



RESEARCH ARTICLE

Mirror symmetry and automorphisms

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Received: 28 August 2020; Revised: 3 March 2022; Accepted: 30 May 2022

2020 Mathematics subject classification: 14J32, 14A20

Abstract

We show that there is an extra grading in the mirror duality discovered in the early nineties by Greene–Plesser and Berglund–Hübsch. Their duality matches cohomology classes of two Calabi–Yau orbifolds. When both orbifolds are equipped with an automorphism s of the same order, our mirror duality involves the weight of the action of s^* on cohomology. In particular it matches the respective s -fixed loci, which are not Calabi–Yau in general. When applied to K3 surfaces with nonsymplectic automorphism s of odd prime order, this provides a proof that Berglund–Hübsch mirror symmetry implies K3 lattice mirror symmetry replacing earlier case-by-case treatments.

1. Introduction

The earliest formulation of mirror symmetry relates pairs of d -dimensional Calabi–Yau manifolds X, X^\vee with mirror Hodge diamonds:

$$h^{p,q}(X) = h^{d-p,q}(X^\vee).$$

In the early 1990s, physicists Greene, Morrison and Plesser found many such mirror pairs [23], starting with a Calabi–Yau (and Fermat) hypersurface in projective space and constructing a mirror, which is a resolution of the quotient of the same hypersurface by a finite group. In 1992, this construction was generalized by Berglund–Hübsch [5], starting with a Calabi–Yau orbifold given as a quotient of a more general hypersurface in weighted projective spaces by a finite group. The hypersurface is a Calabi–Yau orbifold defined as the zero locus of a quasi-homogenous polynomial $W = \sum_{i=0}^n \prod_{j=0}^n x_j^{m_{ij}}$ such that W is nondegenerate and ‘invertible’ (i.e., with as many variables as monomials). After quotienting out by a finite group H of diagonal symmetries within $\mathrm{SL}(n+1; \mathbb{C})$ one obtains the orbifold $\Sigma_{W,H}$. The mirror Σ_{W^\vee, H^\vee} is another such quotient of a hypersurface modulo a finite group. The hypersurface is given by the polynomial W^\vee , defined by transposing the matrix of the exponents $E = [m_{ij}]$ of W . The group H^\vee is a subgroup of $\mathrm{SL}(n+1; \mathbb{C})$ Cartier dual to H and preserving W^\vee ; see equation (19). Then, the mirror duality can be stated in terms of orbifold Chen–Ruan cohomology as

$$H_{\mathrm{CR}}^{p,q}(\Sigma_{W,H}; \mathbb{C}) = H_{\mathrm{CR}}^{d-p,q}(\Sigma_{W^\vee, H^\vee}; \mathbb{C}), \quad (1)$$

which implies the same relation in ordinary cohomology whenever there exists crepant resolutions.

The striking mirror relation above appears more natural when we look at it through the lens of singularity theory or, in physics terminology, the Landau–Ginzburg (LG) model. This happens because mirror symmetry holds for LG models without any Calabi–Yau condition. In this paper, we present this

change of perspective through the LG model via the crepant resolution of a singularity; see Section 5. This not only allows us to simplify previous proofs of LG/CY correspondence by the first author with Ruan [9]; it also yields a new statement of mirror symmetry relating the fixed loci of powers of an isomorphism s of Σ , the Hodge decomposition and the weights the representation s^* in cohomology.

Let $W = x_0^k + f(x_1, \dots, x_n)$ be a nondegenerate, quasi-homogenous, invertible polynomial. Let us consider again the automorphisms groups $H \subseteq \text{Aut } W$ and its dual $H^\vee \in \text{Aut } W$ within $\text{SL}(n + 1; \mathbb{C})$. The Calabi–Yau orbifolds $\Sigma_{W,H}, \Sigma_{W^\vee,H^\vee}$ are equipped with the action by the group μ_k of k th roots of unity spanned by $s: x_0 \mapsto e^{2\pi i/k} x_0$. For i in the group of characters $\mathbb{Z}/k = \text{Hom}(\mu_k; \mathbb{G}_m)$ we consider the weight- i term of cohomology

$$H^*(\text{---}, \mathbb{C})_i = \{x \mid s^*x = i(s)x\}.$$

The first statement is that the s -invariant cohomology mirrors the ‘moving’ cohomology: the sum of all cycles of nonvanishing weight.

Theorem A (see Theorem 44, part 1). *Consider $s: \Sigma_{W,H} \rightarrow \Sigma_{W,H}$ and its mirror partner $s: \Sigma_{W^\vee,H^\vee} \rightarrow \Sigma_{W^\vee,H^\vee}$. We have*

$$H_{orb}^{p,q}(\Sigma_{W,H}; \mathbb{C})_0 \cong \bigoplus_{i=1}^{k-1} H_{orb}^{d-p,q}(\Sigma_{W^\vee,H^\vee}; \mathbb{C})_i,$$

where $d = n - 1$ is the dimension of $\Sigma_{W,H}$.

The locus of geometric points of $\Sigma_{W,H}$ which are fixed by s also exhibits a mirror phenomenon. Since $\Sigma_{W,H}$ is a stack, let us provide a definition for this s -fixed locus. For s a finite order automorphism acting on a smooth Deligne–Mumford orbifold \mathfrak{X} , we consider the graph of $\Gamma_s: \mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$ and its intersection with the graph of the identity (the diagonal morphism)

$$\mathfrak{X} \times_{s, \mathfrak{X} \times \mathfrak{X}, \text{id}} \mathfrak{X},$$

(we write s and id instead of the respective graphs). We recall that orbifold cohomology is the (age-shifted) cohomology of this product for $s = \text{id}$. Then, we define the s -orbifold cohomology as the age-shifted cohomology of the above fibred product in general (see Definition 7). This is a bigraded vector space, and, if the coarse space X of \mathfrak{X} admits a crepant resolution \tilde{X} where s lifts, there is a bidegree-preserving isomorphism $H_s^*(\mathfrak{X}; \mathbb{C}) \cong H_s^*(\tilde{X}; \mathbb{C})$, where the right-hand side is the age-shifted cohomology of the s -fixed locus in \tilde{X} ; see Proposition 9.

We can now state two mirror dualities for s^j -fixed loci in this sense. Under the same conditions on W and H as above, set $\Sigma = \Sigma_{W,H}$ and $\Sigma^\vee = \Sigma_{W^\vee,H^\vee}$. If the order k of s is not prime, then s acts nontrivially on the fixed locus of powers of s . The s -moving cohomology of the fixed locus of powers of s mirrors the same on Σ^\vee , interweaving the weight and the exponent of the power of s .

Theorem B (see Thm 44, part 3). *Let $0 < b, t < k$. Then, we have*

$$H_{s^b}^{p,q}(\Sigma) \left(\frac{b}{k} \right)_t \cong H_{s^{-t}}^{d-p,q}(\Sigma^\vee) \left(\frac{k-t}{k} \right)_{-b},$$

where $d = n - 2$, the largest dimension of the components of the s -fixed locus.

Finally, also the fixed cohomology of each power s^j exhibits a mirror phenomenon but only after adding certain moving cycles in Σ . Namely, the cycles we add are all those whose weight differs from 0 (i.e., moving cycles) and from j (the exponent of s). We denote this group by $\overline{H}_{\text{id},j}^{p,q}(\Sigma)$; see equation (34).

Theorem C (see Theorem 44, part 2). *For $0 < j < k$, we have*

$$\left[H_{s^j}^{p,q}(\Sigma) \left(\frac{j}{k} \right) \right]^s \oplus \overline{H}_{\text{id},j}^{p,q}(\Sigma) \cong \left[H_{s^j}^{d-p,q}(\Sigma^\vee) \left(\frac{j}{k} \right) \right]^s \oplus \overline{H}_{\text{id},j}^{d-p,q}(\Sigma^\vee).$$

The correcting terms \overline{H}^* disappear when $k = 2$ (for $k = 2$, we have $s^j = s$ and there is no positive weight except 1). This shows how the statement above specialises to the construction of Borcea–Voisin mirror pairs, which can be stated as a mirror duality between the s -fixed loci (see [12]).

In dimension 2, and after resolving, these results are about mirror symmetry for K3 surfaces with nonsymplectic automorphisms. Suppose X and X^\vee are crepant resolutions of $\Sigma_{W,H}$ and Σ_{W^\vee,H^\vee} , where W is a polynomial in 4 variables. The above mirror theorems imply that the topological invariants of the fixed locus of the K3 surface X controls that of X^\vee ; we refer to Corollary 51 for simple formulae on the number of fixed points and the genera of the fixed curves. The automorphism s also gives the K3 surface a lattice polarization: $H^2(X, \mathbb{Z})^s$. There is another version of mirror symmetry for lattice polarized K3 surfaces, arising from the work of Nikulin [28], Dolgachev [18], Voisin [35] and Borcea [6]. When the order of s is odd and prime, this lattice is characterised by the invariants (r, a) : the rank and the discriminant. Families of lattice polarized K3 surfaces come in mirror pairs, and in the odd prime case this mirror symmetry takes a lattice with invariants (r, a) to $(20 - r, a)$. The following corollary is a theorem of Bott, Comparin, Lyons, Priddis and Suggs [14, 15, 8] proven by case-by-case analysis. Here, it is shown directly from the above statements (see Theorem 53).

Corollary ([14, 15, 8]). Let p be prime and different from 2. Let $\Sigma_{W,H}$ and Σ_{W^\vee,H^\vee} be mirror K3 orbifolds with order- p automorphisms s, s^\vee , and let Σ and Σ^\vee be crepant resolutions with automorphisms also denoted s, s^\vee . Then Σ and Σ^\vee are mirror as lattice polarized K3 surfaces.

1.1. Relation to previous work

This paper generalises the results of [12]. There, only involutions were considered; here, the mirror theorems apply to automorphism of any order. There, Theorems A and C are simpler (invariant classes mirror anti-invariant classes in Theorem A and no extra terms appear in Theorem C). Theorem B does not apply in the involution case. In the above corollary, we do not consider the order-2 case treated in [12]; in the present paper, this allows us to deduce the lattice mirror symmetry statement of [14] in full.

Section 5 restates and recasts the proof of mirror symmetry through LG models and the correspondence between cohomology and LG models in terms of resolutions of singularities (see Theorem 31). This may be regarded as the outcome of the work of many authors, we refer to [26], [7], [25] [9], [19], [20] and [21] and [13] validating over the years the approach of the physicists Intriligator–Vafa [24] and Witten [36]. It is also worth mentioning that the main object of our study, a polynomial $W = x_0^k + f(x_1, \dots, x_n)$ with the cyclic symmetry group of k th roots of unity acting on x_0 , was used in Varchenko’s proof of semicontinuity of Steenbrink’s spectra of singularities [32] and [31]. We hope that this may lead to further explanations of mirror symmetry in the framework of singularity theory. In particular, our setup only concerns hypersurfaces in weighted projective space; it would be interesting to see if it extends to other contexts where mirror constructions are known.

Finally, it is worth mentioning that the work of Bott, Comparin, Lyons, Priddis and Suggs [14]; earlier work of Artebani, Boissière and Sarti [2] and more generally Nikulin’s classification [28] yield several tables summarising explicit treatments of K3 surfaces via resolution of singularities. Much of these data are now embodied into the s -weighted Hodge numbers of Theorems A, B and C. We provide some examples for this in the tables at the end of §7.

1.2. Structure of the paper

Section 2 states notation and terminology. Section 3 presents the Berglund–Hübsch mirror symmetry construction. Section 4 sets up our generalisation of orbifold cohomology sensing the s -fixed locus: s -orbifold cohomology. Section 5 illustrates and reproves the transition to Landau–Ginzburg models which is crucial in the proof. In particular it provides a straightforward description of the LG/CY correspondence from the crepant resolution conjecture without using the combinatorial model of [9]. Section 6 is the technical heart of the paper; it proves the main theorem (Theorem 43) on the LG side.

Section 7 translates the result from the LG side to the CY side. It contains Theorem 44 proving the statements A, B and C and Theorem 53 specialising to K3 surfaces.

Acknowledgements

We are grateful to Behrang Noohi, whose explanations clarified inertia stacks to us. We are grateful to Baohua Fu, Lie Fu, Dan Israel and Takehiko Yasuda for many helpful conversations. We thank Davide Cesare Veniani, with whom we started studying involutions of Calabi–Yau orbifolds, for continuing to share his insights and expertise. We are extremely grateful to the referee, whose requests made some statements more precise, corrected some mistakes and improved the readability of the paper.

2. Terminology

Deligne–Mumford orbifolds are smooth separated Deligne–Mumford stacks with a dense open subset isomorphic to an algebraic variety.

2.1. Conventions

We work with schemes and stacks over the complex numbers. All schemes are Noetherian and separated. By linear algebraic group we mean a closed subgroup of $GL(m; \mathbb{C})$ for some m .

2.2. Notation

We list here notation that occurs throughout the entire paper.

- v^K is the invariant subspace of a vector space V linearized by a finite group K .
- $\mathbb{P}(\mathbf{w})$ is the quotient stack $[(\mathbb{C}^n \setminus \mathbf{0})/\mathbb{G}_m]$, where \mathbb{G}_m acts with weights \mathbf{w} .
- $Z(f)$ is the variety defined as zero locus of $f \in \mathbb{C}[x_1, \dots, x_n]$.
- $H(a, b)$ is the bigraded vector space with shifted grading: $[H(a, b)]^{p,q} = H^{p+a, q+b}$.

Remark 1 (zero loci). We add the subscript $\mathbb{P}(\mathbf{w})$ when we refer to the zero locus in $\mathbb{P}(\mathbf{w})$ of a polynomial f which is \mathbf{w} -weighted homogeneous. In this way, we have

$$Z_{\mathbb{P}(\mathbf{w})}(f) = [U/\mathbb{G}_m], \quad \text{with } U = Z(f) \subset \mathbb{C}^n \setminus \mathbf{0}.$$

Remark 2 (zero dimensional $\mathbb{P}(d)$ for $d \in \mathbb{N}^*$). For d a positive integer, we write $\mathbb{P}(d)$ for the stack $[\mathbb{C}^x/\mathbb{G}_m]$, isomorphic to $B\mu_d$ if $\lambda \in \mathbb{G}_m$ operates on z as $\lambda \cdot z = \lambda^d z$.

Remark 3 (degree shift). We often write $H(a)$ for $H(a, a)$.

Remark 4 (cohomology coefficients). We only consider cohomology with coefficients in \mathbb{C} ; therefore, we sometimes write $H^*(X; \mathbb{C})$ as $H^*(X)$.

Remark 5 (graphs and maps). Given an automorphism α of \mathfrak{X} , we write Γ_α for the graph $\mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$. However, to simplify formulæ, we often abuse notation and use α for the graph Γ_α as well as the automorphism. In this way, in subscripts, the diagonal $\Delta : \mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$ will be often written as $\text{id}_{\mathfrak{X}}$ or simply id .

3. Setup

We recall the general setup of nondegenerate polynomials P where the theory of Jacobi rings applies. Then we introduce polynomials of the special form

$$W(x_0, x_1, \dots, x_n) = x_0^k + f(x_1, \dots, x_n)$$

for $n > 0$.

3.1. Nondegenerate polynomials

We consider quasi-homogeneous polynomials P of positive degree d and of positive weights w_0, \dots, w_n satisfying

$$P(\lambda^{w_0}x_0, \dots, \lambda^{w_n}x_n) = \lambda^d P(x_0, \dots, x_n),$$

for all $\lambda \in \mathbb{C}$. We assume that the polynomial P is nondegenerate; i.e., the choice of weights and degree is unique and the partial derivatives of W vanish simultaneously only at the origin. We consider the zero locus

$$\Sigma_P = Z_{\mathbb{P}(\mathbf{w})}(P) \subset \mathbb{P}(\mathbf{w})$$

which is, by nondegeneracy, a smooth hypersurface within the weighted projective stack $\mathbb{P}(\mathbf{w}) = [(\mathbb{C}^{n+1} \setminus \mathbf{0})/\mathbb{G}_m]$ with \mathbb{G}_m acting with weights w_0, \dots, w_n . The polynomial is of Calabi-Yau type if

$$\sum_{i=0}^n w_i = d. \tag{2}$$

This implies that the canonical bundle of Σ_P is trivial; we refer to Σ_P as a Calabi–Yau orbifold.

Because P is nondegenerate, the group of its diagonal automorphisms

$$\text{Aut}_P = \{\text{diag}(\alpha_0, \dots, \alpha_n) \mid P(\alpha_0x_0, \dots, \alpha_nx_n) = P(x_0, \dots, x_n)\}$$

is finite. Indeed, the $h \times (n + 1)$ exponent matrix $E = (m_{i,j})$ defined by the condition $P = \sum_{i=1}^h c_i \prod_{j=0}^n x_j^{m_{i,j}}$ is left invertible as a consequence of the uniqueness of the vector $(w_i/d)_{i=0}^n = E^{-1}\mathbf{1}$.

Since we are working over \mathbb{C} , we adopt the notation

$$[a_0, \dots, a_n] = \text{diag}(\exp(2\pi i a_i))_{i=0}^n$$

for $a_i \in \mathbb{Q} \cap [0, 1[$. The age of the diagonal matrix above is

$$\text{age}[a_0, \dots, a_n] = \sum_{i=0}^n a_i. \tag{3}$$

The distinguished diagonal symmetry

$$j_P = \left[\frac{w_0}{d}, \dots, \frac{w_n}{d} \right],$$

usually denoted by j , spans the intersection $\text{Aut}_P \cap \mathbb{G}_m$, where \mathbb{G}_m is the group of automorphisms of the form $\text{diag}(\lambda^{w_0}, \dots, \lambda^{w_n})$. The group element j_P has a natural interpretation as a monodromy operator of the fibration defined by P restricted to the complement in \mathbb{C}^{n+1} of the zero locus $Z(P)$; see, for instance, [17]. We will denote by M_P the generic Milnor fibre

$$\begin{array}{ccc} M_P & \longrightarrow & \mathbb{C}^{n+1} \setminus Z(P) \\ \downarrow & \square & \downarrow P \\ t \neq 0 & \longrightarrow & \mathbb{C}^\times. \end{array} \tag{4}$$

For any subgroup H of Aut_P containing j_P , we consider the Deligne–Mumford stacks

$$\Sigma_{P,H} = [\Sigma_P/H_0], \quad M_{P,H} = [M_P/H],$$

where $H_0 = H/(H \cap \mathbb{G}_m) = H/\langle j \rangle$ and acts faithfully on Σ_P . The orbifold $\Sigma_{P,H}$ is a smooth codimension-1 substack of $[\mathbb{P}(\mathbf{w})/H_0]$

$$\Sigma_{P,H} \subset [\mathbb{P}(\mathbf{w})/H_0],$$

and has trivial canonical bundle if P is Calabi–Yau in the sense of equation (2) and H lies in

$$\mathrm{SL}_P := \mathrm{Aut}_P \cap \mathrm{SL}(n + 1; \mathbb{C}).$$

3.2. Polynomials with automorphism

We focus on polynomials of Calabi–Yau type of the form

$$W(x_0, x_1, \dots, x_n) = (x_0)^k + f(x_1, \dots, x_n).$$

We have $\mathrm{Aut}_W = \mu_k \times \mathrm{Aut}_f$, where, using again the choice $\exp(2\pi i/k)$, the first factor is regarded here as \mathbb{Z}/k , canonically generated by the order- k automorphism

$$s = \left[\frac{1}{k}, 0, \dots, 0 \right]. \tag{5}$$

The second factor is regarded as the subgroup of Aut_W formed by symmetries fixing the first coordinate. We have $j_W = s \cdot j_f$, with $s \in \mathbb{Z}/k$ and $j_f \in \mathrm{Aut}_f$. Notice that the group \mathbb{Z}/k acts on the stack $\Sigma_{W,H}$

$$m: (\mathbb{Z}/k) \times \Sigma_{W,H} \rightarrow \Sigma_{W,H}.$$

Instead of considering all groups $H \subseteq \mathrm{Aut} W \cap \mathrm{SL}(n + 1; \mathbb{C})$ containing j_W , we can equivalently consider their intersections with Aut_f . In this way, since the first coordinate of $(j_W)^h$ equals 1 if and only if $h \in k\mathbb{Z}$, we obtain all the subgroups $K \subset \mathrm{Aut}_f$ satisfying

$$(j_f)^k \in K \subseteq \mathrm{SL}_f.$$

We recover the previous subgroups H of $\mathrm{Aut} W \cap \mathrm{SL}(n + 1; \mathbb{C})$ as the subgroups of Aut_W spanned by j_W and K .

Our mirror duality requires a slightly more general class of groups. Therefore, we consider the subgroup of Aut_W

$$K[j_W, s] = \sum_{a,b=0}^{k-1} (j_W)^a (s)^b K,$$

with its natural $(\frac{1}{k}\mathbb{Z}/\mathbb{Z})$ -gradings

$$d_j = \frac{a}{k}, \quad d_s = \frac{b}{k}.$$

By the condition $\sum_{i=0}^n w_i = d$, the determinant of an element $g \in K[j_W, s]$ equals $\exp(2\pi i d_s(g))$. The condition $d_s = 0$ singles out the groups which we considered initially: $H \subseteq \mathrm{Aut} W \cap \mathrm{SL}(n + 1; \mathbb{C})$ containing j_W .

4. Variants of orbifold Chen–Ruan cohomology

We introduce this section by a short description of the variant of cohomology group needed in this paper and by a motivation via its main application.

For any finite order automorphism g of a stack \mathfrak{X} , we define g -orbifold cohomology, a slight generalization of orbifold Chen–Ruan cohomology; see Definition 7. The setup follows the original definition of orbifold Chen–Ruan cohomology: the cohomology of the inertia stack whose grading is shifted by the locally constant rational age function. Here, we start from the g -inertia stack, which is a g -dependent version of the ordinary inertia stack corresponding to $g = \text{id}_{\mathfrak{X}}$. Then, as in the ordinary case, we take its cohomology with complex coefficients after a degree-shift by the same age function. The treatment provided here improves [12], where the g -inertia stack was obtained via a new *ad hoc* construction. Here, we produce the g -inertia stack as a union of connected components of the inertia stack of $[\mathfrak{X}/G]$, where G is a group acting on \mathfrak{X} and containing g . The definition of the age function is then straightforward via the natural inclusion within the larger inertia stack of $[\mathfrak{X}/G]$ and by restriction of the usual age function.

We recall that the main interest of orbifold Chen–Ruan cohomology is its identification with the ordinary cohomology of certain resolutions of coarse quotients in terms of their orbifold (i.e., stack-theoretic) presentation. Because the g -inertia stack of a space X is simply the g -fixed locus, the present setup allows an analogous statement for g -fixed loci; see Proposition 9. In fact, the orbifold and the crepant resolution can be described as K -equivalent (a relation holding when the canonical sheaves match). Indeed, the canonical bundle of the stack descends from the orbifold to the coarse space and matches the canonical bundle of the resolution via pullback. In this perspective, the identification of cohomology groups follows from a result by Yasuda, [37] on invariance under K -equivalence of motivic cohomology. In Proposition 9, we check that his result applies to our slightly more general setup. We point out that, in [12, Prop. 4.7.2], we proved all this explicitly via a direct argument which holds only in dimension two.

We now introduce the G -inertia stack. The construction parallels the presentation of the inertia stack of a quotient stack $[U/G]$, where U is a scheme and G is a group, which is in turn a quotient stack $[I_G(U)/G]$, where $I_G(U)$ is the G -inertia group scheme (a group scheme referred to as ‘the stabilizer of the groupoid’ in [29, Lem. 70.25.1]) modulo a natural G -action. What follows is the analogue definition when the scheme U is replaced by a Deligne–Mumford stack \mathfrak{X} (see [12, §4.3] for an earlier version of this). This is again a special case of the above-mentioned G -inertia group scheme construction of [29, Lem. 70.25.1]. Here, we illustrate how these notions can be made more explicit in the present setup.

We consider a finite group G acting on a Deligne–Mumford orbifold \mathfrak{X}

$$m: G \times \mathfrak{X} \rightarrow \mathfrak{X}.$$

The G -inertia stack $I_G(\mathfrak{X})$ fits in the following fibre diagram

$$\begin{array}{ccc} I_G(\mathfrak{X}) & \longrightarrow & \mathfrak{X} \\ \downarrow & \square & \downarrow \Delta \\ G \times \mathfrak{X} & \xrightarrow{(m, \text{pr}_2)} & \mathfrak{X} \times \mathfrak{X}. \end{array}$$

When G is a trivial group, $I\mathfrak{X}$ is the ordinary inertia of \mathfrak{X} .

In this way, we have

$$I_G(\mathfrak{X}) = (G \times \mathfrak{X}) \times_{(m, \text{pr}_2), \mathfrak{X} \times \mathfrak{X}, \Delta} \mathfrak{X} = \bigsqcup_{g \in G} I_g(\mathfrak{X}),$$

where $I_g(\mathfrak{X})$ is the g -inertia orbifold

$$I_g(\mathfrak{X}) := \mathfrak{X} \times_{\mathfrak{X} \times \mathfrak{X}, \text{id}} \mathfrak{X}. \tag{6}$$

The G -inertia stack of \mathfrak{X} can be naturally related to the ordinary inertia stack of $[\mathfrak{X}/G]$. Indeed, we have the fibre diagram

$$\begin{array}{ccc}
 I_G(\mathfrak{X}) & \longrightarrow & \mathfrak{X} \\
 \downarrow p & \square & \downarrow \\
 I[\mathfrak{X}/G] & \longrightarrow & [\mathfrak{X}/G],
 \end{array}
 \tag{7}$$

and we may regard p as a G -torsor by pullback of $\mathfrak{X} \rightarrow [\mathfrak{X}/G]$.

The G -action on $\bigsqcup_{g \in G} I_g \mathfrak{X}$ is given by conjugation on the indices and by $F_h : I_g \mathfrak{X} \rightarrow I_{hgh^{-1}} \mathfrak{X}$ on the components, where F_h acts by the effect of h on the first factor of equation (6) and by the identity on the second factor.

Remark 6. In this paper, we only consider quotient stacks of the form $[U/H]$, where H is an abelian group with finite stabilizers. Then, as in [12, Defn. 4.3.2], unraveling the above definitions yields a presentation of $I_g[U/H]$ (denoted by ‘ $\mathfrak{I}_{[U/H]}^g$ ’ there) as the quotient stack $[I_{gH}(U)/H]$, where $I_{gH}(U)$ equals $\{(gh, x) \mid gh \cdot x = x\}$ and the group H operates on the second factor.

4.1. A g -orbifolded cohomology

The g -orbifold cohomology is the cohomology of $I_g(\mathfrak{X})$ defined in equation (6) shifted by the locally constant function ‘age’ given by $\mathfrak{a}_g = \mathfrak{a} \circ p$

$$\mathfrak{a}_g : I_g(\mathfrak{X}) \xrightarrow{p} I[\mathfrak{X}/G] \xrightarrow{\mathfrak{a}} \mathbb{Q},
 \tag{8}$$

where p is the fibre diagram (7) and where \mathfrak{a} assigns to each geometric point of the ordinary inertia stack $(x, g \in \text{Aut}(x))$ the rational number $\text{age}(g)$ from equation (3) applied to the diagonalization of the action of $g \in \text{Aut}(x)$ on the tangent bundle $T(\mathfrak{X})$ at x .

We assume that \mathfrak{X} is smooth so that $I_g(\mathfrak{X})$ is smooth and all coarse spaces are quasi-smooth; in particular cohomology groups admit a Hodge decomposition. Starting from a Hodge decomposition of weight n , for any $r \in \mathbb{Q}$, we can produce a new decomposition of weight $n - 2r$ via $H(r)^{p,q} = H^{p+r, p+r}$. We will systematically use this notation (r) for bi-graded vector spaces.

Definition 7 (g -orbifold cohomology). For any $g \in G$ the g -orbifold cohomology is defined as

$$H_g^*(\mathfrak{X}; \mathbb{C}) = H^*(I_g(\mathfrak{X}); \mathbb{C})(-\mathfrak{a}_g).$$

We point out the slight abuse of notation: The age function is not constant in general, but, since it is locally constant, the shift operates independently on each cohomology group arising from each connected component. A precise notation should read

$$H^{p,q}(-; \mathbb{C})(\mathfrak{a}) = \bigoplus_{r \in \mathbb{Q}_{\geq 0}} H^{p,q}(\mathfrak{a}^{-1}(r); \mathbb{C})(-r).$$

For $g = \text{id} = 1_G$, g -orbifold cohomology coincides with the cohomology of the inertia stack shifted by the age function. By definition, this is Chen–Ruan orbifold cohomology

$$H_{\text{id}}^*(\mathfrak{X}; \mathbb{C}) = H_{\text{CR}}^*(\mathfrak{X}; \mathbb{C}).$$

In this paper, we often consider the relative version of orbifold Chen–Ruan cohomology; indeed when \mathfrak{Z} is a substack of \mathfrak{X} then $I(\mathfrak{Z})$ is a substack of $I(\mathfrak{X})$, and we set

$$H_{\text{id}}^*(\mathfrak{X}, \mathfrak{Z}; \mathbb{C}) = H^*(I(\mathfrak{X}), I(\mathfrak{Z}); \mathbb{C})(-\mathfrak{a}_{\text{id}}),$$

where \mathfrak{a}_{id} is the age function on $I(\mathfrak{X})$.

Remark 8. Beside the grading, orbifold cohomology is merely the cohomology of quasi-smooth schemes locally presented as complex varieties modulo finite groups. Therefore, the above cohomology and relative cohomology groups are not new and inherit the same Hodge decomposition holding at the level of schemes. The values of the rational age function identify distinct connected components. There, the relative cohomology groups are taken in the ordinary sense, and the relative cohomology sequence does not need to be generalized. In fact, it is merely the ordinary sequence shifted identically on each term by the age function (we will apply this in §5.1).

Yasuda [37] proves the identity of the dimensions of each term in the Hodge decomposition of orbifold Chen–Ruan cohomology under the following definition of K -equivalence: Two smooth Deligne–Mumford stacks \mathfrak{X} and \mathfrak{Y} are K -equivalent whenever there exists a smooth and proper Deligne–Mumford stack \mathfrak{Z} with birational morphisms $\mathfrak{Z} \rightarrow \mathfrak{X}$ and $\mathfrak{Z} \rightarrow \mathfrak{Y}$ with $\omega_{\mathfrak{Z}/\mathfrak{X}} \cong \omega_{\mathfrak{Z}/\mathfrak{Y}}$. This equivalence holds for our Gorenstein orbifolds and for the crepant resolution of their coarse space by a simple argument which we detail in the proof of Proposition 9.

The hypotheses in our setup are slightly stronger and yield the statement below. Indeed, we consider a smooth Gorenstein orbifold \mathfrak{X} : The local picture at each point $x \in \mathfrak{X}$ is $[\mathbb{C}^{\dim \mathfrak{X}}/H]$ with $H \in \mathrm{SL}(\dim \mathfrak{X}; \mathbb{C})$. In particular, the orbifold cohomology groups $H^{p,q}$ of \mathfrak{X} vanish if $(p, q) \notin \mathbb{Z}^2$.

Let $G = \langle s \rangle$ be a cyclic group of order k whose generator s acts on each tangent space T_x of the Gorenstein orbifold \mathfrak{X} with age $a_x \in 1/k + \mathbb{Z}$ (the age is defined up to automorphisms of x whose age is integer). Let us assume that the coarse space X of \mathfrak{X} admits a crepant resolution \tilde{X} where we can lift the G -action induced by \mathfrak{X} on X . By Yasuda’s theorem we have a bidegree-preserving isomorphism

$$H^*(\mathfrak{X}; \mathbb{C}) \cong H^*(\tilde{X}; \mathbb{C}).$$

The aim of the following proposition, is to point out how this isomorphism relating \mathfrak{X} and \tilde{X} extends to g -orbifold cohomology.

Proposition 9. *We assume the above conditions on \mathfrak{X} (smooth Gorenstein Deligne–Mumford stack), on X (the coarse space) and on \tilde{X} (the crepant resolution). In particular, we consider the order- k cyclic group $G = \langle s \rangle$ acting compatibly on \mathfrak{X} , X , and \tilde{X} . The age a_x of the action of s on each tangent space T_x of \mathfrak{X} satisfies $a_x \in 1/k + \mathbb{Z}$. Then, for any $g \in G$, we have an isomorphism preserving the bidegree*

$$H_g^*(\mathfrak{X}; \mathbb{C}) \cong H_g^*(\tilde{X}; \mathbb{C}).$$

In particular, the isomorphism identifies $H_g^(\mathfrak{X}; \mathbb{C})$ with $H^*(\tilde{X}_g; \mathbb{C})(-\tilde{\alpha}_g)$, where $\tilde{\alpha}_g$ is the composite of $\tilde{X}_g \rightarrow [\tilde{X}/G]$ and of the age function $[\tilde{X}/G] \rightarrow \mathbb{Q}$.*

Proof. Following the definition of K -equivalence, let us show the existence of a smooth and proper Deligne–Mumford stack \mathfrak{Z} mapping to the stack \mathfrak{X} and its resolution \tilde{X} . We consider the fibred product $\mathfrak{Z} = \mathfrak{X} \times_X \tilde{X}$ and the associated reduced stack. Then, there exists a proper birational morphism $\mathfrak{Z}' \rightarrow \mathfrak{Z}$ such that \mathfrak{Z}' is smooth. The existence of this resolution is explained in Section 4.5, §2, of Yasuda’s paper [37] (and is essentially due to Villamayor papers [33] and [34] showing the existence of resolutions compatible with smooth, in particular étale, morphisms). Actually, in his recent generalization [38], Yasuda proves that it suffices to consider the reduction and the normalization of \mathfrak{Z} , without any resolution. This happens because his new statements allows us to extend the definition of orbifold cohomology to singular or wild (in positive characteristic) Deligne–Mumford stacks.

We consider the cyclic group $G = \langle s \rangle$. Then $\mathfrak{U}' = [\mathfrak{X}/G]$ and $\mathfrak{U}'' = [\tilde{X}/G]$ are K -equivalent by the same argument. Indeed, the action of G descends to the coarse space X and we can consider the stack $\mathfrak{U} = [X/G]$ and the morphisms $\mathfrak{U}' \rightarrow \mathfrak{U}$ and $\mathfrak{U}'' \rightarrow \mathfrak{U}$. Then, the reduced stack associated to the fibred product $\mathfrak{U}' \times_{\mathfrak{U}} \mathfrak{U}''$ can be resolved and yields a smooth Deligne–Mumford stack \mathfrak{Z} mapping to \mathfrak{U}' and \mathfrak{U}'' . As above, the fact that the canonical bundles of \mathfrak{X} and \tilde{X} are the pullback of ω_X is enough to show that $\mathfrak{Z} \rightarrow \mathfrak{U}' = [\mathfrak{X}/G]$ and $\mathfrak{Z} \rightarrow \mathfrak{U}'' = [\tilde{X}/G]$ is a K -equivalence. Indeed, $\omega_{\mathfrak{U}'}$ is merely the G -equivariant line bundle ω_X , which pulls back to $\omega_{\mathfrak{U}'}$ and $\omega_{\mathfrak{U}''}$.

Then, [37, Cor. 4.8] affirms that the Chen–Ruan orbifold cohomology of \mathfrak{X}' and that of \mathfrak{X}'' are isomorphic for all bidegrees (p, q) . The desired claim in our statement is just a restriction of this claim. Indeed, for $g = s^b$ and $b \in \{0, \dots, k - 1\}$, the cohomologies of $I_g \mathfrak{X}$ and $I_g \tilde{X}$ arise as the summands of the Chen–Ruan cohomology groups of $[\mathfrak{X}/G]$ and of $[\tilde{X}/G]$ whose bidegree lie in $(b/k, b/k) + \mathbb{Z}^2$. By definition, they are the cohomology groups of the sectors attached to g , which are given by g -invariant classes of $I_g(\mathfrak{X})$ and $I_g(\tilde{X})$. Since g operates trivially on these sectors, we can regard these contributions as $H^*(I_g(\mathfrak{X}); \mathbb{C})$ and $H^*(I_g(\tilde{X}); \mathbb{C})$. Finally, we obtain an identification at the level of the age-shifted g -orbifolded cohomology $H_g^*(\text{---}; \mathbb{C})$ due to the fact that the age is a rational function factoring through the usual age function of $[\mathfrak{X}/H]$ and of $[\tilde{X}/H]$. \square

Remark 10. The above statement uses a condition on the age of the automorphism s in order to deduce an isomorphism from a restriction of Yasuda’s isomorphism. It is possible, but we did not check it, that Yasuda’s identification preserves the G -grading in general, even when the G -grading cannot be reconstructed from the bigrading.

In special cases where $\tilde{\alpha}_g$ is constant, the above theorem allows us to relate the g -orbifold cohomology to the cohomology of the g -fixed locus of the resolution via a constant shift by $\tilde{\alpha}_g$. The following example generalises the case of antisymplectic involutions of orbifold K3 surfaces considered in [12] (this case occurs in Section 7 for $k = 2$ and 4).

Example 11. Consider a proper, smooth, Gorenstein, Deligne–Mumford orbifold \mathfrak{X} of dimension 2 satisfying the Calabi–Yau condition $\omega \cong \mathcal{O}$. We refer to this as a K3 orbifold because there exists a minimal resolution \tilde{X} which is a K3 surface. Consider the volume form Ω of \tilde{X} , which descends on \mathfrak{X} . We assume that g is an order- k automorphism of \mathfrak{X} whose induced action on Ω is multiplication by $e^{2\pi i(k-1)/k}$. Then, g naturally lifts to the minimal resolution \tilde{X} ; furthermore, locally at each fixed point of \tilde{X} , the action of g can be diagonalized and expressed as $(x, y) \mapsto (\xi_k^a x, \xi_k^b y)$ where $a + b \equiv k - 1 \pmod k$. In fact, $a + b$ equals $k - 1$ without reduction mod k (this happens because the case $a + b = 2k - 1$ is impossible for $a, b \in \{0, \dots, k - 1\}$). In this way, the age shift α_g at the fixed loci always equals $1 - \frac{1}{k}$

$$H_g^*(\mathfrak{X}; \mathbb{C}) = H^*(\tilde{X}_g; \mathbb{C})\left(\frac{1}{k} - 1\right).$$

5. Landau–Ginzburg state space

The expression ‘Landau–Ginzburg’ comes from physics and is often used for \mathbb{C} -valued functions defined on vector spaces possibly equipped with the action of a group. More generally the definition is extended to vector bundles on a stack. In this paper, we only use it for the above setup $P: [\mathbb{C}^{n+1}/H] \rightarrow \mathbb{C}$, where P is a nondegenerate polynomial and $j \in H \subseteq \text{Aut}_P$. Indeed, this may be regarded as a \mathbb{C} -valued function defined on a rank- $(n + 1)$ vector bundle on the stack $BH = [\text{Spec } \mathbb{C}/H]$. We show how this geometric setup is naturally connected to $\Sigma_{W,H}$ via K -equivalence.

The structure of this section is the following. In §5.1, we setup the K -equivalence relating the g -orbifold cohomology of a vector bundle \mathbb{V} to the the g -orbifold cohomology of a line bundle \mathbb{L} on a weighted projective stack $\mathbb{P}(\mathbf{w})$. We deduce from this equivalence the entire proof of Theorem 24 stated in §5.4 relating a g -orbifold variant of the Landau–Ginzburg state space to the g -orbifold cohomology of $\Sigma_{W,H}$. On the one hand, in §5.2, the line bundle \mathbb{L} is related to the hypersurface $\Sigma_{W,H}$ via a generalization of the Thom isomorphism. On the other hand, in §5.3, the vector bundle \mathbb{V} is related to the Landau–Ginzburg state space (the vector space underlying a Jacobi ring) introduced in its earliest orbifold formulation by Fan, Jarvis and Ruan. In §5.4, based on the previous sections, we choose an isomorphism relating \mathbb{V} and \mathbb{L} yielding the desired result: Thm. 24. We then comment on previous proofs and related results.

5.1. *K-equivalence*

Consider the rank- $(n + 1)$ vector bundle

$$\mathbb{V} = \mathcal{O}_{\mathbb{P}(d)}(-w_0) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}(d)}(-w_n) = [\mathbb{C}^{n+1}/\langle j \rangle],$$

where we recall that $\mathbb{P}(d)$ is simply the special case of a zero-dimensional projective stack isomorphic to $B\mu_d$ (see Remark 2). Its coarse space is $X = \mathbb{C}^{n+1}/\langle j \rangle$ with a singularity given by the μ_d -action $[\frac{w_0}{d}, \frac{w_1}{d}, \dots, \frac{w_n}{d}]$.

We also consider the smooth Deligne–Mumford stack

$$\mathbb{L} = \mathcal{O}_{\mathbb{P}(\mathbf{w})}(-d)$$

defined as the (stack-theoretic) total space \mathbb{L} of the line bundle of degree $-d$ on $\mathbb{P}(\mathbf{w})$. Because \mathbb{L} is a weighted blow up of X in the stack-theoretic sense, there is a natural map from \mathbb{L} to X .

The stacks \mathbb{V} and \mathbb{L} are the two stack-theoretic GIT quotients of $\mathbb{C} \times \mathbb{C}^{n+1}$ modulo \mathbb{G}_m operating with weights $(-d, w_0, \dots, w_{n+1})$ on the two open subsets obtained by removing the origin of the left- and right-hand side factors of the product $\mathbb{C} \times \mathbb{C}^{n+1}$. Notice that \mathbb{V} without the origin coincides with the line bundle \mathbb{L} without the zero section: $\mathbb{V}^\times = \mathbb{L}^\times$.

As in equation (4), we consider the generic fibre M_P of $P: \mathbb{C}^{n+1} \rightarrow \mathbb{C}$. The morphism P descends to a morphism from \mathbb{V} to \mathbb{C} , whose generic fibre is isomorphic to $\mathbb{F} = [M_P/\langle j \rangle]$, a substack of \mathbb{V}^\times which we may regard, via the above identification, as a substack of \mathbb{L}^\times .

We assume the Calabi–Yau condition $\sum_{i=0}^n w_i = d$. Then, the canonical bundle of \mathbb{V} is j -invariant and descends to X and its pullback to \mathbb{L} coincides with $\omega_{\mathbb{L}}$. Following the above arguments, i.e., by Proposition 9 and ultimately by Yasuda [37], we have

$$\Phi: H_g^{p,q}(\mathbb{V}; \mathbb{C}) \xrightarrow{\cong} H_g^{p,q}(\mathbb{L}; \mathbb{C}) \tag{9}$$

for any $p, q \in \mathbb{Q}$ and for any $g = s^b$ for $b \in \{0, \dots, k - 1\}$ and $g = [0, \frac{1}{k}, 0, \dots, 0]$ (notice that we extended g trivially on the fibre of \mathbb{L}).

The isomorphism $H_g^*(\mathbb{V}; \mathbb{C}) \rightarrow H_g^*(\mathbb{L}; \mathbb{C})$ is not canonical, the claim of its existence is simply an identity between dimensions of vector spaces. The argument here consists in choosing an isomorphism Φ in such a way that it commutes with the restrictions $H_g^r(\mathbb{V}; \mathbb{C}) \rightarrow H_g^r(\mathbb{F}; \mathbb{C})$ and $H_g^r(\mathbb{L}; \mathbb{C}) \rightarrow H_g^r(\mathbb{F}; \mathbb{C})$ and with the identity on $H_g^r(\mathbb{F}; \mathbb{C})$ in the diagram below in all degrees $r \in \mathbb{Q}$

$$\begin{array}{ccccccc} \dots & \rightarrow & H_g^r(\mathbb{V}, \mathbb{F}; \mathbb{C}) & \rightarrow & H_g^r(\mathbb{V}; \mathbb{C}) & \rightarrow & H_g^r(\mathbb{F}; \mathbb{C}) & \rightarrow & H_g^{r+1}(\mathbb{V}, \mathbb{F}; \mathbb{C}) & \rightarrow & \dots \\ & & \downarrow & & \downarrow \Phi & & \downarrow = & & \downarrow & & \\ \dots & \rightarrow & H_g^r(\mathbb{L}, \mathbb{F}; \mathbb{C}) & \rightarrow & H_g^r(\mathbb{L}; \mathbb{C}) & \rightarrow & H_g^r(\mathbb{F}; \mathbb{C}) & \rightarrow & H_g^{r+1}(\mathbb{L}, \mathbb{F}; \mathbb{C}) & \rightarrow & \dots \end{array} \tag{10}$$

We detail this isomorphism Φ in §5.4 after studying more carefully the two horizontal exact sequences involving \mathbb{L} and \mathbb{V} in §5.2 and §5.3, respectively. This allows us to conclude that there is a bidegree-preserving isomorphism

$$H_g^*(\mathbb{V}, \mathbb{F}; \mathbb{C}) \cong H_g^*(\mathbb{L}, \mathbb{F}; \mathbb{C})$$

making the above diagram commute. Based on §5.2 (using [13, Prop. 3.5]), the right-hand side is naturally identified via the Thom isomorphism to the Chen–Ruan cohomology of $\Sigma_{P,H}$ up to a (-1) -shift. Based on §5.3, the left-hand side is naturally identified to an orbifold version of the Jacobi ring known as the FJRW or Landau–Ginzburg state space.

5.2. Thom isomorphism in orbifold cohomology

Consider $P: \mathbb{L} \rightarrow \mathbb{C}$ and its generic fibre $\mathbb{F} = P^{-1}(t)$ for $t \neq 0$. We have an isomorphism of Hodge structures

$$H^*(\mathbb{L}, \mathbb{F}; \mathbb{C}) \cong H^*(\Sigma_P; \mathbb{C})(-1). \tag{11}$$

Indeed, the left-hand side can be regarded after retraction as

$$H^*(\mathbb{P}(\mathbf{w}), \mathbb{P}(\mathbf{w}) \setminus \Sigma_P; \mathbb{C})$$

which is isomorphic to the (-1) -shifted cohomology of Σ_P by the Thom isomorphism

$$H^*(\mathbb{P}(\mathbf{w}), \mathbb{P}(\mathbf{w}) \setminus \Sigma_P; \mathbb{C}) \cong H^*(\Sigma_P; \mathbb{C})(-1). \tag{12}$$

Equation (11) suggests that the orbifold cohomology $H_{\text{id}}^{p,q}(\mathbb{L}_H, \mathbb{F}_{P,H}; \mathbb{C})$ is related to the orbifold cohomology of $\Sigma_{P,H}$. However, due to the age shift, the argument above does not yield an isomorphism in orbifold cohomology. In fact, the two isomorphisms

$$H^*(\mathbb{L}_H, \mathbb{F}_{P,H}; \mathbb{C}) \cong H^*(\mathbb{P}(\mathbf{w}), \mathbb{P}(\mathbf{w}) \setminus \Sigma_{P,H}; \mathbb{C}) \cong H^*(\Sigma_{P,H}; \mathbb{C})(-1) \tag{13}$$

may not respect the orbifold cohomology bidegree. However, Jan Nagel and the first author proved that the first and the third term of equation (13) match in orbifold cohomology with their respective bigrading even when the second isomorphism (the ordinary Thom isomorphism) does not hold in orbifold cohomology. For these reasons, we regard the special case where $g = \text{id}$

$$H_{\text{id}}^*(\mathbb{L}, \mathbb{F}; \mathbb{C}) \cong H_{\text{id}}^*(\Sigma_P; \mathbb{C})(-1)$$

as the correct formulation of Thom isomorphism in orbifold cohomology.

Theorem 12 (the orbifold Thom isomorphism, [13]). *For any $H \subseteq \text{Aut}_P$ containing j_P , for any g in Aut_P and for any $p, q \in \mathbb{Q}$, we have*

$$H_g^{p,q}(\mathbb{L}_H, \mathbb{F}_{P,H}; \mathbb{C}) \cong H_g^{p,q}(\Sigma_{P,H}; \mathbb{C})(-1). \tag{14}$$

Before proving the statement, we illustrate it with a simple example allowing us to describe the two cases of the proof.

Example 13. We provide an elementary and yet very rich example of a Calabi–Yau orbifold embedded in a non-Gorenstein $\mathbb{P}(\mathbf{w})$.

The isomorphism above matches the orbifold cohomology of a hypersurface Σ_P and the relative cohomology of (\mathbb{L}, \mathbb{F}) . Therefore, on the one side, we consider the hypersurface Σ_P defined by $P = x^3 + xy = 0$ in $\mathbb{P}(1, 2)$. There are two components in the ambient orbifold $\mathbb{P}(1, 2)$: the untwisted sector (labelled with u) attached to the identity and the twisted sector (labelled with t) attached to the nontrivial symmetry $(-1, 1)$.

Within the untwisted sector $\mathbb{P}(1, 2)_u = \mathbb{P}(1, 2)$, the equation $x^3 + xy = 0$ cuts out a codimension-one hypersurface which can be described as the disjoint union of an ordinary point, the \mathbb{G}_m -orbit $p = (-\lambda, \lambda^2)$ and a point with nontrivial μ_2 -automorphism: the locus where the first coordinate vanishes ($x = 0$) $\cong \mathbb{P}(2)$.

Within the twisted sector $\mathbb{P}(1, 2)_t = \mathbb{P}(2)$, the equation $x^3 + xy = 0$ identifies a hypersurface which is not transversal to $\mathbb{P}(2)$. In other words, the twisted sector of Σ_P is the entire twisted sector $\mathbb{P}(1, 2)_t = \mathbb{P}(2)$.

These two conditions, $\Sigma_{P,\beta} \subsetneq \mathbb{P}(\mathbf{w}_\beta)$ and $\Sigma_{P,\beta} = \mathbb{P}(\mathbf{w}_\beta)$, play two different roles in the orbifold Thom isomorphism as we illustrate in this example and detail further in general in the proof.

The right-hand side of the orbifold Thom isomorphism reads $H_{\text{orb}}^*(\Sigma_P; \mathbb{C})(-1)$ (for $g = \text{id}$) and can be easily computed in this example: $\mathbb{C}^{\oplus 3}(-1)$. This happens because the inertia stack consists of three points

$$I(\Sigma_P) = \Sigma_{P,u} \sqcup \Sigma_{P,t} = (2 \text{ pts}) \sqcup \mathbb{P}(2), \tag{15}$$

(the labels u and t stand again for untwisted and twisted). The age-shift plays no role for a zero dimensional stack (the tangent bundle vanishes). We get

$$H_{\text{orb}}^*(\Sigma_P; \mathbb{C})(-1) \cong \mathbb{C}^{\oplus 3}(-1). \tag{16}$$

The left-hand side of the orbifold Thom isomorphism above reads $H_{\text{orb}}^*(\mathbb{L}, \mathbb{F}_P; \mathbb{C})$, which, unlike the orbifold cohomology group $H_{\text{orb}}^*(\mathbb{P}(1, 2), \mathbb{P}(1, 2) \setminus \Sigma)$, matches equation (16). We compute both $H_{\text{orb}}^*(\mathbb{P}(1, 2), \mathbb{P}(1, 2) \setminus \Sigma)$ and $H_{\text{orb}}^*(\mathbb{L}, \mathbb{F}_P; \mathbb{C})$.

For $H_{\text{orb}}^*(\mathbb{P}(1, 2), \mathbb{P}(1, 2) \setminus \Sigma)$, we have $I(\mathbb{P}(1, 2)) = \mathbb{P}(1, 2)_u \sqcup \mathbb{P}(1, 2)_t = \mathbb{P}(1, 2) \sqcup \mathbb{P}(2)$. Using equation (15) and writing $\mathbb{P}(\mathbf{w}) = \mathbb{P}(1, 2)$ for short, we get

$$\begin{aligned} H_{\text{orb}}^*(\mathbb{P}(\mathbf{w}), \mathbb{P}(\mathbf{w}) \setminus \Sigma) &= H^*(\mathbb{P}(\mathbf{w})_u, \mathbb{P}(\mathbf{w})_u \setminus \Sigma_{P,u}) \oplus H^*(\mathbb{P}(\mathbf{w})_t, \mathbb{P}(\mathbf{w})_t \setminus \Sigma_{P,t})(-\tfrac{1}{2}) \\ &= H^*(\mathbb{P}(\mathbf{w}), \mathbb{P}(\mathbf{w}) \setminus \{2 \text{ pts}\}) \oplus H^*(\mathbb{P}(2), \emptyset)(-\tfrac{1}{2}) \\ &= H^*(2 \text{ pts})(-1) \oplus H^0(\mathbb{P}(2))(-\tfrac{1}{2}) \\ &= \mathbb{C}^{\oplus 2}(-1) \oplus \mathbb{C}(-\tfrac{1}{2}). \end{aligned}$$

On the other hand, for $H_{\text{orb}}^*(\mathbb{L}, \mathbb{F})$, the ambient orbifold \mathbb{L} induces an extra $\frac{1}{2}$ age-shift on the twisted sector because the stabilizer of $\mathbb{P}(2)$ acts nontrivially in the direction of the fibre of the line bundle $\mathbb{L} \rightarrow \mathbb{P}(1, 2)$. Therefore, we get

$$H_{\text{id}}^*(\mathbb{L}, \mathbb{F}) = \mathbb{C}^2(-1) \oplus H^*(\mathbb{P}(2))(-1) \cong \mathbb{C}^{\oplus 3}(-1)$$

as in equation (16).

Proof. We study $H_g^*(\mathbb{L}, \mathbb{F}; \mathbb{C})$, $H_g^*(\mathbb{P}(\mathbf{w}), \mathbb{P}(\mathbf{w}) \setminus \Sigma_P; \mathbb{C})$, and $H_g^*(\Sigma_P; \mathbb{C})(-1)$ sector-by-sector, i.e., by restricting to each symmetry $\beta = g(\lambda^{w_0}, \lambda^{w_1}, \dots, \lambda^{w_n})$ operating on \mathbb{C}^{n+1} diagonally. We distinguish two cases: (1) the ordinary sectors where $\Sigma_{P,\beta}$ is a codimension-1 hypersurface in $\mathbb{P}(\mathbf{w}_\beta)$ (this happens, for instance, for the untwisted sector in Example 13); (2) the case where $\mathbb{P}(\mathbf{w}_\beta)$ and $\Sigma_{P,\beta}$ coincide (as in the twisted sector of Example 13). The two cases above may also be distinguished as follows: In case (1) we consider diagonal automorphisms β fixing points of Σ_P and the fibre of the normal bundle $N_{\Sigma_P/\mathbb{P}(\mathbf{w})}$ (β fixes the points of $\mathbb{P}(\mathbf{w}_\beta)$); in case (2) we consider automorphisms β fixing points of Σ_P and operating nontrivially on the normal bundle $N_{\Sigma_P/\mathbb{P}(\mathbf{w})}$. Notice also the following simple reformulation; since the restriction of $\mathbb{L} = \mathcal{O}_{\mathbb{P}(\mathbf{w})}(d)$ to Σ_P is dual to the normal bundle of Σ_P within $\mathbb{P}(\mathbf{w})$, in case (1) we have $\Sigma_{P,\beta} \subseteq \mathbb{P}(\mathbf{w}_\beta) \subseteq \mathbb{L}_\beta$ and each inclusion has codimension 1, whereas in case (2) we have $\Sigma_{P,\beta} = \mathbb{P}(\mathbf{w}_\beta) = \mathbb{L}_\beta$.

For ordinary sectors where $\Sigma_{P,\beta} \subseteq \mathbb{P}(\mathbf{w}_\beta) \subseteq \mathbb{L}_\beta$, we have $H^*(\mathbb{P}(\mathbf{w}_\beta), \mathbb{P}(\mathbf{w}_\beta) \setminus \Sigma_{P,\beta}; \mathbb{C}) \cong H^*(\Sigma_{P,\beta}; \mathbb{C})(-1)$ and the age shift of $\mathbb{P}(\mathbf{w}_\beta)$ coincides with that of $\Sigma_{P,\beta}$ since β acts trivially on the normal bundle $N_{\Sigma_{P,\beta}/\mathbb{P}(\mathbf{w}_\beta)}$. No age-shift difference occurs also when we pass to \mathbb{L}_β , which is a vector bundle on $\mathbb{P}(\mathbf{w}_\beta)$: The age-shifted classes of $H^*(\mathbb{L}_\beta, \mathbb{F}_\beta)$ match those of $H^*(\mathbb{P}(\mathbf{w}_\beta), \mathbb{P}(\mathbf{w}_\beta) \setminus \Sigma_{P,\beta})$ and, by the above argument, those of $H^*(\Sigma_{P,\beta}; \mathbb{C})(-1)$.

For the remaining sectors, when $\Sigma_{P,\beta} = \mathbb{P}(\mathbf{w}_\beta) = \mathbb{L}_\beta$, the cohomology classes of the sectors labelled by β within $\mathbb{L}, \mathbb{P}(\mathbf{w})$ and Σ_P match via the trivial identities

$$\begin{aligned}
 H^*(\mathbb{L}_\beta, \mathbb{F}_\beta) &= H^*(\mathbb{P}(\mathbf{w}_\beta), \mathbb{P}(\mathbf{w}_\beta) \setminus \Sigma_{P,\beta}; \mathbb{C}) \\
 &= H^*(\mathbb{P}(\mathbf{w}_\beta), \emptyset; \mathbb{C}) \\
 &= H^*(\mathbb{P}(\mathbf{w}_\beta); \mathbb{C}) \\
 &= H^*(\Sigma_{P,\beta}; \mathbb{C}).
 \end{aligned}$$

Hence, in ordinary cohomology, the desired equation (14) holds without any (-1) -shift on the right-hand side. The (-1) -shift arises when we compare the age-shifts of the sector \mathbb{L}_β and that of $\Sigma_{P,\beta}$; i.e., when we compare the age of the action of β on the normal bundles $N_{\Sigma_{P,\beta}/\Sigma_P}$ and $N_{\mathbb{L}_\beta/\mathbb{L}}$ restricted to $\mathbb{L}_\beta = \Sigma_{P,\beta}$. Let $q \in]0, 1[$ be the nontrivial character of the action of β on the normal bundle $N_{\Sigma_P/\mathbb{P}(\mathbf{w})}$. In these cases, we have $H^*_{\text{orb}}(\mathbb{P}(\mathbf{w}_\beta), \mathbb{P}(\mathbf{w}_\beta) \setminus \Sigma_{P,\beta}; \mathbb{C}) \cong H^*_{\text{orb}}(\Sigma_{P,\beta}; \mathbb{C})(-q)$. Since the restriction of $\mathbb{L} = \mathcal{O}_{\mathbb{P}(\mathbf{w})}(d)$ to Σ_P is dual to the normal bundle $N_{\Sigma_P/\mathbb{P}(\mathbf{w})}$, the age of β acting on \mathbb{L} at a point p of $\Sigma_{P,\beta} = \mathbb{L}_\beta$ differs from the age of the representation on the tangent bundle $T_{\Sigma_P,p}$ by $q + (1 - q)$. We have

$$H^*(\mathbb{L}_\beta, \mathbb{F}_\beta; \mathbb{C})(\mathbf{a}^\mathbb{L}) \cong H^*(\Sigma_{P,\beta}; \mathbb{C})(\mathbf{a}^\Sigma - q - (1 - q)) = H^*(\Sigma_{P,\beta}; \mathbb{C})(\mathbf{a}^\Sigma - 1)$$

as desired (we wrote $\mathbf{a}^\mathbb{L}$ and \mathbf{a}^Σ for the locally constant age functions on the orbifolds \mathbb{L} and Σ_P). □

Remark 14. The previous result is proven in [13, Prop. 3.4] in the more general setup of complete intersections. Notice, however, that the present setup of g -orbifold cohomology required an independent treatment.

Remark 15 (definition of ambient and primitive cohomology). For a hypersurface Σ within a projective space \mathbb{P} , we usually refer to the Poincaré dual of the image of the homology of Σ within the homology of \mathbb{P} as the ambient cohomology of Σ . In orbifold cohomology, we say that the ambient cohomology of $\Sigma_{P,H}$ is Poincaré dual to the image of the homology of $I_g(\Sigma_{P,H})$ within the homology of $I_g(\mathbb{P}(\mathbf{w}))$. The identification above allows us to provide another description. We may regard the ambient cohomology of $\Sigma_{P,H}$ as the image of the morphism

$$H^r_g(\mathbb{L}_H, \mathbb{F}_{P,H}; \mathbb{C}) \rightarrow H^r_g(\mathbb{L}_H; \mathbb{C}). \tag{17}$$

Similarly, we can consider the primitive cohomology, which is the kernel after Poincaré duality of the direct image mapping the homology of Σ within the cohomology of \mathbb{P} . In orbifold cohomology, the primitive cohomology of $\Sigma_{P,H}$ is the kernel of the morphism (17) in $H^*(\Sigma_{P,H}; \mathbb{C})(1)$. We summarize these notations as follows.

Definition 16.

$$\begin{aligned}
 H^r_{g,\text{amb}}(\Sigma_{W,H}; \mathbb{C})(-1) &= \text{im} \left(H^r_g(\mathbb{L}_H, \mathbb{F}_{P,H}; \mathbb{C}) \rightarrow H^r_g(\mathbb{L}_H; \mathbb{C}) \right), \\
 H^r_{g,\text{prim}}(\Sigma_{W,H}; \mathbb{C})(-1) &= \text{ker} \left(H^r_g(\mathbb{L}_H, \mathbb{F}_{P,H}; \mathbb{C}) \rightarrow H^r_g(\mathbb{L}_H; \mathbb{C}) \right).
 \end{aligned}$$

We now turn to the LG side, where the image of the analogue morphism allows us to describe the so called ‘narrow’ and ‘broad’ sectors.

5.3. Jacobi ring

The Jacobi ring

$$\text{Jac } P = dx_0 \wedge \cdots \wedge dx_n \mathbb{C}[x_0, \dots, x_n] / (\partial_0 P, \dots, \partial_n P),$$

regarded as a \mathbb{C} -vector space, has dimension $\prod_j \frac{d-w_j}{w_j}$ (due to the nondegeneracy of the polynomial P) and is isomorphic to $H^*(\mathbb{C}^{n+1}, M_P; \mathbb{C})$. The natural monodromy action of $\boldsymbol{\mu}_d = \langle j \rangle$ from equation (4), and more generally the action of any $\text{diag}(\alpha_0, \dots, \alpha_n) \in \text{Aut}_P$,

$$\text{diag}(\alpha_0, \dots, \alpha_n) \cdot \left(\prod_{j=0}^n x_j^{b_j-1} \bigwedge_{j=0}^n dx_j \right) = \prod_{j=0}^n \alpha_j^{b_j} \left(\prod_{j=0}^n x_j^{b_j-1} \bigwedge_{j=0}^n dx_j \right),$$

allows us to write

$$[\text{Jac}(P)^{j_P}]_{p,q} = H^{p,q}(\mathbb{V}; \mathbb{F}),$$

where the subscript p, q denote the subgroup spanned by elements of the form

$$\left(\prod_{j=0}^n x_j^{b_j-1} \bigwedge_{j=0}^n dx_j \right) \quad \text{with} \quad (p, q) = \left(n - \sum_j b_j \frac{w_j}{d}, \sum_j b_j \frac{w_j}{d} \right).$$

The above claim is due to Steenbrink [30] in the present weighted homogenous setup; see also [11, Appendix A].

Remark 17. The action of Aut_P on $\text{Jac}(P)$ is well defined because any automorphism $\text{diag}(\alpha_0, \dots, \alpha_n)$ operates on each monomial in $\partial_i P$ by multiplication by α_i^{-1} . This happens because $\text{diag}(\alpha_0, \dots, \alpha_n)$ fixes $x_i \partial_i P$ since it fixes P .

Furthermore, the grading (p, q) is well defined simply because $\text{deg}(x_i) = \frac{w_i}{d}$ yields a \mathbb{Q} -grading on $\mathbb{C}[x_0, \dots, x_n]$, which descends to a \mathbb{Q} -grading of $\text{Jac}(P)$ because the Jacobi ideal $(\partial_0 P, \dots, \partial_n P)$ is homogeneous (each monomial in $\partial_i P$ has degree $d - \frac{w_i}{d}$).

We can introduce the following slight generalization of the FJRW state space.

Definition 18. For a quasi-homogenous polynomial P of degree d and weight w_0, \dots, w_n and for any $H \subseteq \text{Aut}_P$ containing j_P , the g -orbifolded Landau–Ginzburg state space is

$$\mathcal{H}_{P,H,g} = \bigoplus_{h \in gH} (\text{Jac } P_h)^H(-\text{age}(h)),$$

where, for any diagonal symmetry $h \in H$, we consider the Jacobi ring $\text{Jac } P_h$, where P_h is the restriction of P to the ring of polynomials in the h -fixed variables (and the age is given by equation (3)).

Remark 19. Notice that, as a consequence of the nondegeneracy of P , the restriction P_g is still a nondegenerate polynomial.

Remark 20. When g is the identity we recover the FJRW state space $\mathcal{H}_{P,H}$.

Remark 21. We have

$$\mathcal{H}_{P,H,g}^{p,q} = H_g^{p,q}(\mathbb{V}_H; \mathbb{F}_{P,H}). \tag{18}$$

Indeed for $H = \langle j_P \rangle$ this is the well-known identification between the middle cohomology of the fibre \mathbb{F} and the monodromy invariant part of the cohomology of the Milnor fibre (see, for instance, [16]). The general statement follows from taking the invariant classes on the two sides with respect to the action of the finite group $H_0 = H/\langle j_P \rangle$.

Remark 22 (definition of narrow and broad sectors). The elements h for which no variables are fixed yield a summand $\text{Jac}(P_h) = \mathbb{C}$; these are special elements in the FJRW state space; they span the subspace of the so-called narrow classes. In FJRW theory, the remaining summands are referred to as broad classes (see [22]). In complete analogy with the notation for ambient and primitive cohomology, we can identify the narrow and broad classes to the image and the kernel of the morphism

$$H_g^r(\mathbb{V}_H, \mathbb{F}_{P,H}; \mathbb{C}) \rightarrow H_g^r(\mathbb{V}_H; \mathbb{C}).$$

Definition 23. We have

$$\begin{aligned} \mathcal{H}_{P,H,g}^{r,\text{nar}} &= \text{im} (H_g^r(\mathbb{V}_H, \mathbb{F}_{P,H}; \mathbb{C}) \rightarrow H_g^r(\mathbb{V}_H; \mathbb{C})), \\ \mathcal{H}_{P,H,g}^{r,\text{broad}} &= \text{ker} (H_g^r(\mathbb{V}_H, \mathbb{F}_{P,H}; \mathbb{C}) \rightarrow H_g^r(\mathbb{V}_H; \mathbb{C})). \end{aligned}$$

5.4. Landau–Ginzburg/Calabi–Yau correspondence

The above equations (14) and (18) add up to a simple proof of the so-called Landau–Ginzburg/Calabi–Yau correspondence based on Yasuda’s theorem and K -equivalence (ensured by the Calabi–Yau condition).

Theorem 24 ([10, 13]). *For any nondegenerate quasi-homogeneous polynomial P of Calabi–Yau type, for any group $H \subseteq \text{Aut}_P$ containing j_P and for any $g \in \text{Aut}_P$, we have*

$$\Phi: H_g^{P,q}(\Sigma_{P,H}; \mathbb{C})(-1) \xrightarrow{\sim} \mathcal{H}_{P,H,g}^{P,q}.$$

Proof. Let us consider the case $H = \langle j \rangle$ for simplicity first. In the statement, the right-hand side is isomorphic to $H_g^*(\mathbb{L}, \mathbb{F})$ and the left-hand side is isomorphic to $H_g^*(\mathbb{V}, \mathbb{F})$; hence, we only need to define an isomorphism $H_g^*(\mathbb{V}) \rightarrow H_g^*(\mathbb{L})$ making the diagram (10) commute. The relative cohomology sequence allows us to split $H_g^*(\mathbb{V})$ and $H_g^*(\mathbb{L})$ into two direct summands:

$$H_g^*(A) = \text{coker}(H_g^*(A, \mathbb{F}) \rightarrow H_g^*(A)) \oplus \text{ker}(H_g^*(A) \rightarrow H_g^*(\mathbb{F})),$$

for $A = \mathbb{V}$ and \mathbb{L} . We show that there is a bidegree-preserving isomorphism matching the above cokernels within $H^*(\mathbb{V})$ and $H^*(\mathbb{L})$. Since any extension on the kernel of the restriction morphism makes equation (10) commute, the claim follows.

We need to treat separately two cases depending on whether the isomorphism $\gamma = g(\lambda^{-d}, \lambda^{w_0}, \lambda^{w_1}, \dots, \lambda^{w_n})$ yields an empty sector \mathbb{F}_γ or not. If \mathbb{F}_γ is empty, then the restrictions morphisms $H^r(\mathbb{V}_\gamma) \rightarrow H^r(\mathbb{F}_\gamma)$ and $H^r(\mathbb{L}_\gamma) \rightarrow H^r(\mathbb{F}_\gamma)$ vanish. Therefore, these sectors only contribute to the above mentioned kernel of the restrictions morphisms mapping to $H_g^*(\mathbb{F})$.

In the remaining cases, as argued in the proof of Theorem 12, the sector $\Sigma_{P,\gamma}$ of Σ_P is a codimension-1 subspace of the γ -fixed coordinate subspace $\mathbb{P}(\mathbf{w}_\gamma)$ and \mathbb{L}_γ is the bundle $\mathcal{O}(-d)$ over $\mathbb{P}(\mathbf{w}_\gamma)$. In these cases, the restriction morphism is nonzero only on the fundamental classes of \mathbb{V}_γ and \mathbb{L}_γ (these vanishing property are obvious for \mathbb{V}_γ , whereas for \mathbb{L}_γ we may refer to a more detailed study of the Lefschetz hyperplane theorem in this setup in [13, Lem. 3.6]). We notice that, in these cases, the fundamental classes of \mathbb{V}_γ and \mathbb{L}_γ share the same bidegree ($\text{age}(\gamma)$, $\text{age}(\gamma)$).

The existence of a bidegree-preserving isomorphism $H_g^*(\mathbb{V}) \cong H_g^*(\mathbb{L})$, alongside with the bidegree-preserving identification $\mathbf{1}_{\mathbb{V}_\gamma} \mapsto \mathbf{1}_{\mathbb{L}_\gamma}$ (for $\mathbb{F}_\gamma \neq \emptyset$), which matches the cokernels within $H_g^*(\mathbb{V})$ and $H_g^*(\mathbb{L})$, ensures the existence of an isomorphism $\Phi: H_g^*(\mathbb{V}) \cong H_g^*(\mathbb{L})$ extending $\mathbf{1}_{\mathbb{V}_\gamma} \mapsto \mathbf{1}_{\mathbb{L}_\gamma}$ on the kernels of the restriction morphisms. By construction Φ is compatible with equation (10), and, as a consequence, there exists a bidegree-preserving isomorphism $H_g^d(\mathbb{V}, \mathbb{F}; \mathbb{C}) \cong H_g^d(\mathbb{L}, \mathbb{F}; \mathbb{C})$.

The complete statement involves a more general group action with H replacing $\langle j \rangle$. This is a minor generalization. Indeed, we note that \mathbb{F} can be regarded as the generic fibre of the two morphisms induced by the polynomial $P: \mathbb{V} \rightarrow \mathbb{C}$ and $\mathbb{L} \rightarrow \mathbb{C}$. If we consider any group $H \subseteq \text{Aut}_P$ containing j , we can apply the above claim to $\mathbb{V}_H = [\mathbb{V}/H_0]$, $\mathbb{L}_H = [\mathbb{L}/H_0]$ and $\mathbb{F}_{P,H} = [\mathbb{F}/H_0]$. We get $H_g^{P,q}(\mathbb{V}_H, \mathbb{F}_{P,H}; \mathbb{C}) \cong H_g^{P,q}(\mathbb{L}_H, \mathbb{F}_{P,H}; \mathbb{C})$, for any p, q and for any $g \in \text{Aut}_W$. The above dichotomy distinguishing two types of sectors applies without changes to isomorphisms of the form $g(\lambda^{-d}, \alpha_0 \lambda^{w_0}, \dots, \alpha_n \lambda^{w_n})$, for $\alpha_i, \lambda \in \mathbb{G}_m$. □

Remark 25. A shortcoming of the above approach is the lack of a preferred isomorphism. Since the above isomorphism follows from K -equivalence, it is not explicitly given. In [10], we provide an explicit isomorphism, generalized in [13] to complete intersections. In [12], we make the isomorphism explicit

in dimension two at the level of ordinary cohomology classes of the (crepant and minimal) scheme-theoretic resolution of \mathbb{L} .

Finally, let us point out that, by a slight abuse of notation, we adopt the same notation Φ for the isomorphism in the statement as well as for $\Phi: H_g^*(\mathbb{V}; \mathbb{C}) \rightarrow H_g^*(\mathbb{L}; \mathbb{C})$ from equation (9).

6. Unprojected mirror symmetry

6.1. Mirror duality

In this section, we review the mirror construction due to Berglund and Hübsch [5]. The mirror construction appeared first in the form given below in Krawitz [26]. Other key references are to Berglund and Henningson [4] for the group duality, Kreuzer and Skarke [27] for a systematic study and Borisov [7] for a reinterpretation of the setup and further generalizations in terms of vertex algebræ.

The Berglund–Hübsch mirror construction [5] applies to nondegenerate polynomials P of invertible type, i.e., having as many monomials as variables. Up to rescaling the variables, these polynomials are entirely encoded by the exponent matrix $E = (m_{i,j})$ and are paired to a second polynomial of the same type

$$P(x_0, \dots, x_n) = \sum_{i=0}^n \prod_{j=0}^n x_j^{m_{i,j}}, \quad P^\vee(x_0, \dots, x_n) = \sum_{i=0}^n \prod_{j=0}^n x_j^{m_{j,i}},$$

by transposing the matrix of exponents E .

Remark 26. The square matrix E is invertible because the vector $(w_i/d)_{i=0}^n$ is uniquely determined as a consequence of the nondegeneracy condition. The inverse matrix $E^{-1} = (m^{i,j})$ allows a simple description of Aut_P : The columns express the symmetries

$$\rho_j = [m^{0,j}, m^{1,j}, \dots, m^{n,j}]$$

spanning Aut_P . It is also easy to see that the columns of E express all the relations $\sum_i m_{i,j} \rho_i$ among these generators. Naturally, the rows of E^{-1} provide an expression for the symmetries $\bar{\rho}_i$ generating Aut_{P^\vee} under the relations provided by the rows $\sum_j m_{i,j} \bar{\rho}_j$ of E . In particular, we have a canonical isomorphism

$$\text{Aut}_{P^\vee} = (\text{Aut}_P)^*,$$

where $(G)^*$ denotes the Cartier dual $\text{Hom}(G, \mathbb{G}_m)$. The identification matches the symmetry $[q_0, \dots, q_n]$ to the homomorphism mapping ρ_i to $\exp(2\pi i q_i) \in \mathbb{Q}/\mathbb{Z}$. Based on this identification, for any subset $S \subseteq \text{Aut}_P$, we set $S^\vee \subseteq \text{Aut}_{P^\vee}$ as follows

$$S^\vee = \{\varphi \in (\text{Aut}_P)^* \mid \varphi|_S = 0\}.$$

This is a duality exchanging subgroups of Aut_P and subgroups of Aut_{P^\vee} ; for any group $H \subseteq \text{Aut}_P$, we can write

$$H^\vee = \ker(\text{Aut}_P \rightarrow \text{Hom}(H; \mathbb{G}_m)). \tag{19}$$

The dual of the group generated by J_P is SL_{P^\vee} . The duality reverses inclusions: If $H_1 \subset H_2$, then $H_2^\vee \subset H_1^\vee$.

The unprojected state space U_P is defined [26] as

$$U_P = \bigoplus_{h \in \text{Aut}_P} \text{Jac}(P_h)(-\text{age}(h)),$$

where the sum is over all diagonal symmetries and without taking any invariant [26]. For each summand $\text{Jac}(P_h)$, there exists an Aut_{P^\vee} -grading defined by

$$\ell_h : \text{Jac}(P_h) \rightarrow \text{Aut}_{P^\vee} \\ \prod_j x_j^{b_j-1} \bigwedge_j dx_j \mapsto \prod_j \bar{\rho}_j^{b_j}, \tag{20}$$

where all the products run over the set F_h of labels of variables fixed by h .

Lemma 27. *This grading is well-defined.*

Proof. It suffices to show that $\partial_i P_h$ is homogeneous under this grading. To do this, we show that each monomial in $\partial_i P_h$ is graded by $J_{P^\vee}(\bar{\rho}_i)^{-1}$. Note that each monomial appearing in $\partial_i P_h dx_i$ maps to the same automorphism as each monomial appearing in P . Furthermore, the automorphism obtained in this way is the identity by the relation $\sum_j m_{i,j} \bar{\rho}_j$ discussed above. We can finally conclude that $\partial_i P_h \bigwedge_j dx_j$ maps to the same automorphism as $\bigwedge_{j \neq i} dx_j$: namely, $\prod_{j \neq i} \bar{\rho}_j = J_{P^\vee}(\bar{\rho}_i)^{-1}$. \square

In this way, the unprojected state space admits a double decomposition

$$U_P = \bigoplus_{h \in \text{Aut}_P} \text{Jac}(P_h)(-\text{age}(h)) = \bigoplus_{h \in \text{Aut}_P} \bigoplus_{k \in \text{Aut}_{P^\vee}} U_h^k(P), \tag{21}$$

where $U_h^k(P)$ is the k -graded component of $\text{Jac}(P_h)(-\text{age}(h))$. We write

$$U_H^K(P) = \bigoplus_{h \in H} \bigoplus_{k \in K} U_h^k(P),$$

for any set $H \subseteq \text{Aut}_P$ and $K \subseteq \text{Aut}_{P^\vee}$. When a subscript H or a superscript K is omitted, we assume that H or K equal Aut_P or Aut_{P^\vee} . When no ambiguity may occur, we omit the polynomial P in the notation.

Proposition 28. *The vector space U_H^K is the K^\vee -invariant subspace of U_H*

$$U_H^K(P) = [U_H(P)]^{K^\vee}.$$

In particular, we have

$$\mathcal{H}_{P,H,g} = U_{gH}^{H^\vee}(P).$$

Proof. By reinterpreting the definition of K^\vee , we can see that, for any form f within $\text{Jac}(P_h)(-\text{age}(h)) \subseteq U_P$, we have $\ell_h(f) \in K$ if and only if f is invariant with respect to K^\vee . The claim follows immediately. \square

Theorem 29 (Krawitz [26], Borisov [7]). *For any $h \in \text{Aut}_P$ and $k \in \text{Aut}_{P^\vee}$, we have an explicit isomorphism*

$$U_h^k(P) \cong U_k^h(P^\vee),$$

yielding an explicit isomorphism

$$\text{Mirror}_P : U_P \longrightarrow U_{P^\vee}$$

mapping (p, q) -classes to $(n + 1 - p, q)$ classes.

We illustrate the isomorphism explicitly in the special case where $P = x^k$. It is elementary, and it plays a crucial role in this paper.

Example 30. Let $P = x^k$. Then Aut_P equals \mathbb{Z}_k (because we fix a primitive k th root ξ). For $1 \in \mu_k = \mathbb{Z}_k$, we have $P_1 = P$, so

$$\text{Jac}(P_1) = dx\mathbb{C}[x]/(x^{k-1}) = \sum_{h=1}^{k-1} U_1^{\xi^h}$$

with $U_1^{\xi^h}$ spanned by $x^{h-1}dx$. Furthermore, for $i \neq 0$, we have $P_{\xi^i} = 0$, so

$$\text{Jac}(P_{\xi^i}) = \mathbb{C} = U_{\xi^i}^1 \quad (\text{for } i = 1, \dots, k - 1).$$

By mapping $x^{i-1}dx \in \text{Jac}(P_1)$ to the generator 1_{ξ^i} of $\text{Jac}(P_{\xi^i})$, we have defined a map matching $(1 - \frac{i}{k}, \frac{i}{k})$ -classes to $(\frac{i}{k}, \frac{i}{k})$ -classes.

If P is a polynomial which can be expressed as the sum of two invertible and nondegenerate polynomials P' and P'' involving disjoint sets of variables, we clearly have $\text{Aut}_P = \text{Aut}_{P'} \times \text{Aut}_{P''}$. This and the theorem above imply the following crucial properties of the mirror map Mirror_P .

Thom–Sebastiani. If P is a polynomial which can be expressed as the sum of two invertible and nondegenerate polynomials

$$P = P'(x'_0, \dots, x'_{n_1}) + P''(x''_0, \dots, x''_{n_2})$$

involving two disjoint sets of variables, then we have

$$\text{Mirror}_P = \text{Mirror}_{P'} \otimes \text{Mirror}_{P''}.$$

Group actions. For any $H, K \subseteq \text{Aut}_P$, the restriction of Mirror_P yields an isomorphism

$$\text{Mirror}_P : [U_H(P)]^K \cong [U_{H^\vee}(P^\vee)]^{K^\vee}. \tag{22}$$

There are many consequences of the existence of Mirror_P and of its properties with respect to group actions. We list a few of them, starting from the first, most transparent, application. It appeared in [10], and it should be regarded as a combination of the mirror map Mirror_P of Krawitz and Borisov [26, 7] and of the LG/CY isomorphism Φ of the first named author with Ruan [10]. There the isomorphism is deduced by a combinatorial model. Here, we presented both sides of the correspondence in terms of relative orbifold cohomology, and we deduced the main theorem from K -equivalence between the ambient orbifolds. In this sense, the present setup clarifies [10].

Theorem 31 (mirror symmetry for CY models). *For any invertible, nondegenerate P of Calabi–Yau type and for any $H \subseteq \text{Aut}_P$ satisfying $j_P \in H \subseteq \text{SL}_P$, we have an isomorphism*

$$H_{\text{id}}^{p,q}(\Sigma_{P,H}; \mathbb{C}) \cong H_{\text{id}}^{n-1-p,q}(\Sigma_{P^\vee,H^\vee}; \mathbb{C}) \quad (p, q \in \mathbb{Q}).$$

Proof. Recall Definition 18:

$$\mathcal{H}_{P,H,\text{id}} = \bigoplus_{h \in H} (\text{Jac } P_h)^H(-\text{age}(h)).$$

We get

$$\mathcal{H}_{P,H,\text{id}} = [U_H(P)]^H \text{ and } \mathcal{H}_{P^\vee,H^\vee,\text{id}} = [U_{H^\vee}(P)]^{H^\vee}$$

Therefore, by equation (22), Mirror_P gives an isomorphism $\text{Mirror}_P(\mathcal{H}_{P,H,\text{id}}) \cong \mathcal{H}_{P^\vee,H^\vee,\text{id}}$. By assumption, P satisfies the Calabi–Yau condition. We also have that $H \ni j_P$ and $H^\vee \ni j_{P^\vee}$ —the latter

follows from the assumption that $H \subseteq \mathrm{SL}_P$. We can therefore apply Theorem 24 to both sides to obtain the claim, at least without the grading. The fact that a (p, q) -graded element is mapped to an $(n - 1 - p, q)$ -graded element follows from Theorem 29 and by the fact that the LG/CY correspondence in Theorem 24 preserves the bidegree. \square

The Thom–Sebastiani property applies to $P' = x_0^k$ and $P'' = f$ adding up to

$$W = x_0^k + f(x_1, \dots, x_n).$$

The aim of this paper is to study the relation between the above cohomological mirror symmetry and the symmetry $s = [\frac{1}{k}, 0, \dots, 0]$. Note that $s = \rho_0$ in the notation of Remark 26.

Proposition 32. *Let (ϕ, h) be a monomial element*

$$(\phi, h) = \left(\prod_{j=0}^n z_j^{b_j-1} \bigwedge_{j=0}^n dz_j \right)$$

in $\mathrm{Jac}(P_h^\vee)(-\mathrm{age}(h))$. Let $\xi = \exp(2\pi i/k)$. Then, $s^*(\phi, h) = \xi^i(\phi, h)$ if and only if $\mathrm{Mirror}_W(\phi, h)$ is of the form (ϕ', h') with $h' = [\frac{1}{k}, a_1, \dots, a_n]$. In particular, Mirror_W maps invariant elements to noninvariant elements.

Proof. This happens because s spans $\mathrm{Aut}_{x_0^k}$, whose dual group is trivial. The claim follows by $\mathrm{Mirror}_W = \mathrm{Mirror}_{x_0^k} \otimes \mathrm{Mirror}_f$ (see Example 30). \square

6.2. Unprojected states and automorphisms

We study the behaviour of U_W with respect to s . We begin by restricting to a conveniently large state space \mathbb{H}_W within U_W .

Let us consider $W = x_0^k + f(x_1, \dots, x_n)$ and, as in §3.2, a subgroup $K \subset \mathrm{Aut}_f$ satisfying

$$(j_f)^k \in K \subseteq \mathrm{SL}_f .$$

We define

$$\mathbb{H}_{K[j_W, s]}^K(W) = [U_{K[j_W, s]}(W)]^K = \bigoplus_{h \in K[j_W, s]} \mathrm{Jac}(P_h)^K(-\mathrm{age}(h)). \tag{23}$$

If no ambiguity arises, when the polynomial W and the group K are fixed, we write simply \mathbb{H} .

Remark 33. We now have two important state spaces: \mathbb{H} and $\mathcal{H}_{P, H, g}$. The description of $\mathcal{H}_{P, H, g}$ is given in Definition 18; it is the Landau–Ginzburg state space which matches with the state space on the Calabi–Yau side. \mathbb{H} , on the other hand, is a larger space (in particular containing \mathcal{H}) that allows us to see the full mirror symmetry isomorphism on the Landau–Ginzburg side.

In the above setup, we have three groups: K , $K[j_W]$ and $K[j_W, s]$. Only $K[j_W]$ satisfies the two conditions of mirror symmetry theorems: Namely, it contains j_W and is contained in SL_W . Its mirror group $K[j_W]^\vee$ has the same properties. The following proposition describes how K and $K[j_W, s]$ behave with respect to the group duality. We omit the proof as it is identical to the proof in Proposition 6.2.2 in [12].

Proposition 34. *Consider $K \in \mathrm{Aut}_f$ satisfying $(j_f)^k \in K \subseteq \mathrm{SL}_f$. Then we have*

$$(j_{f^\vee})^k \in (K[j_W, s])^\vee \subseteq \mathrm{SL}_{f^\vee} \quad \text{and} \quad K^\vee = (K[j_W, s])^\vee[j_W^\vee, s].$$

Furthermore, we have a mirror isomorphism

$$\text{Mirror}_W : \left(\mathbb{H}_{K[j_W, s]}^K(W) \right)^{p, q} = \left(\mathbb{H}_{K^\vee[j_W, s]^\vee}(W^\vee) \right)^{n+1-p, q}. \tag{24}$$

The unprojected state space projects to the sum of state spaces of the form $\mathcal{H}_{W, H, g}$ after taking j_W -invariant elements.

Corollary 35. *We have*

$$\mathbb{H}^{j_W} = \bigoplus_{b=0}^{k-1} \mathcal{H}_{W, K[j_W], s^b}.$$

In particular, if W is Calabi–Yau, we have

$$\mathbb{H}^{j_W} = \bigoplus_{b=0}^{k-1} H_{s^b}^*(\Sigma_{W, K[j_W]}; \mathbb{C}),$$

where $(\frac{b}{k} + p, \frac{b}{k} + q)$ -classes in $H_{s^b}^*(\Sigma_{W, K[j_W]}; \mathbb{C})$ match $(\frac{b}{k} + p, \frac{b}{k} + q)$ -classes in $\mathcal{H}_{W, K[j_W], s^b}$ for any $p, q \in \mathbb{Z}$.

Proof. Tracking through the definition,

$$\mathbb{H}_{K[j_W, s]}^K(W) = \bigoplus_{h \in K[j_W, s]} \text{Jac}(P_h)^{K[j_W]}(-\text{age}(h)),$$

which is precisely

$$\bigoplus_{b=0}^{k-1} \mathcal{H}_{W, K[j_W], s^b}.$$

Now we can apply equation (22) to obtain the claim. The statement about grading follows from the age shift. □

6.3. The twist and the elevators

Throughout this section, the polynomial W and the group K will be fixed; we simplify the notation and write

$$\mathbb{H} := \mathbb{H}_{K[j_W, s]}^K(W), \quad j := j_W, \quad \mathbb{H}^\vee := \mathbb{H}_{K^\vee[j_W, s]^\vee}^K(W^\vee), \quad j^\vee = j_{W^\vee}.$$

We also write M for the mirror map Mirror_W .

Note that the monomial element $(\phi, h) \in \mathbb{H}$ with

$$\phi = \prod_{i \in I} x_i^{b_i-1} \bigwedge_{i \in I} dx_i \quad \text{and} \quad I = \{i \mid h \cdot x_i = x_i\}$$

is an eigenvector with respect to the diagonal symmetry $\alpha = [p_0, \dots, p_n]$: The eigenvalue is $\exp(2\pi i \sum_j b_j p_j)$. It is natural to attach to each (ϕ, g) and α the so-called α -charge of the form ϕ defined on the g -fixed space:

$$Q_\alpha : (\phi, g) \mapsto Q_\alpha(\phi, g) = \sum_{j \in J} b_j p_j \pmod{\mathbb{Z}}.$$

Definition 36. We define several important \mathbb{Q}/\mathbb{Z} -valued gradings on \mathbb{H} . To do this, we decompose \mathbb{H} as

$$\mathbb{H} = \bigoplus_{a=0}^{k-1} \bigoplus_{b=0}^{k-1} \bigoplus_{g \in j^a s^b K} (\text{Jac } W_g)^K.$$

The four \mathbb{Q}/\mathbb{Z} -valued gradings on the set of generators

$$\left(\phi = \prod_{i \in I} x_i^{b_i-1} \bigwedge_{i \in I} dx_i, \quad g = [p_0, p_1, \dots, p_n] \in j^a s^b K \right)$$

are defined as

1. the *j*-charge $Q_j = Q_j: (\phi, g) \mapsto Q_j(\phi, g)$;
2. the *j*-degree $d_j = \frac{a}{k}$;
3. the *s*-charge $Q_s = Q_s: (\phi, g) \mapsto Q_s(\phi, g)$;
4. the *s*-degree $d_s = \frac{b}{k}$.

We can now decompose

$$\mathbb{H} = \bigoplus_{0 \leq a, b, c, d \leq k-1} \left[\mathbb{H} \mid (d_j, d_s, Q_j, Q_s) = \frac{1}{k}(a, b, c, d) \right].$$

The following proposition further simplifies the decomposition. By definition, the condition $Q_s = 0$ identifies the fixed space with respect to the action of *s*. The condition $Q_s \neq 0$ identifies the space spanned by the moving states. We make the following observation.

Proposition 37 (the moving subspace and the fixed subspace). *For any element (ϕ, g) , we have either (i) $d_s = -d_j \pmod{\mathbb{Z}}$, or (ii) $Q_s \neq 0$.*

Proof. This happens because $Q_s = 0$ if and only if $g \cdot x_0 = x_0$, i.e. $0 \in I$. By definition of d_s and d_j , we have $g \in j^{kd_j} s^{kd_s} K$ and $g \cdot x_0 = \exp(2\pi i(d_j + d_s))x_0$. We conclude that $Q_s = 0$ if and only if $d_s + d_j \in \mathbb{Z}$. □

In other words, \mathbb{H} decomposes into an *s*-moving part $\mathbb{H}^m (Q_s \neq 0)$ and an *s*-fixed part $\mathbb{H}^f (Q_s = 0)$

$$\mathbb{H} = \mathbb{H}^m \oplus \mathbb{H}^f = [\mathbb{H} \mid Q_s \neq 0] \oplus [\mathbb{H} \mid Q_s = 0],$$

and the first summand is $[\mathbb{H} \mid d_j + d_s = 0]$; hence, the three parameters d_j, Q_j, Q_s suffice for decomposing \mathbb{H}^m , and the three parameters d_j, d_s, Q_j suffice for decomposing \mathbb{H}^f . We write

$$\begin{aligned} \mathbb{H}^m &= \bigoplus_{\substack{0 \leq X, Y < k \\ 0 < Z < k}} \left[\mathbb{H} \mid (d_j, Q_s - Q_j, Q_s) = \frac{1}{k}(X, Y, Z) \right] = \bigoplus_{\substack{0 \leq X, Y < k \\ 0 < Z < k}} \mathbb{H}^m_{X, Y, Z}, \\ \mathbb{H}^f &= \bigoplus_{\substack{0 \leq X, Y < k \\ 0 < Z < k}} \left[\mathbb{H} \mid (d_j, -Q_j, d_j + d_s) = \frac{1}{k}(X, Y, Z) \right] = \bigoplus_{\substack{0 \leq X, Y < k \\ 0 < Z < k}} \mathbb{H}^f_{X, Y, Z}, \end{aligned}$$

where the choice of the three parameters in $\{0, \dots, k - 1\}$ modulo $k\mathbb{Z}$ are given by

$$X = kd_j, \quad Y = \begin{cases} k(Q_s - Q_j) & \text{in } \mathbb{H}^m \\ k(Q_s - Q_j) = -kQ_j & \text{in } \mathbb{H}^f \end{cases}, \quad Z = \begin{cases} kQ_s & \text{in } \mathbb{H}^m \\ k(d_j + d_s) & \text{in } \mathbb{H}^f \end{cases}. \tag{25}$$

Note that the definition of *Z* depends on the sector. These choices are motivated by the following proposition.

Proposition 38 (twist). For $Z = 1, \dots, k - 1$, we have an isomorphism

$$\begin{aligned} \tau : \mathbb{H}_{X,Y,Z}^m &\rightarrow \mathbb{H}_{X,Y,Z}^f \\ (x_0^{Z-1} dx_0 \wedge \phi, g) &\mapsto (\phi, s^Z g) \end{aligned}$$

transforming (p, q) -classes into $(p - 1 + 2Z/k, q)$ -classes.

Proof. Indeed, the above homomorphism exchanges $d_j + d_s$ with Q_s and preserves d_j and $Q_s - Q_j$. Note that τ only affects the x_0 part of W : The fact that τ is an isomorphism follows from the Thom–Sebastiani principle and the simple description of the state space for $P' = x_0^k$. \square

There are natural isomorphisms matching $\mathbb{H}_{X,Y,1}^f \cong \mathbb{H}_{X,Y,2}^f \cong \dots \cong \mathbb{H}_{X,Y,k-1}^f$ and $\mathbb{H}_{X,Y,1}^m \cong \mathbb{H}_{X,Y,2}^m \cong \dots \cong \mathbb{H}_{X,Y,k-1}^m$. We refer to them as ‘elevators’.

Proposition 39 (elevators). For any $0 \leq X, Y < k$ and $0 < Z' < Z'' < k$, we have the isomorphisms

$$\begin{aligned} e_{Z',Z''}^m : \mathbb{H}_{X,Y,Z'}^m &\rightarrow \mathbb{H}_{X,Y,Z''}^m & e_{Z',Z''}^f : \mathbb{H}_{X,Y,Z'}^f &\rightarrow \mathbb{H}_{X,Y,Z''}^f \\ (\phi, g) &\mapsto (x_0^{Z''-Z'} \phi, g) & (\phi, g) &\mapsto (\phi, s^{Z''-Z'} g), \end{aligned}$$

with $e_{Z',Z''}^m$ and $e_{Z',Z''}^f$ transforming (p, q) -classes into classes whose bidegrees equal $(p - (Z'' - Z')/k, q + (Z'' - Z')/k)$ and $(p + (Z'' - Z')/k, q + (Z'' - Z')/k)$, respectively. \square

For $0 < Z' < Z'' < k$, we set $e_{Z',Z'}^m = (e_{Z',Z''}^m)^{-1}$ and $e_{Z'',Z'}^f = (e_{Z',Z''}^f)^{-1}$.

Proposition 34 specializes to the following statement.

Proposition 40. The mirror isomorphism M yields isomorphisms

$$M : \mathbb{H}_{X,Y,Z}^f \xrightarrow{\sim} (\mathbb{H}_{Y,X,Z}^\vee)^m, \quad M : \mathbb{H}_{X,Y,Z}^m \xrightarrow{\sim} (\mathbb{H}_{Y,X,Z}^\vee)^f.$$

Proof. Let us consider M as a morphism mapping U_W to U_{W^\vee} . For $W = (x_0)^k + f$, we have $W^\vee = ((x_0)^k)^\vee + f^\vee = (x_0)^k + f^\vee$. Using equation (21) and Thom–Sebastiani, every state of the form $(\phi, g) \in U_W$ can be regarded as an element of

$$(\phi, g) \in U_{a_1}^{b_1} \otimes U_{a_2}^{b_2}$$

with $a_1 \in \text{Aut}_{(x_0)^k} = \mathbb{Z}/k$, $b_1 \in \text{Aut}_{((x_0)^k)^\vee} = \mathbb{Z}/k$, $a_2 \in \text{Aut}_f$ and $b_2 \in \text{Aut}_{f^\vee}$. Example 30 shows that there are only two possibilities: (1) b_1 is the identity element or (2) a_1 is the identity element. More precisely, in case (1), (ϕ, g) is in \mathbb{H}^f , it is fixed by s , b_1 is the trivial symmetry $1 \in \text{Aut}_{(x_0)^k}$ and a_1 is the nontrivial character corresponding to $kd_s \in \mathbb{Z}/k \setminus \{0\}$. In case (2), (ϕ, g) is in \mathbb{H}^m , it is not fixed by s and a_1 is trivial whereas b_1 is the nontrivial character $kQ_s \in \mathbb{Z}/k \setminus \{0\}$. Since M exchanges a_1 and b_1 , this proves that M exchanges \mathbb{H}^m and \mathbb{H}^f and preserves the coordinate Z which coincides with kd_s and kQ_s within \mathbb{H}^m and \mathbb{H}^f .

Furthermore, M maps $U_{a_2}^{b_2}(f)$ to $U_{b_2}^{a_2}(f^\vee)$ with $a_2 \in \text{SL}_f[j_f]$ and $b_2 \in \text{SL}_{f^\vee}[j_{f^\vee}]$. We recall that $j_f^k \in \text{SL}$ on both sides; therefore, $\det a_2$ and $\det b_2$ are μ_k -characters. The claim $(X, Y, Z) \mapsto (Y, X, Z)$ follows from

$$\det a_2 = -kd_j, \quad \det b_2 = -kQ_s + kQ_j,$$

where μ_k -characters are identified with elements of \mathbb{Z}/k . The first identity is immediate: a_2 is related to $(\phi, g) \in U_W$ by $a_2 = g|_{x_0=0}$. The identity follows from $\det(j|_{x_0=0}) = \xi_k^{-1}$ by the Calabi–Yau condition. The second identity follows from the definition of

$$\ell_{a_2} : \text{Jac}(f_{a_2}) \rightarrow \text{Aut}(f^\vee), \quad \prod_j x_j^{b_j-1} \bigwedge_j dx_j \mapsto \prod_j \bar{\rho}_j^{b_j}$$

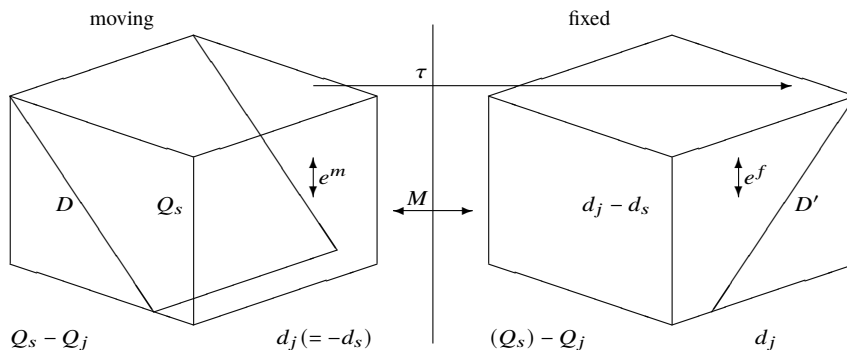


Figure 1. Two blocks (with elevators) representing the coordinates of the moving subspace \mathbb{H}^m and of the fixed subspace \mathbb{H}^f . The condition $Q_j = 0$ defines a plane cutting the diagonal D of the left-hand side face of the moving block; D represents the moving part of $H_{\text{id}}^*(\Sigma_{W,H}; \mathbb{C})$. On the fixed block, the same condition $Q_j = 0$ defines the face on the right-hand side; within it, the diagonal D' is symmetrical to D and represents the fixed part of $H_{\text{id}}^*(\Sigma_{W,H}; \mathbb{C})$.

from equation (20): The determinant of $\bar{\rho}_j$ is $\xi_d^{w_j}$; hence, $\det(\prod_j \bar{\rho}_j^{b_j})$ is identified with the j_f -charge Q_{j_f} of the form ϕ restricted to $(x_0 = 0)$. This yields an identification between $\det b_2$ and the μ_k -character $kQ_j - kQ_s$. \square

In view of the above proposition, mirror symmetry operates as a plane symmetry exchanging the two blocks; see Figure 1.

Remark 41. For Fermat potentials, all the above discussion can be carried out more explicitly because the group elements $\mathbf{a} = \prod_{i=1}^n \rho_i^{a_i}$ coincide with $\frac{1}{d}[a_1 w_1, \dots, a_n w_n]$. By adopting this notation, the space $U_a^b(W)$ may be regarded as the one-dimensional space spanned by

$$\left(\phi = \prod_{b_i > 0} x_i^{b_i - 1} \bigwedge_{b_i > 0} dx_i, \mathbf{a} = \prod_{i=0}^n \rho_i^{a_i} \right),$$

where

$$a_i = 0 \Leftrightarrow b_i \neq 0. \tag{26}$$

Mirror symmetry is simply an exchange of the $w\mathbb{Z}/d\mathbb{Z}$ -valued vectors \mathbf{a} and \mathbf{b} . By unravelling Definition 23, the bidegree (p, q) is given by

$$\left(\#(\mathbf{b}) - \sum_{i=0}^n b_i \frac{w_i}{d} + \sum_{i=0}^n a_i \frac{w_i}{d}, \sum_{i=0}^n b_i \frac{w_i}{d} + \sum_{i=0}^n a_i \frac{w_i}{d} \right),$$

where $\#(\mathbf{b})$ is the number of elements i such that $b_i \neq 0$. Notice that Q_s is b_0/k and $d_s + d_j$ is a_0/k ; therefore, the equivalence in Proposition 37 reads $b_0 = 0$ is a special case of equation (26). Furthermore, we have

$$(X, Y, Z) = \begin{cases} (a_0 - |\mathbf{a}|, b_0 - |\mathbf{b}|, b_0) & \text{on the moving side,} \\ (a_0 - |\mathbf{a}|, b_0 - |\mathbf{b}|, a_0) & \text{on the fixed side.} \end{cases}$$

It is now clear that M exchanges the moving side with the fixed side, (X, Y, Z) with (Y, X, Z) and (p, q) with $(n + 1 - p, q)$.

In view of Proposition 35, we obtain the j -invariant contribution by setting $Q_j = 0$. By equation (25), this amounts to imposing $Y = Z$ within \mathbb{H}^m and $Y = 0$ within \mathbb{H}^f . We get

$$\left(\bigoplus_{\substack{0 \leq X < k \\ 0 < Z < k}} \mathbb{H}^m_{X,Z,Z} \right) \oplus \left(\bigoplus_{\substack{0 \leq X < k \\ 0 < Z < k}} \mathbb{H}^f_{X,0,Z} \right)$$

We get a picture of the j -invariant state space $\mathcal{H}_{W,K[jw],id}$ by setting $X = 0$ within \mathbb{H}^m and $X = Z$ within \mathbb{H}^f

$$\mathcal{H}_{W,K[jw],id} = \left(\bigoplus_{0 < t < k} \mathbb{H}^m_{0,t,t} \right) \oplus \left(\bigoplus_{0 < t < k} \mathbb{H}^f_{t,0,t} \right)$$

(we refer to Figure 1). More generally, the $(d_s = b)$ -part of \mathbb{H}^{jw} is the state space $\mathcal{H}_{W,K[jw],s^b}$ (see Proposition 35). By equation (25), we obtain it by setting $X = -b$ within \mathbb{H}^m and $Z = X + b$ within \mathbb{H}^f

$$\mathcal{H}_{W,K[jw],s^b} = \left(\bigoplus_{0 < t < k} \mathbb{H}^m_{-b,t,t} \right) \oplus \left(\bigoplus_{\substack{0 \leq t < k \\ t+b \neq 0}} \mathbb{H}^f_{t,0,t+b} \right). \tag{27}$$

Notice that the second summand only depends on $\mathbb{H}^f_{0,0,1}$ and $\mathbb{H}^f_{1,0,1}, \dots, \mathbb{H}^f_{k-1,0,k-1}$ since, for $b \neq 0$, it equals

$$\mathcal{H}_{W,K[jw],s^b} = \left(\bigoplus_{0 < t < k} \mathbb{H}^m_{-b,t,t} \right) \oplus \left(e^f_{1,b}(\mathbb{H}^f_{0,0,1}) \oplus \bigoplus_{\substack{0 < t < k \\ t+b \neq 0}} e^f_{t,t+b}(\mathbb{H}^f_{t,0,t}) \right), \tag{28}$$

with the convention $(e^f_{i,j}) = (e^f_{j,i})^{-1}$ if $j < i$. By Proposition 35, the above data correspond to $H^*_{s^b}(\Sigma_{W,K[jw]}; \mathbb{C})$ under the Calabi–Yau condition.

Proposition 9 relates it to the cohomology of an s^b -fixed locus within a crepant resolution. Using this geometric picture, we can predict some vanishing conditions, which we prove in general, without relying on any Calabi–Yau condition in the next proposition. The first guess is immediate: Since $\langle s \rangle$ operates trivially on an s -fixed locus, it is natural to expect that $\mathbb{H}^m_{-1,t,t}$ vanishes for all t . More generally, since $\langle s^b \rangle$ operates trivially on an s^b -fixed locus, we expect that $\mathbb{H}^m_{-b,t,t}$ vanishes if $tb \equiv 0 \pmod{\mathbb{Z}}$. We prove that this holds true regardless of any Calabi–Yau condition or existence of crepant resolution.

Proposition 42. *Let $b \in \{0, \dots, k - 1\}$. We have*

$$\mathbb{H}^m_{b,t,t} = 0,$$

unless t is a multiple of $k/\gcd(b, k)$ in $k\mathbb{Z}$.

Proof. We prove that

$$\mathbb{H}^m_{b,t,t} \neq 0$$

implies $bt \in k\mathbb{Z}$. Recall that $\frac{t}{k}$ equals $Q_s - Q_j$. For $\mathbb{H}_{b,t,t}^m \neq 0$, we can compute $Q_s - Q_j$ explicitly using $(\phi, j^b g) \in \mathbb{H}_{b,t,t}^m$ with $g \in \text{Aut}_f$ and ϕ a g -invariant form

$$\phi = x_0^{kQ_s-1} \prod_{l \in I'} x^{b_l-1} dx_0 \wedge \bigwedge_{l \in I'} dx_l$$

with $I' = \{l \geq 1 \mid j^b g \cdot x_l = x_l\} \subseteq \{1, \dots, N\}$. Using $w_0/d = 1/k$, we get

$$Q_s - Q_j = Q_s - kQ_s \frac{1}{k} - \sum_{l \in I'} b_l \frac{w_l}{d} = - \sum_{l \in I'} b_l \frac{w_l}{d}.$$

Let us write g as $[0, p_1 \dots, p_n] \in \text{Aut}_f$; then we have $l \in I'$ if and only if

$$b \frac{w_l}{d} + p_l \in \mathbb{Z}.$$

Now note that

$$b \frac{t}{k} = -b \sum_{l \in I} b_l \frac{w_l}{d} = - \sum_{l \in I} b_l b \frac{w_l}{d}.$$

Up to an element of \mathbb{Z} , this is

$$\sum_{l \in I} b_l p_l \in \mathbb{Z},$$

where the last relation holds since ϕ is g -invariant. So k divides bt . □

6.4. Mirror symmetry on the Landau–Ginzburg side

In this section, we derive an interpretation of Proposition 40 in terms of the Landau–Ginzburg state space. This amounts to expressing both sides of the isomorphism $\mathbb{H}_{X,Y,Z}^m \cong \mathbb{H}_{Y,X,Z}^f$ in terms of j_W -invariant spaces.

Consider the j_W -invariant summands

$$\mathbb{H}_{X,Y,Y}^m \subset \mathbb{H}^m \text{ and } \mathbb{H}_{X,0,Z}^f \subset \mathbb{H}^f.$$

Their mirrors (i.e., their image under M) are $(\mathbb{H}^\vee)_{Y,X,Y}^f$ and $(\mathbb{H}^\vee)_{0,X,Z}^m$ and lie in the j_W -invariant part if and only if $X = 0$ and $X = Z$. This happens if and only if we consider the mirror of $[\mathbb{H} \mid Q_j = d_s = 0]$ (imposing $X = 0$ in \mathbb{H}^m and $X = Z$ in \mathbb{H}^f is the same as requiring $d_s = 0$).

We obtain the first consequence of Proposition 40. Let

$$[\mathcal{H}_{W,H,\text{id}}]_{\chi_s=i}^{p,q}$$

be the eigenspace on which s operates as the character $i \in \mathbb{Z}/k\mathbb{Z}$. For any $H \in \text{Aut}_W$ containing j_W , we have

$$M : [\mathcal{H}_{W,H,\text{id}}]_{\chi_s=0}^{p,q} \xrightarrow{\cong} \bigoplus_{i=1}^{k-1} [\mathcal{H}_{W^\vee,H^\vee,\text{id}}]_{\chi_s=i}^{n+1-p,q}, \tag{29}$$

where $H = K[j]$.

We now study $\mathbb{H}_{0,0,i}^f$. By Proposition 40, the subspace $(\mathbb{H}^\vee)_{0,0,i}^f$ mirrors $\mathbb{H}_{0,0,i}^m$ under M . By applying the twist τ from Proposition 38, we land again on $(\mathbb{H}^\vee)_{0,0,i}^f$ which is a part of the j -invariant state space $\mathcal{H}_{W^\vee, H^\vee, s^i}$.

Recall that the elevator maps for $t \neq 0, k - i$

$$e_{t,t+i}^f : \mathbb{H}_{t,0,t}^f \longrightarrow \mathbb{H}_{t,0,t+i}^f$$

can be viewed as mapping into components of \mathcal{H}_{W,H,s^i}^s (see equation (27)). We can conclude from equation (28) that the homomorphism

$$el_i := \bigoplus_{t \neq 0, k-i} e_{t,t+i}^f : \bigoplus_{t \neq 0, k-i} \mathbb{H}_{t,0,t}^f \longrightarrow [\mathcal{H}_{W,H,s^i}]^s \tag{30}$$

has cokernel $\mathbb{H}_{0,0,i}^f$. Here $t + i$ is understood to be mod k . This means that the effect of the map on grading can be described as

$$\text{im}(el_i)\left(\frac{i}{k}\right) \cong \bigoplus_{0 < t < k-i} \mathbb{H}_{t,0,t}^f \oplus \bigoplus_{k-i < t < k} \mathbb{H}_{t,0,t}^f(1).$$

We write el_i^\vee for the same construction on the mirror. We conclude

$$M : \left[\frac{\mathcal{H}_{W,H,s^i}\left(\frac{i}{k}\right)^s}{\text{im}(el_i)\left(\frac{i}{k}\right)} \right]^{p,q} \xrightarrow{\cong} \left[\frac{\mathcal{H}_{W^\vee, H^\vee, s^i}\left(\frac{i}{k}\right)^s}{\text{im}(el_i^\vee)\left(\frac{i}{k}\right)} \right]^{n-p,q}, \tag{31}$$

where the bidegrees have been computed using $\mathcal{H}\left(\frac{i}{k}\right)^{p,q} = \mathcal{H}^{p+i/k, q+i/k}$, the fact that M_W transforms (p, q) -classes to $(n + 1 - p, q)$ -classes and the twist τ maps (p, q) -classes to $(p - 1 + \frac{2i}{k}, q)$ -classes.

Note that using equation (29) (recalling that d_j and d_s switch under mirror symmetry) we can write

$$[\text{im}(el_i^\vee)\left(\frac{i}{k}\right)]^{n-p,q} = \bigoplus_{0 < j < k-i} [\mathcal{H}_{W,H,\text{id}}(1, 0)]_{\chi_s=j}^{p,q} \oplus \bigoplus_{k-i < j < k} [\mathcal{H}_{W,H,\text{id}}(0, 1)]_{\chi_s=j}^{p,q}.$$

Write \mathcal{H}_g for $\mathcal{H}_{W,H,g}$ and \mathcal{H}_g^\vee for $\mathcal{H}_{W^\vee, H^\vee, g}$. We obtain

$$M : [\mathcal{H}_{s^i}\left(\frac{i}{k}\right)]_{\chi_s=0}^{p,q} \oplus \bigoplus_{j < k-i} [\mathcal{H}_{\text{id}}(1, 0)]_{\chi_s=j}^{p,q} \oplus \bigoplus_{j > k-i} [\mathcal{H}_{\text{id}}(0, 1)]_{\chi_s=j}^{p,q} \xrightarrow{\cong} [\mathcal{H}_{s^i}^\vee\left(\frac{i}{k}\right)]_{\chi_s=0}^{n-p,q} \oplus \bigoplus_{j < k-i} [\mathcal{H}_{\text{id}}^\vee(1, 0)]_{\chi_s=j}^{n-p,q} \oplus \bigoplus_{j > k-i} [\mathcal{H}_{\text{id}}^\vee(0, 1)]_{\chi_s=j}^{n-p,q}, \tag{32}$$

where $0 < j < k$. Finally, we focus on the moving part of $\mathcal{H}_{W,K[jW],s^b}$ which, by equation (28) can be written as $\bigoplus_{0 < t < k} \mathbb{H}_{k-b,t,t}^m$. This is the decomposition of $\mathcal{H}_{W,K[j],s^b}$ into eigenspaces corresponding to the s -action operating as the character $t \in \mathbb{Z}/k$. By applying the mirror map M_W , the twist τ^{-1} and the elevator $e_{t,k-b}^f$ we get

$$\mathbb{H}_{k-b,t,t}^m \xrightarrow{M_W} (\mathbb{H}^\vee)_{t,k-b,t}^f \xrightarrow{\tau^{-1}} (\mathbb{H}^\vee)_{t,k-b,t}^m \xrightarrow{e_{t,k-b}^m} (\mathbb{H}^\vee)_{t,k-b,k-b}^m.$$

Therefore, we have

$$\left[\mathcal{H}_{W,H,s^b}\left(\frac{b}{k}\right) \right]_{\chi_s=t}^{p,q} \cong \left[\mathcal{H}_{W^\vee, H^\vee, s^t}\left(\frac{k-t}{k}\right) \right]_{\chi_s=k-b}^{n-p,q}. \tag{33}$$

Notice that the map on the bidegrees is the composite of

1. a shift $(p, q) \mapsto (p + b/k, q + b/k)$,
2. mirror symmetry $(p, q) \mapsto (n + 1 - p, q)$,
3. τ^{-1} yielding $(p, q) \mapsto (p + 1 - 2t/k, q)$,
4. the elevator yielding $(p, q) \mapsto (p + (b - (k - t))/k, q + (b + (k - t))/k)$,
5. a shift backwards $(p, q) \mapsto (p - (k - t)/k, q - (k - t)/k)$,

inducing $(p, q) \mapsto (n - p, q)$.

We now are almost ready to state the theorem, which will follow from applying equations (29), (32) and (33) to the geometric interpretation (18) of the Landau–Ginzburg state space in terms of relative cohomology of $(\mathbb{V}_H, \mathbb{F}_{W,H})$ (provided in §5.3).

Let $W = x_0^k + f(x_1, \dots, x_n)$ be a quasi-homogeneous nondegenerate polynomial of degree d and weights w_1, \dots, w_n . Assume $j_W \in H \subseteq \text{SL}_W$ (in particular, $\sum_j w_j$ is a positive multiple of $d\mathbb{N}$). Then W descends to $\mathbb{V}_H = [\mathbb{V}/H_0] \rightarrow \mathbb{C}$ and its generic fibre is $\mathbb{F}_{W,H}$. Consider the automorphism $s = [\frac{1}{k}, 0, \dots, 0]: \mathbb{V}_H \rightarrow \mathbb{V}_H$, the orbifold cohomology groups $H_{\text{id}}^*(\mathbb{V}_H, \mathbb{F}_{W,H})$ and $H_s^*(\mathbb{V}_H, \mathbb{F}_{W,H})$.

For $0 < i < k$, define $\overline{H}_{\text{id},i}^*(\mathbb{V}_H, \mathbb{F}_{W,H})$ to be the bigraded vector space

$$\bigoplus_{j < k-i} [H_{\text{id}}^*(\mathbb{V}_H, \mathbb{F}_{W,H})(1, 0)]_{\chi_s=j} \oplus \bigoplus_{j > k-i} [H_{\text{id}}^*(\mathbb{V}_H, \mathbb{F}_{W,H})(0, 1)]_{\chi_s=j};$$

here $j \in \{1, \dots, k - 1\}$. This is the padding needed to state the mirror theorem.

Theorem 43 (mirror theorem for Landau–Ginzburg models). *Let $W = x_0^k + f(x_1, \dots, x_n)$ be a quasi-homogeneous, nondegenerate, invertible polynomial and H a group of symmetries satisfying $j_W \in H \subseteq \text{SL}_W$. As above, the polynomial W descends to $\mathbb{V}_H = [\mathbb{V}/H_0] \rightarrow \mathbb{C}$, and its generic fibre is $\mathbb{F}_{W,H}$.*

Then, for b and $t \neq 0$, we have

1. $H_{\text{id}}^{p,q}(\mathbb{V}_H, \mathbb{F}_{W,H})_{\chi_s=0} \cong \bigoplus_{i=1}^{k-1} H_{\text{id}}^{n+1-p,q}(\mathbb{V}_{H^\vee}, \mathbb{F}_{W^\vee, H^\vee})_{\chi_s=i}$;
2. Let $\mathbb{F} := \mathbb{F}_{W,H}$ and $\mathbb{F}^\vee := \mathbb{F}_{W^\vee, H^\vee}$.

For $0 < i < k$,

$$\left[H_{s^i}^{p,q}(\mathbb{V}_H, \mathbb{F})\left(\frac{i}{k}\right) \right]^s \oplus \overline{H}_{\text{id},i}^{p,q}(\mathbb{V}_H, \mathbb{F}) \cong \left[H_{s^i}^{n-p,q}(\mathbb{V}_{H^\vee}, \mathbb{F}^\vee)\left(\frac{i}{k}\right) \right]^s \oplus \overline{H}_{\text{id},i}^{n-p,q}(\mathbb{V}_{H^\vee}, \mathbb{F}^\vee);$$

$$3. H_{s^b}^{p,q}(\mathbb{V}_H, \mathbb{F}_{W,H})\left(\frac{b}{k}\right)_{\chi_s=t} \cong H_{s^{-t}}^{n-p,q}(\mathbb{V}_{H^\vee}, \mathbb{F}_{W^\vee, H^\vee})\left(\frac{k-t}{k}\right)_{\chi_s=-b}.$$

Proof. Since H equals $K[j_W]$ for a suitable $K \subseteq \text{SL}_f$ containing j_W^k , we can conclude that $\mathcal{H}_{P,H,g}^{p,q} = H_g^{p,q}(\mathbb{V}_H; \mathbb{F}_{P,H})$. Using this result, we can translate the results derived above into the form stated in the theorem. More precisely, equation (29) directly gives the first part, equation (32) the second and equation (33) the third. □

7. Geometric mirror symmetry

If W is of Calabi–Yau type, via the Landau–Ginzburg/Calabi–Yau correspondence of Theorem 24 based on $\Phi: H^*(\mathbb{V}; \mathbb{C}) \rightarrow H^*(\mathbb{L}; \mathbb{C})$, we provide an equivalent statement on the Calabi–Yau side.

The existence of the isomorphism Φ is guaranteed by the Calabi–Yau condition (ensuring K -equivalence). As before, for $0 < i < k$, define $\overline{H}_{\text{id},i}^*(\Sigma_{W,H})$ to be the bigraded vector space

$$\bigoplus_{j < k-i} [H_{\text{id}}^*(\Sigma_{W,H})(1, 0)]_{\chi_s=j} \oplus \bigoplus_{j > k-i} [H_{\text{id}}^*(\Sigma_{W,H})(0, 1)]_{\chi_s=j}, \tag{34}$$

where again, j runs between 1 and $k - 1$. Then we have the following statement.

Theorem 44 (mirror theorem for CY orbifolds with automorphism s). Let $W = x_0^k + f(x_1, \dots, x_n)$ be a quasi-homogeneous, nondegenerate, invertible, Calabi–Yau polynomial and H a group of symmetries satisfying $j_W \in H \subseteq \text{SL}_W$. Let $\Sigma = \Sigma_{W,H}$ and $\Sigma^\vee = \Sigma_{W^\vee, H^\vee}$. Then the following holds for $b, t \neq 0$.

1. Let $d = n - 1$. Then $H_{\text{id}}^{p,q}(\Sigma)_{\chi_s=0} \cong \bigoplus_{i=1}^{k-1} H_{\text{id}}^{d-p,q}(\Sigma^\vee)_{\chi_s=i}$;
2. Let $d = n - 2$. For $0 < i < k$,

$$\left[H_{s^i}^{p,q}(\Sigma) \left(\frac{i}{k} \right) \right]^s \oplus \overline{H}_{\text{id},i}^{p,q}(\Sigma) \cong \left[H_{s^i}^{d-p,q}(\Sigma^\vee) \left(\frac{i}{k} \right) \right]^s \oplus \overline{H}_{\text{id},i}^{d-p,q}(\Sigma^\vee);$$

3. Let $d = n - 2$. Then $H_{s^b}^{p,q}(\Sigma)_{\chi_s=t} \left(\frac{b}{k} \right) \cong H_{s^{-t}}^{d-p,q}(\Sigma^\vee) \left(\frac{k-t}{k} \right)_{\chi_s=-b}$.

Proof. This follows immediately from Theorem 43 and the LG/CY correspondence (Theorem 24). \square

Remark 45. In the theorem, d denotes the maximum of the dimensions of the components of the inertia stack considered in each case.

For $k = 2$, the second equation of the statement of Theorem 44 can be stated as a mirror symmetry statement involving the cohomology groups H_s^* . Notice that the first statement says that Berglund–Hübsch mirror symmetry exchanges invariant (p, q) -classes for $\Sigma_{W,H}$ and anti-invariant $(n - 1 - p, q)$ -classes of Σ_{W^\vee, H^\vee} (and vice versa). Finally, the third statement is trivial because both sides vanish by Proposition 42. In this way, we recover the main theorem of [12].

Corollary 46. Let $W = x_0^2 + f(x_1, \dots, x_n)$ be a quasi-homogeneous, nondegenerate, invertible, Calabi–Yau polynomial and H a group of symmetries satisfying $j_W \in H \subseteq \text{SL}_W$. Then, we have

$$H_{\text{id}}^{p,q}(\Sigma_{W,H})^\pm \cong H_{\text{id}}^{n-1-p,q}(\Sigma_{W^\vee, H^\vee})^\mp;$$

$$H_s^{p,q}(\Sigma_{W,H}) \left(\frac{1}{2} \right) \cong H_s^{n-2-p,q}(\Sigma_{W^\vee, H^\vee}) \left(\frac{1}{2} \right).$$

Example 47. Let us consider $E = (x^6 + y^3 + z^2 = 0)$ within $\mathbb{P}(1, 2, 3)$ with its order-6 symmetry $s = [\frac{1}{6}, 0, 0]$. In this case, the ‘Calabi–Yau orbifold’ is represented by an elliptic curve. The cohomology groups $H_{s^b}^*$ describe the cohomology of the s^b -fixed loci E_b , shifted by $(\frac{b}{6}, \frac{b}{6})$. Furthermore, the mirror of E coincides with E because the defining equation is of Fermat type and J equals SL (the order of SL is $w_x w_y w_z / \text{deg}$ and equals the order deg of J). This example allows us to test \mathbb{H} as a state space computing the cohomology of E and the cohomology of its fixed spaces satisfying $E_1 = E_2 \cap E_3$ and $E_2 = E_4$. Since E is the elliptic curve with order-6 complex multiplication, E_1 is the origin and the fixed spaces $E_3 (= E[2])$ and E_2 are, respectively, a set of four points and three points intersecting at the origin. Clearly, $E_3 \setminus E_1$ is the unique order-3 orbit and $E_2 \setminus E_1$ is the unique order-2 orbit.

The b th row in the table below represents the ranks of contributions of $\mathbb{H}[d_s = \frac{b}{6}]$, whereas the a th column represents the contributions to the state space of $\mathbb{H}[d_j = \frac{a}{6}]$. Notice that, by means of the elevators, all rows are identical except for the antidiagonal entries of the form $\mathbb{H}[d_j + d_s = 0]$, which we underlined.

$\dim(H_{\text{id}}) =$	2	1	0	0	0	1
$\dim(H_s) =$	0	1	0	0	0	0
$\dim(H_{s^2}) =$	0	1	0	0	1	1
$\dim(H_{s^3}) =$	0	1	0	2	0	1
$\dim(H_{s^4}) =$	0	1	1	0	0	1
$\dim(H_{s^5}) =$	0	0	0	0	0	1

The 0th row is the four-dimensional cohomology of the elliptic curve E organised in its two-dimensional primitive part (spanned by the forms $dx \wedge dy \wedge dz$ and $x^4 y dx \wedge dy \wedge dz$) and its two-dimensional ambient part arising in the state space $\mathbb{H}[d_s = 0, d_j = a/6]$ for $a = 1$ and $a = 5$ (j and j^5

		0	0	0	1							
$\dim(H_{\text{id}}) =$	1	0+1+1	1	0	6	0	0	7	0	0	6	0
		0		1		0		0		0		0
$\dim(H_s) =$		$\binom{3}{0}$	0	$\binom{6}{0}$	0	$\binom{7}{0}$	0	$\binom{0}{0}$	0	$\binom{0}{0}$	0	$\binom{0}{0}$
		$\binom{3}{3}$		1		0		0		0		0
$\dim(H_{s^2}) =$		$\binom{3}{0}$	0	$\binom{6}{0}$	0	$\binom{0}{0}$	0	$\binom{1}{0}$	0	$\binom{0}{0}$	0	$\binom{1}{0}$
		$\binom{3}{3}$		1		0		0		0		6
$\dim(H_{s^3}) =$		$\binom{3}{0}$	0	$\binom{0}{0}$	0	$\binom{0}{0}$	0	$\binom{1}{0}$	0	$\binom{0}{0}$	0	$\binom{1}{0}$
		$\binom{3}{3}$		0		0		7		0		6

From this, we see that the mirror s -fixed locus is four projective curves and 12 isolated fixed points. The mirror Hodge diamond for statement (2) is

$$\begin{array}{c}
 16 \\
 0+1 \quad 0+1 \\
 4
 \end{array}$$

Note that despite being an order-2 automorphism, the Hodge diamonds of the fixed loci of s^2 do not mirror each other.

Clearly, mirror symmetry should also yield a relation between the quantum invariants of the primitive classes of the curve and the orbifold quantum invariants of these sectors.

The structure of the of the above example is shared by all K3 orbifolds of this type with order-4 automorphism. Combining the mirror theorem with the fact that s and s^3 have the same fixed locus (and hence cohomology of the same dimension), we can see that, for any $W = x_0^4 + f(x_1, x_2, x_3)$ and group G , the table for $\Sigma_{W,G}$ is given by

		0	0	0	1							
$\dim(H_{\text{id}}) =$	1	$a-1+b+a-1$	1	0	$a'-1$	0	0	b'	0	0	$a'-1$	0
		0		1		0		0		0		0
$\dim(H_s) =$		$\binom{g}{g}$	$\binom{g}{g}$	$\binom{a'-1}{0}$	0	$\binom{b'}{0}$	0	$\binom{0}{0}$	0	$\binom{0}{0}$	0	$\binom{0}{0}$
		$\binom{g}{g}$	$\binom{g}{g}$	1		0		0		0		0
$\dim(H_{s^2}) =$		$\binom{g}{g}$	$\binom{g}{g}$	$\binom{a'-1}{0}$	0	$\binom{c}{c}$	$\binom{c'}{c}$	$\binom{1}{0}$	0	$\binom{0}{0}$	0	$\binom{1}{0}$
		$\binom{g}{g}$	$\binom{g}{g}$	1		$\binom{c'}{c}$	$\binom{c}{c}$	$\binom{0}{0}$		0		$\binom{0}{0}$
$\dim(H_{s^3}) =$		$\binom{g}{g}$	$\binom{g}{g}$	0	0	$\binom{0}{0}$	$\binom{0}{b'}$	$\binom{1}{0}$	0	$\binom{0}{0}$	0	$\binom{1}{0}$
		$\binom{g}{g}$	$\binom{g}{g}$	0		0		$\binom{0}{a'-1}$		0		$\binom{0}{a'-1}$

The table for the Σ_{W^\vee, G^\vee} is obtained from this table by replacing $x \mapsto x^\vee$.

Using this table, we can find relationships between the topological invariants of the fixed loci of crepant resolutions of $\mathfrak{X} := \Sigma_{W,G}$ and its mirror \mathfrak{X}^\vee . Example 11 shows that there is an isomorphism between the s^3 -orbifold cohomology of \mathfrak{X} and the cohomology of the s fixed locus in the resolution \tilde{X} . Recall that this is because for K3 surfaces, the age function is constant (of $3/4$) on the s^3 -orbifold cohomology of the resolution. By similar reasoning, the s^2 -orbifold cohomology of \tilde{X} also has a constant age function (of $1/2$).

Now consider the following invariants for $i = 1, 2$:

- f_i , the number of isolated fixed points of s^i ;
- g_i , the sum of the genera of the fixed curves of s^i ;
- N_i , the number of curves in the fixed locus of s^i .

A superscript \vee indicates the invariants of the mirror K3.

Corollary 49. *We have*

1. $N_1 = g_1^\vee + 1$;
2. $N_2 + g_2 + f_1 = 20 - N_2^\vee - g_2^\vee - f_1^\vee$.

Proof. The table above implies that $2a + b + 2a^\vee + b^\vee = 24$ and that $g_1 = g$, $N_1 = g^\vee + 1$, and $f_1 = a^\vee + b^\vee - 2$. The statements for the mirror invariants are obtained by $x \leftrightarrow x^\vee$: For example, $N_1^\vee = g + 1$.

Similarly, $N_2 - g_2^\vee = (g^\vee + c + a^\vee) - (g^\vee + c) = a^\vee$, which implies the statement. □

The same analysis also works for K3 surfaces with prime order automorphisms. Let W be a Calabi–Yau polynomial of the form $W = x_0^p + f(x_1, x_2, x_3)$ for p prime. Then the Landau–Ginzburg state space breaks down as

	0	0	0	1
$\dim(H_{\text{id}}) =$	1	$(p-1)a-2$	1	0
	0	$a^\vee-1$	0	0
	0	1	\dots	0
$\dim(H_s) =$	g	g^\vee	g	0
	g^\vee	0	$a^\vee-1$	0
	g^\vee	1	\dots	0
$\dim(H_{s^2}) =$	g	g^\vee	g	0
	g^\vee	0	$a^\vee-1$	0
	g^\vee	0	0	0
	\vdots	\vdots	\vdots	0
$\dim(H_{s^{p-1}}) =$	g	g^\vee	g	0
	g^\vee	0	0	0
	g^\vee	0	a^\vee	0

The following lemma follows immediately from considering this table.

Lemma 50. *Suppose $\Sigma_{W,H}$ is a K3 orbifold with $W = x_0^p + f(x_1, x_2, x_3)$. Then $p - 1 \mid 24$.*

Let \tilde{X} be a crepant resolution of $\mathfrak{X} = \Sigma(W, G)$ and \tilde{X}^\vee a crepant resolution of the mirror. The fixed locus of s is a disjoint union of curves and isolated fixed points. As before, let f_1 be the number of isolated fixed points, N_1 the number of curves, and g_1 the sum of the genera of the curves.

Corollary 51. *Suppose $p > 2$. Then $N_1 = g_1^\vee + 1$ and*

$$f_1 + f_1^\vee + 4 = \frac{(p - 2)}{(p - 1)} 24.$$

Proof. Using the table, it is easy to see

$$N_1 = g^\vee + 1, g_1 = g.$$

Additionally,

$$f_1 = (p - 2)a^\vee - 2.$$

Combining this with $(p - 1)a + (p - 1)a^\vee = 24$, we obtain the statement in the theorem. □

This corollary implies that Berglund–Hübsch mirror symmetry agrees with mirror symmetry for lattice polarised K3 surfaces. We briefly recall the latter.

Given a smooth K3 surface Σ , $\Lambda = H^2(\Sigma, \mathbb{Z})$ is equipped with a lattice structure via the cup product taking values in $H^4(\Sigma; \mathbb{Z}) = \mathbb{Z}$. Let $S_\Sigma := \Lambda \cap H^{1,1}(\Sigma; \mathbb{C})$ be the Picard lattice of Σ .

Let M be a hyperbolic lattice with signature $(1, t - 1)$. A K3 surface Σ is called *M-polarized* if there exists a primitive embedding $M \hookrightarrow S_\Sigma$. Given a nonsymplectic automorphism s of Σ , the invariant sublattice $S(s) := \Lambda^s$ is in fact a primitive sublattice of the Picard lattice.

Definition 52. Given M a primitive hyperbolic sublattice of $\Lambda = H^2(\Sigma, \mathbb{Z})$ of rank at most 19 such that

$$M^\perp = U \oplus M^\vee,$$

M^\vee is defined to be the *mirror* lattice to M .

Recall that we have restricted to the case where s has prime order $p > 2$ (we have discussed $p = 2$ in [12]). We now show that if two K3 surfaces with prime order automorphisms arise as crepant resolutions of a mirror pair of Berglund–Hübsch orbifolds, they have mirror lattices. In this case, $M := S(s)$ is p -elementary. That is, $M^*/M = (\mathbb{Z}/p\mathbb{Z})^{\oplus a}$, and it is completely classified by its rank r and a . Then, by [3], the fixed locus of s is either just isolated points or a disjoint union of N curves, of which $N - 1$ are rational and the remaining one has genus g and f isolated points. Set $m = \frac{22-r}{p-1}$. Moreover, [3] states (in a slightly different form) that, for $p = 3, 5, 7, 13$, if the fixed locus contains a curve,

$$\circ m = 2g + a, -g + N = \frac{r-11+p}{p-1}.$$

Notice that these are the only prime orders we need to consider as we have shown that $p - 1 | 24$.

Lattice mirror symmetry exchanges (r, a) with $(20 - r, a)$.

Theorem 53. Let $\Sigma_{W,H}$ and Σ_{W^\vee,H^\vee} be mirror K3 orbifolds with prime order $p > 2$ automorphisms s, s^\vee , and let Σ and Σ^\vee be crepant resolutions with automorphisms also denoted s, s^\vee . Then Σ and Σ^\vee are mirror as lattice polarized K3 surfaces.

Proof. Corollary 51 relates the invariants (g, N, f) and (g^\vee, N^\vee, f^\vee) . It is enough to show that these relations give the mirror relations on (r, a) , namely that

$$(r^\vee, a^\vee) = (20 - r, a).$$

Notice that there is always a fixed curve when the K3 is a hypersurface in weighted projective space of this form. Therefore, we see that

$$\begin{aligned} r^\vee &= (-g^\vee + N^\vee)(p - 1) + (11 - p) = (-N + g + 2)(p - 1) + (11 - p) \\ &= (p - 1)\left(2 - \frac{r - 11 + p}{p - 1}\right) + 11 - p = 20 - r. \end{aligned}$$

Finally, this implies

$$a^\vee = 2g^\vee - \frac{22 - r^\vee}{p - 1} = 2(N - 1) - \frac{2 + r}{p - 1}.$$

Using that $N = \frac{r-11+p}{p-1} + g$, we obtain that

$$a^\vee = 2g - \frac{22 - r}{p - 1} = a. \quad \square$$

Acknowledgements. Alessandro Chiodo is supported by the ANR project ‘Categorification in Algebraic Geometry’, CANR-17-CE40-0014 and the ANR project ‘Enumerative Geometry’, PRC ENUMGEOM. Elana Kalashnikov is supported by Discovery Grant RGPIN-2022-03013 from the Natural Sciences and Engineering Research Council of Canada.

Conflicts of Interest. none

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