$. For $t \in (0, 2)$,

$$\begin{aligned} r'(t) + \frac{\alpha}{\beta} r^2(t) &= \frac{1}{20}((t-2)^4 + 8(t-2)^2 + 10(t-2) + 16) \\ &= \frac{1}{20}\left((t-2)^4 + 8\left[(t-2) + \frac{5}{8}\right]^2 + \frac{103}{8}\right) \\ &> 0 \quad \text{for all } t \in (0, 2). \end{aligned}$$

Therefore, condition (18) is satisfied. It is now clear that the optimal T^* is given by the unique solution of the equation

$$Tr(T) - 1.25 = 0,$$

and T^* cannot be greater than 2.0. Accordingly, there exists the optimal replacement time T^* in the interval $(0, 2.0)$. The long run average cost rate $C(T)$ is given in Figure 1.

It is generally known that, when the failure rate function is decreasing, preventive maintenance of the system is not necessary. This is true if the repair type performed on each failure is the minimal repair. However, as shown in this example, if the repair type is the GPP repair, it may be necessary to apply the preventive maintenance policy even if the failure rate function is decreasing.

6. Concluding remarks

In this paper we have characterized the GPP by deriving some of its properties. Even though the GPP has neither the independent increments nor the stationary increments property, it possesses very 'nice' properties, which simplifies relevant analysis in many applications. The most important characteristic of the GPP is the 'restarting property' discussed in this paper. With the help of this property, many properties of the GPP have been successfully derived. This important property can also be used in many applications to simplify the analysis.

As illustrated in the introduction, counting processes have been applied in many areas, such as reliability, queueing analysis, insurance risk analysis, biology, and telecommunications. As illustrated in detail in this paper, the results could be directly applied to various stochastic models in queueing and insurance. In the reliability area, numerous new repair types have been suggested, generalizing the 'minimal repair'. However, few repair types yield mathematically tractable results in relevant studies on maintenance policy. On the contrary, the repair type based on the GPP given in this paper allows a clear interpretation of the effect of the repair and yields mathematically tractable results. This will be a crucial contribution of this work especially in the area of reliability.

Appendix A. Proof of Theorem 1

We first derive $\mathbb{P}(N(t)=n)$. Let T_i, $i=1,\dots,N(t)$, be the sequential arrival times of the events in $(0,t]$. Then the joint distribution of $(T_1, T_2, \dots, T_{N(t)}, N(t))$ is given by

$$\begin{aligned}
&f_{T_i,\,1\le i\le N(t),N(t)}(t_1,t_2,\dots,t_n,n)\\
&\quad=[\beta\lambda(t_1)\exp\{-\beta\Lambda(t_1)\}(\beta+\alpha)\lambda(t_2)\exp\{-(\beta+\alpha)[\Lambda(t_2)-\Lambda(t_1)]\}\\
&\qquad\times(\beta+2\alpha)\lambda(t_3)\exp\{-(\beta+2\alpha)[\Lambda(t_3)-\Lambda(t_2)]\}\\
&\qquad\times\cdots\times(\beta+(n-1)\alpha)\lambda(t_n)\exp\{-(\beta+(n-1)\alpha)[\Lambda(t_n)-\Lambda(t_{n-1})]\}\\
&\qquad\times\exp\{-(\beta+n\alpha)[\Lambda(t)-\Lambda(t_n)]\}]\\
&\quad=\bigg[\frac{\beta}{\alpha}\bigg(1+\frac{\beta}{\alpha}\bigg)\bigg(2+\frac{\beta}{\alpha}\bigg)\cdots\bigg((n-1)+\frac{\beta}{\alpha}\bigg)\\
&\qquad\times\alpha\lambda(t_1)\exp\{\alpha\Lambda(t_1)\}\alpha\lambda(t_2)\exp\{\alpha\Lambda(t_2)\}\cdots\alpha\lambda(t_n)\exp\{\alpha\Lambda(t_n)\}\\
&\qquad\times\exp\{-(\beta+n\alpha)\Lambda(t)\}\bigg]\\
&\quad=\frac{\Gamma(\beta/\alpha+n)}{\Gamma(\beta/\alpha)}\bigg(\prod_{i=1}^{n}\alpha\lambda(t_i)\exp\{\alpha\Lambda(t_i)\}\bigg)\exp\{-(\beta+n\alpha)\Lambda(t)\},
\end{aligned}$$

where $0\le t_1\le t_2\le\cdots\le t_n\le t$. Thus, the marginal distribution of $N(t)$ can be obtained as

$$\begin{aligned}
\mathbb{P}(N(t)=n)&=\int_0^t\cdots\int_0^{t_3}\int_0^{t_2} f_{T_i,\,1\le i\le N(t),N(t)}(t_1,t_2,\dots,t_n,n)\,\mathrm{d}t_1\,\mathrm{d}t_2\cdots\mathrm{d}t_n\\
&=\bigg[\frac{\Gamma(\beta/\alpha+n)}{\Gamma(\beta/\alpha)}\int_0^t\cdots\int_0^{t_3}\int_0^{t_2}\bigg(\prod_{i=1}^{n}\alpha\lambda(t_i)\exp\{\alpha\Lambda(t_i)\}\bigg)\mathrm{d}t_1\,\mathrm{d}t_2\cdots\mathrm{d}t_n\\
&\qquad\times\exp\{-(\beta+n\alpha)\Lambda(t)\}\bigg]\\
&=\frac{\Gamma(\beta/\alpha+n)}{\Gamma(\beta/\alpha)n!}(\exp\{\alpha\Lambda(t)\}-1)^n\exp\{-(\beta+n\alpha)\Lambda(t)\}
\end{aligned}$$

https://doi.org/10.1239/aap/1418396247 Published online by Cambridge University Press

$$
\begin{aligned}
&= \frac{\Gamma(\beta/\alpha+n)}{\Gamma(\beta/\alpha)n!}(\exp\{\alpha\Lambda(t)\}-1)^n(\exp\{-\alpha\Lambda(t)\})^{n+\beta/\alpha} \\
&= \frac{\Gamma(\beta/\alpha+n)}{\Gamma(\beta/\alpha)n!}(1-\exp\{-\alpha\Lambda(t)\})^n(\exp\{-\alpha\Lambda(t)\})^{\beta/\alpha}. \qquad (19)
\end{aligned}
$$

We now obtain the conditional distribution of $\mathbb{P}(N(u_2)-N(u_1)=n_2 \mid N(u_1)=n_1)$. From Proposition 1, given that $N(u_1)=n_1$, the future process $\{N_{u_1}(t), t\geq 0\}$ is the GPP with parameter set $(\lambda(u_1+t), \alpha, \beta+n_1\alpha)$. Therefore, by replacing β in (19) with $\beta+n_1\alpha$ and $\Lambda(t)$ with $\int_0^{u_2-u_1}\lambda(u_1+s)\,\mathrm{d}s=\Lambda(u_2)-\Lambda(u_1)$, we have

$$
\begin{aligned}
&\mathbb{P}(N(u_2)-N(u_1)=n_2 \mid N(u_1)=n_1) \\
&\quad= \frac{\Gamma(\beta/\alpha+n_1+n_2)}{\Gamma(\beta/\alpha+n_1)n_2!}(1-\exp\{-\alpha[\Lambda(u_2)-\Lambda(u_1)]\})^{n_2} \\
&\qquad\times(\exp\{-\alpha[\Lambda(u_2)-\Lambda(u_1)]\})^{n_1+\beta/\alpha}. \qquad (20)
\end{aligned}
$$

Next we derive the marginal distribution of $N(u_2)-N(u_1)$. Observe that, for $r>0$ and $0<q<1$,

$$
\sum_{k=0}^{\infty}\binom{k+r-1}{k}q^k=(1-q)^{-r}. \qquad (21)
$$

Then it is clear that

$$
\begin{aligned}
&\mathbb{P}(N(u_2)-N(u_1)=n_2) \\
&\quad= \sum_{n_1=0}^{\infty}\mathbb{P}(N(u_2)-N(u_1)=n_2 \mid N(u_1)=n_1)\mathbb{P}(N(u_1)=n_1) \\
&\quad= \sum_{n_1=0}^{\infty}\frac{\Gamma(\beta/\alpha+n_2+n_1)}{\Gamma(\beta/\alpha+n_2)n_1!}(\exp\{-\alpha[\Lambda(u_2)-\Lambda(u_1)]\}-\exp\{-\alpha\Lambda(u_2)\})^{n_1} \\
&\qquad\qquad\times\frac{\Gamma(\beta/\alpha+n_2)}{\Gamma(\beta/\alpha)n_2!}(1-\exp\{-\alpha[\Lambda(u_2)-\Lambda(u_1)]\})^{n_2}(\exp\{-\alpha\Lambda(u_2)\})^{\beta/\alpha}. \\
&\quad= \frac{\Gamma(\beta/\alpha+n_2)}{\Gamma(\beta/\alpha)n_2!}\bigg(\frac{1-\exp\{-\alpha[\Lambda(u_2)-\Lambda(u_1)]\}}{1+\exp\{-\alpha\Lambda(u_2)\}-\exp\{-\alpha[\Lambda(u_2)-\Lambda(u_1)]\}}\bigg)^{n_2} \\
&\qquad\times\bigg(\frac{\exp\{-\alpha\Lambda(u_2)\}}{1+\exp\{-\alpha\Lambda(u_2)\}-\exp\{-\alpha[\Lambda(u_2)-\Lambda(u_1)]\}}\bigg)^{\beta/\alpha}.
\end{aligned}
$$

Now the joint distribution of $\mathbb{P}(N(u_i)-N(u_{i-1})=n_i,\ i=1,2,\dots,m)$ will be derived. From (19) and (20), the joint distribution of $\mathbb{P}(N(u_1)=n_1,\ N(u_2)-N(u_1)=n_2)$ is given by

$$
\begin{aligned}
&\mathbb{P}(N(u_1)=n_1,\ N(u_2)-N(u_1)=n_2) \\
&\quad= \frac{\Gamma(\beta/\alpha+n_1)}{\Gamma(\beta/\alpha)n_1!}(1-\exp\{-\alpha\Lambda(u_1)\})^{n_1}(\exp\{-\alpha\Lambda(u_1)\})^{\beta/\alpha}\frac{\Gamma(\beta/\alpha+n_1+n_2)}{\Gamma(\beta/\alpha+n_1)n_2!} \\
&\qquad\times(1-\exp\{-\alpha[\Lambda(u_2)-\Lambda(u_1)]\})^{n_2}(\exp\{-\alpha[\Lambda(u_2)-\Lambda(u_1)]\})^{n_1+\beta/\alpha}.
\end{aligned}
$$

From the GPP property, we have

$$
\begin{aligned}
&\mathbb{P}(N(u_3)-N(u_2)=n_3 \mid N(u_1)=n_1,\ N(u_2)-N(u_1)=n_2) \\
&\quad= \mathbb{P}(N(u_3)-N(u_2)=n_3 \mid N(u_2)=n_1+n_2),
\end{aligned}
$$

https://doi.org/10.1239/aap/1418396247 Published online by Cambridge University Press

and, by Proposition 1, given that $N(u_2) = n_1 + n_2$, the future process $\{N_{u_2}(t),\ t \geq 0\}$ is the GPP with parameter set $(\lambda(u_2 + t), \alpha, \beta + (n_1 + n_2)\alpha)$. Thus, the conditional distribution of $\mathbb{P}(N(u_3) - N(u_2) = n_3 \mid N(u_1) = n_1,\ N(u_2) - N(u_1) = n_2)$ is given by

$$\begin{aligned}
&\mathbb{P}(N(u_3) - N(u_2) = n_3 \mid N(u_1) = n_1,\ N(u_2) - N(u_1) = n_2) \\
&\quad = \frac{\Gamma(\beta/\alpha + n_1 + n_2 + n_3)}{\Gamma(\beta/\alpha + n_1 + n_2) n_3!} (1 - \exp\{-\alpha[\Lambda(u_3) - \Lambda(u_2)]\})^{n_3} \\
&\quad\quad \times (\exp\{-\alpha[\Lambda(u_3) - \Lambda(u_2)]\})^{n_1 + n_2 + \beta/\alpha}.
\end{aligned}$$

From this, the corresponding joint distribution can be obtained as

$$\begin{aligned}
&\mathbb{P}(N(u_1) = n_1,\ N(u_2) - N(u_1) = n_2,\ N(u_3) - N(u_2) = n_3) \\
&\quad = \mathbb{P}(N(u_3) - N(u_2) = n_3 \mid N(u_1) = n_1,\ N(u_2) - N(u_1) = n_2) \\
&\quad\quad \times \mathbb{P}(N(u_1) = n_1,\ N(u_2) - N(u_1) = n_2).
\end{aligned}$$

Applying recursive procedure, we finally have

$$\begin{aligned}
&\mathbb{P}(N(u_i) - N(u_{i-1}) = n_i,\ i = 1, 2, \dots, m) \\
&\quad = \prod_{i=1}^{m} \Bigg[\frac{\Gamma(\beta/\alpha + \sum_{k=1}^{i} n_k)}{\Gamma(\beta/\alpha + \sum_{k=1}^{i-1} n_k) n_i!} (1 - \exp\{-\alpha[\Lambda(u_i) - \Lambda(u_{i-1})]\})^{n_i} \\
&\quad\quad \times (\exp\{-\alpha[\Lambda(u_i) - \Lambda(u_{i-1})]\})^{\sum_{k=1}^{i-1} n_k + \beta/\alpha} \Bigg].
\end{aligned}$$

Appendix B. Proof of Theorem 2

Let $\mathcal{H}_{[0,u)} \equiv \mathcal{H}_{u-}$ be the history of the process in $[0, u)$. Then it can be equivalently defined in terms of $N(u-)$ and the sequential arrival points of the events $0 < T_1 < T_2 < \cdots < T_{N(u-)} < t$ in $[0, u)$: $\mathcal{H}_{[0,u)} = \{N(u-), T_1, T_2, \dots, T_{N(u-)}\}$. Similarly, the history of the process in $[u, u + t)$, denoted by $\mathcal{H}_{[u,u+t)}$, can be specified as $\mathcal{H}_{[u,u+t)} = \{N((u + t)-) - N(u-), T_{u1}, T_{u2}, \dots, T_{u(N((u+t)-)-N(u-))}\}$. Observe that

$$\begin{aligned}
\lambda_t^u &\equiv \lim_{\Delta t \to 0} \frac{\mathbb{P}(N_u(t, t + \Delta t) = 1 \mid \mathcal{H}_{[u,u+t)})}{\Delta t} \\
&= \mathbb{E}_{\mathcal{H}_{[0,u)} \mid \mathcal{H}_{[u,u+t)}} \left[\lim_{\Delta t \to 0} \frac{\mathbb{P}(N_u(t, t + \Delta t) = 1 \mid \mathcal{H}_{[0,u)}, \mathcal{H}_{[u,u+t)})}{\Delta t} \right], \qquad (22)
\end{aligned}$$

where $\mathbb{E}_{\mathcal{H}_{[0,u)} \mid \mathcal{H}_{[u,u+t)}}$ stands for the expectation with respect to the conditional distribution of $(\mathcal{H}_{[0,u)} \mid \mathcal{H}_{[u,u+t)})$. Furthermore, by the definition of the stochastic intensity in (1),

$$\begin{aligned}
&\lim_{\Delta t \to 0} \frac{\mathbb{P}(N_u(t, t + \Delta t) = 1 \mid \mathcal{H}_{[0,u)}, \mathcal{H}_{[u,u+t)})}{\Delta t} \\
&\quad = \lambda_{u+t} \\
&\quad = (\alpha(N(u-) + [N((u + t)-) - N(u-)]) + \beta)\lambda(u + t). \qquad (23)
\end{aligned}$$

Now, in order to take the conditional expectation in (22), we derive the conditional joint distribution of

$$\begin{aligned}
\mathcal{H}_{[0,u)} \mid \mathcal{H}_{[u,u+t)} = (&T_1, T_2, \dots, T_{N(u-)}, N(u-) \mid T_{u1}, T_{u2}, \dots, \\
&T_{u(N((u+t)-)-N(u-))}, N((u + t)-) - N(u-)).
\end{aligned}$$

https://doi.org/10.1239/aap/1418396247 Published online by Cambridge University Press

The joint distribution of

$$(T_1, T_2, \dots, T_{N(u-)}, T_{u1}, T_{u2}, \dots, T_{u(N((u+t)-)-N(u-))}, N(u-), N((u+t)-)-N(u-))$$

is given by

$$\begin{aligned}
&f_{T_i,\,1\le i\le N(u-),T_{uj},\,1\le j\le N((u+t)-)-N(u-),N(u-),N((u+t)-)-N(u-)}(t_1,t_2,\dots,t_{n_1},\\
&\qquad\qquad\qquad\qquad\qquad\qquad t_{u1},t_{u2},\dots,t_{un_2},n_1,n_2)\\
&= [\beta\lambda(t_1)\exp\{-\beta\Lambda(t_1)\}(\beta+\alpha)\lambda(t_2)\exp\{-(\beta+\alpha)[\Lambda(t_2)-\Lambda(t_1)]\}\\
&\quad\times(\beta+2\alpha)\lambda(t_3)\exp\{-(\beta+2\alpha)[\Lambda(t_3)-\Lambda(t_2)]\}\\
&\quad\times\cdots\times(\beta+(n_1-1)\alpha)\lambda(t_{n_1})\exp\{-(\beta+(n_1-1)\alpha)[\Lambda(t_{n_1})-\Lambda(t_{n_1-1})]\}\\
&\quad\times\exp\{-(\beta+n_1\alpha)[\Lambda(u)-\Lambda(t_{n_1})]\}]\\
&\quad\times[(\beta+n_1\alpha)\lambda(t_{u1})\exp\{-(\beta+n_1\alpha)[\Lambda(t_{u1})-\Lambda(u)]\}\\
&\qquad\times(\beta+n_1\alpha+\alpha)\lambda(t_{u2})\exp\{-(\beta+n_1\alpha+\alpha)[\Lambda(t_{u2})-\Lambda(t_{u1})]\}\\
&\qquad\times\cdots\times(\beta+n_1\alpha+(n_2-1)\alpha)\lambda(t_{un_2})\\
&\qquad\times\exp\{-(\beta+n_1\alpha+(n_2-1)\alpha)[\Lambda(t_{un_2})-\Lambda(t_{u(n_2-1)})]\}\\
&\qquad\times\exp\{-(\beta+n_1\alpha+n_2\alpha)[\Lambda(u+t)-\Lambda(t_{un_2})]\}]\\
&= \bigg[\frac{\beta}{\alpha}\bigg(1+\frac{\beta}{\alpha}\bigg)\bigg(2+\frac{\beta}{\alpha}\bigg)\cdots\bigg((n_1-1)+\frac{\beta}{\alpha}\bigg)\\
&\quad\times\alpha\lambda(t_1)\exp\{\alpha\Lambda(t_1)\}\alpha\lambda(t_2)\exp\{\alpha\Lambda(t_2)\}\cdots\alpha\lambda(t_{n_1})\exp\{\alpha\Lambda(t_{n_1})\}\bigg]\\
&\quad\times\bigg[\bigg(n_1+\frac{\beta}{\alpha}\bigg)\bigg(n_1+1+\frac{\beta}{\alpha}\bigg)\bigg(n_1+2+\frac{\beta}{\alpha}\bigg)\cdots\bigg(n_1+(n_2-1)+\frac{\beta}{\alpha}\bigg)\\
&\quad\times\alpha\lambda(t_{u1})\exp\{\alpha\Lambda(t_{u1})\}\alpha\lambda(t_{u2})\exp\{\alpha\Lambda(t_{u2})\}\cdots\alpha\lambda(t_{un_2})\exp\{\alpha\Lambda(t_{un_2})\}\\
&\quad\times\exp\{-(\beta+n_1\alpha+n_2\alpha)\Lambda(u+t)\}\bigg]\\
&= \bigg[\frac{\Gamma(\beta/\alpha+n_1)}{\Gamma(\beta/\alpha)}\bigg(\prod_{i=1}^{n_1}\alpha\lambda(t_i)\exp\{\alpha\Lambda(t_i)\}\bigg)\\
&\quad\times\frac{\Gamma(\beta/\alpha+n_1+n_2)}{\Gamma(\beta/\alpha+n_1)}\bigg(\prod_{j=1}^{n_2}\alpha\lambda(t_{uj})\exp\{\alpha\Lambda(t_{uj})\}\bigg)\\
&\quad\times\exp\{-(\beta+n_1\alpha+n_2\alpha)\Lambda(u+t)\}\bigg].
\end{aligned}\tag{24}$$

From (24), the joint distribution of

$$(T_{u1}, T_{u2}, \dots, T_{u(N((u+t)-)-N(u-))}, N(u-), N((u+t)-)-N(u-))$$

is given by

$$\begin{aligned}
&\bigg[\frac{\Gamma(\beta/\alpha+n_1)}{\Gamma(\beta/\alpha)}\int_0^u\cdots\int_0^{t_3}\int_0^{t_2}\bigg(\prod_{i=1}^{n_1}\alpha\lambda(t_i)\exp\{\alpha\Lambda(t_i)\}\bigg)\,\mathrm{d}t_1\,\mathrm{d}t_2\cdots\mathrm{d}t_{n_1}\\
&\quad\times\frac{\Gamma(\beta/\alpha+n_1+n_2)}{\Gamma(\beta/\alpha+n_1)}\bigg(\prod_{j=1}^{n_2}\alpha\lambda(t_{uj})\exp\{\alpha\Lambda(t_{uj})\}\bigg)\\
&\qquad\times\exp\{-(\beta+n_1\alpha+n_2\alpha)\Lambda(u+t)\}\bigg]
\end{aligned}$$

$$= \left[\frac{\Gamma(\beta/\alpha + n_1)}{\Gamma(\beta/\alpha) n_1!} (\exp\{\alpha\Lambda(u)\} - 1)^{n_1} \right.$$
$$\times \frac{\Gamma(\beta/\alpha + n_1 + n_2)}{\Gamma(\beta/\alpha + n_1)} \left(\prod_{j=1}^{n_2} \alpha\lambda(t_{uj}) \exp\{\alpha\Lambda(t_{uj})\} \right)$$
$$\left. \times \exp\{-(\beta + n_1\alpha + n_2\alpha)\Lambda(u+t)\} \right].$$

Thus, by applying (21), the joint distribution of

$$(T_{u1}, T_{u2}, \dots, T_{u(N((u+t)-)-N(u-))}, N((u+t)-) - N(u-))$$

is given by

$$(1 + \exp\{-\alpha\Lambda(u+t)\} - \exp\{-\alpha(\Lambda(u+t) - \Lambda(u))\})^{-(\beta/\alpha + n_2)}$$
$$\times \left[\frac{\Gamma(\beta/\alpha + n_2)}{\Gamma(\beta/\alpha)} \left(\prod_{j=1}^{n_2} \alpha\lambda(t_{uj}) \exp\{\alpha\Lambda(t_{uj})\} \right) \exp\{-(\beta + n_2\alpha)\Lambda(u+t)\} \right].$$

Finally, the conditional joint distribution of

$$(T_1, T_2, \dots, T_{N(u-)}, N(u-) \mid T_{u1}, T_{u2}, \dots, T_{u(N((u+t)-)-N(u-))}, N((u+t)-) - N(u-))$$

is given by

$$\frac{\Gamma(\beta/\alpha + n_1 + n_2)}{\Gamma(\beta/\alpha + n_2)} \left(\prod_{i=1}^{n_1} \alpha\lambda(t_i) \exp\{\alpha\Lambda(t_i)\} \right) (\exp\{-\alpha\Lambda(u+t)\})^{n_1}$$
$$\times (1 + \exp\{-\alpha\Lambda(u+t)\} - \exp\{-\alpha(\Lambda(u+t) - \Lambda(u))\})^{\beta/\alpha + n_2}. \tag{25}$$

Note that

$$\lim_{\Delta t \to 0} \frac{\mathbb{P}(N_u(t, t+\Delta t) = 1 \mid \mathcal{H}_{[0,u)}, \mathcal{H}_{[u,u+t)})}{\Delta t}$$

given in (23) contains only $N(u-)$ among the elements of the history

$$\mathcal{H}_{[0,u)} = \{N(u-), T_1, T_2, \dots, T_{N(u-)}\}$$

and, thus, the conditional expectation in (22) should be taken only with respect to the random variable $N(u-)$. Accordingly, from (25), we now have to obtain the conditional distribution of

$$(N(u-) \mid T_{u1}, T_{u2}, \dots, T_{u(N((u+t)-)-N(u-))}, N((u+t)-) - N(u-)),$$

which is given by the following negative binomial distribution:

$$\frac{\Gamma(\beta/\alpha + n_2 + n_1)}{\Gamma(\beta/\alpha + n_2) n_1!} (\exp\{-\alpha(\Lambda(u+t) - \Lambda(u))\} - \exp\{-\alpha\Lambda(u+t)\})^{n_1}$$
$$\times (1 + \exp\{-\alpha\Lambda(u+t)\} - \exp\{-\alpha(\Lambda(u+t) - \Lambda(u))\})^{\beta/\alpha + n_2}.$$

Finally, from (22),

$$\lambda_t^u = (\alpha(\mathbb{E}[N(u-) \mid T_{u1}, T_{u2}, \dots, T_{u(N((u+t)-)-N(u-))}, N((u+t)-) - N(u-)]$$
$$+ [N((u+t)-) - N(u-)]) + \beta)\lambda(u+t),$$

https://doi.org/10.1239/aap/1418396247 Published online by Cambridge University Press

where

$$\begin{aligned}\mathbb{E}[N(u-) \mid T_{u1}, T_{u2}, \dots, T_{u(N((u+t)-)-N(u-))}, N((u+t)-) - N(u-)]\\ = \left(\frac{\beta}{\alpha} + [N((u+t)-) - N(u-)]\right)\\ \times \frac{\exp\{-\alpha(\Lambda(u+t) - \Lambda(u)) - \exp\{-\alpha\Lambda(u+t)\}}{1 + \exp\{-\alpha\Lambda(u+t)\} - \exp\{-\alpha(\Lambda(u+t) - \Lambda(u))\}}.\end{aligned}$$

Therefore, we have

$$\lambda_t^u = (\alpha[N((u+t)-) - N(u-)] + \beta)\psi(t, u),$$

where

$$\psi(t, u) \equiv \frac{\lambda(u+t)\exp\{\alpha\Lambda(u+t)\}}{1 + \exp\{\alpha\Lambda(u+t)\} - \exp\{\alpha\Lambda(u)\}}.$$

Acknowledgements

The author thanks the anonymous referee for a short comment, which has improved the readability of this paper. This work was supported by the National Research Foundation of Korea funded by the Korean government (MEST) (grant number 2011-0017338). This work was also supported by the Priority Research Centers Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (grant number 2009-0093827). The author would also like to thank Mrs Sul Ja Choi for helpful and pleasant discussions on some practical applications of the theoretical results in this paper.

References

Aven, T. (1996). Condition based replacement policies—A counting process approach. *Reliab. Eng. System Safety* **51,** 275–281.

Aven, T. and Jensen, U. (1999). *Stochastic Models in Reliability*. Springer, New York.

Aven, T. and Jensen, U. (2000). A general minimal repair model. *J. Appl. Prob.* **37,** 187–197.

Badía, F. G., Berrade, M. D. and Campos, C. A. (2001). Optimization of inspection intervals based on cost. *J. Appl. Prob.* **38,** 872–881.

Barbu, V. S. and Limnios, N. (2008). *Semi-Markov Chains and Hidden Semi-Markov Models Toward Applications. Their Use in Reliability and DNA Analysis*. Springer, New York.

Cha, J. H. (2001). Burn-in procedures for a generalized model. *J. Appl. Prob.* **38,** 542–553.

Cha, J. H. (2003). A further extension of the generalized burn-in model. *J. Appl. Prob.* **40,** 264–270.

Cha, J. H. and Finkelstein, M. (2009). On a terminating shock process with independent wear increments. *J. Appl. Prob.* **46,** 353–362.

Cha, J. H. and Finkelstein, M. (2011a). On new classes of extreme shock models and some generalizations. *J. Appl. Prob.* **48,** 258–270.

Cha, J. H. and Finkelstein, M. (2011b). Stochastic intensity for minimal repairs in heterogeneous populations. *J. Appl. Prob.* **48,** 868–876.

Cox, D. R. (1962). *Renewal Theory*. Methuen, London.

Cox, D. R. and Isham, V. (2000). *Point Processes*. Chapman & Hall, London.

Ebrahimi, N. (1997). Multivariate age replacement. *J. Appl. Prob.* **34,** 1032–1040.

Finkelstein, M. and Cha, J. H. (2013). *Stochastic Modelling for Reliability. Shocks, Burn-in and Heterogeneous Populations*. Springer, London.

Kao, E. P. C. (1997). *An Introduction to Stochastic Processes*. Duxbury, Belmont, CA.

Konno, H. (2010). On the exact solution of a generalized Pólya process. *Adv. Math. Phys.* **2010,** 504267.

Limnios, N. and Oprişan, G. (2001). *Semi-Markov Processes and Reliability*. Birkhäuser, Boston, MA.

Mi, J. (1994). Burn-in and maintenance policies. *Adv. Appl. Prob.* **26,** 207–221.

Nakagawa, T. (2005). *Maintenance Theory of Reliability*. Springer, London.

Ross, S. M. (1996). *Stochastic Processes*, 2nd edn. John Wiley, New York.

Ross, S. M. (2003). *Introduction to Probability Models*, 8th edn. Academic Press, San Diego, CA.

SHERIF, Y. S. AND SMITH, M. L. (1981). Optimal maintenance models for systems subject to failure—a review. *Naval Res. Logistics Quart.* **28,** 47–74.

TADJ, L., OUALI M.-S., YACOUT, S. AND AIT-KADI, D. (eds) (2011). *Replacement Models with Minimal Repair*. Springer, London.

VALDEZ-FLORES, C. AND FELDMAN, R. M. (1989). A survey of preventive maintenance models for stochastically deteriorating single-unit systems. *Naval Res. Logistics* [Quart. to **36,**] 419–446.

WANG, H. (2002). A survey of maintenance policies of deteriorating systems. *Europ. J. Operat. Res.* **139,** 469–489.