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A NOTE ON MARKOV BRANCHING PROCESSES

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Abstract

We present a simple proof of Zolotarev's representation for the Laplace transform of the normalized limit of a Markov branching process and relate it to the Harris representation.

GALTON–WATSON: BACKWARD KOLMOGOROV EQUATION; LAPLACE TRANSFORM

Let $Z(t)$ denote a supercritical Markov branching process with infinitesimal generating function $u(x)=a(f(x)-x)$ where $f(x)=\sum f_i x^i$ is the offspring p.g.f. with mean m and where a^{-1} is the mean particle lifetime.

Zolotarev (1957) showed the existence of a normalizing function $\gamma(t)$ making $\gamma(t)Z(t)$ converge in distribution to a proper limit W whose Laplace transform $\phi(s)=E[\exp(-sW)\mid Z(0)=1]$ satisfies

$$\phi^{-1}(x)=\exp\int_\Delta^x \frac{m-1}{f(r)-r}\,dr \tag{1}$$

for a specified constant Δ.

Here is a very quick proof of (1) relying on the Poincaré functional equation (familiar in the Galton–Watson setting) for the embedded discrete skeleton. Integrate the backward Kolmogorov equation $\partial F/\partial t=u(F(x,t))$ to obtain

$$t=\int_\Delta^{F(\Delta,t)} \frac{dr}{u(r)} \tag{2}$$

where $F(x,t)=E[x^{Z(t)}\mid Z(0)=1]$ and Δ is an arbitrary constant. For each t the process $\{Z(nt), n\geqq 0\}$ is Galton–Watson so, using the constants of Seneta (1968), we find that

$$\phi(\exp(\lambda t)s)=F(\phi(s),t)$$

where $\lambda=a(m-1)$ is the Malthusian parameter. Setting $s=1$ and defining $\Delta=\phi(1)$ we get $F(\Delta,t)=\phi(\exp\lambda t)$ which, when substituted into (2) with $x=\phi(\exp\lambda t)$, results in

$$\phi^{-1}(x)=\exp\lambda t=\exp\lambda\int_\Delta^x \frac{dr}{u(r)}=\exp\int_\Delta^x \frac{m-1}{f(r)-r}\,dr,$$

completing the proof.

Next, we separate the integral in (1) into a sum

$$\int_\Delta^x \left(\frac{m-1}{f(r)-r}+\frac{1}{1-r}\right)dr-\int_\Delta^x \frac{1}{1-r}\,dr.$$

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If (and only if) the familiar logarithmic moment condition $\sum (i \log i) f_i < \infty$ holds, the first integral is known to remain bounded as $x \to 1$ (essentially Theorem 4.3 of Seneta (1969)) whence it is permissible to split the integration into one part running from Δ to 1 and a second part from 1 to x, which results in

$$\int_1^x \left(\frac{m-1}{f(r)-r} + \frac{1}{1-r}\right) dr - \int_1^\Delta \left(\frac{m-1}{f(r)-r} + \frac{1}{1-r}\right) dr + \log \frac{1-x}{1-\Delta}$$

and culminates in

$$\phi^{-1}(x) = (1-x) \exp \int_1^x \left(\frac{m-1}{f(r)-r} + \frac{1}{1-r}\right) dr \Big/ (1-\Delta) \exp \int_1^\Delta \left(\frac{m-1}{f(r)-r} + \frac{1}{1-r}\right) dr.$$

Under the classical norming $W = \lim \exp(-\lambda t) Z(t)$ and $E[W] = 1$, and hence $\lim (x \to 1) \phi^{-1}(x)/(1-x) = 1$. This shows that the denominator in the previous line equals 1 and therefore

$$\phi^{-1}(x) = (1-x) \exp \int_1^x \left(\frac{m-1}{f(r)-r} + \frac{1}{1-r}\right) dr,$$

a representation first derived by Harris (1951) assuming the finiteness of the second moment $\sum i^2 f_i$. There is a proof by Karlin and McGregor (1968) also only requiring $\sum (i \log i) f_i < \infty$ which is based on a different approach.

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