On Non-Associative Systems

By A. ROBINSON

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1. *Introduction.*

The present paper is concerned with the "logarithmetic", or arithmetic of shapes of non-associative combinations as defined by Etherington in ref. (1). The shape of a non-associative product is defined as "the manner of association of its factors without regard to their identity". Thus, for a binary non-communicative operation, the products $((AB)C)D$ and $((BA)C)D$ and $((AA)A)A$ all have the same shape, while $D((AB)C)$ has a different shape. The sum of the two shapes a and b is defined as the shape of the product of two expressions, of shapes a and b respectively, in the original system of non-associative combination. The product of two shapes a and b is defined as the shape of any expression obtained by replacing every factor in an expression of shape b by an expression of shape a. It is readily shown that these definitions are unambiguous.

As an alternative to the synthetic, or genetic, approach to the arithmetic of shapes, as given above, we may try to define the arithmetic system of shapes by a set of axioms, and then verify that any realisation of the set of axioms is *isomorphic* with the synthetic system of shapes as defined above. This is done in §2 below. Realisations of the set of axioms specified there will be called "simple forests", in deference to the fact that "shapes" have also been called "trees" from a different point of view (Cayley, ref. 2). The wider concept of a general "forest" is considered in §3.

In §4 we deal with systems corresponding to commutative operations. In §5 our considerations are extended to n-ary operations.

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2. *Axioms and Fundamental Properties of "Simple Forests".*

A single binary operation called addition is taken as primary concept, $c = a + b$. Any realisation of the set of axioms detailed below will be called a "simple forest", F. An element $a \epsilon F$ will be called a component of an element $c \epsilon F$, if there exists an element $b \epsilon \mathrm{F}$, such that either $a + b = c$ or $b + a = c$.

The following axioms are supposed to be satisfied:

I. The operation of addition is unambiguous, and F is closed with respect to it; *i.e.* given $a \in F$ and $b \in F$ there is a unique $c \in F$ such that $c = a + b$.

II. If $a + b = c + d$, then $a = c$ and $b = d$.

III. There is not more than one element in F without component.

IV. In every non-empty subset F' of F, there is at least one element without component in F'.

According to III and IV there is just one element without components in F. It will be denoted by e.

THEOREM 1. (Principle of mathematical induction.) *If e belongs to a subset F' of F, and if $a \in F'$, $b \in F'$ implies $(a + b) \in F'$, then $F' = F$.*

Proof: Assume, on the contrary, that $F'' = F - F'$ is non-empty. Then F'' contains an element f'' without components in F'', by IV. However, f'' must have components in F, otherwise $f'' = e$, by III, and that would be contrary to the assumption that $e \in F'$. Hence $f'' = a + b$, say, where both a and b belong to F'. But in that case, f'' also belongs to F', by assumption. Hence F'' is in fact empty and the theorem is proved.

We now define multiplication by induction: $a \cdot e = a$, and if $a \cdot b$ and $a \cdot c$ are defined, then $a \cdot (b + c)$ is defined as $a \cdot b + a \cdot c$. It can be shown that this definition is unique (unambiguous).

THEOREM 2. *$e \cdot a = a$ for all a.*

The proof is by induction. In fact $e \cdot e = e$ and $e \cdot a = a$, $e \cdot b = b$ implies $e \cdot (a + b) = e \cdot a + e \cdot b = (a + b)$. Hence, by Theorem 1, $e \cdot a = a$ for all a.

THEOREM 3. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Proof: Keeping a and b fixed, we are going to show that the set F' of elements c for which $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ coincides with F.

In fact $(a \cdot b) \cdot e = a \cdot b = a \cdot (b \cdot e)$, so that $e \in F'$. Also if $c_1 \in F'$, $c_2 \in F'$, $c = c_1 + c_2$, then

$$\begin{aligned}(a \cdot b) \cdot c = (a \cdot b) \cdot (c_1 + c_2) = (a \cdot b) \cdot c_1 + (a \cdot b) \cdot c_2 &= a \cdot (b \cdot c_1) + a \cdot (b \cdot c_2) \\ &= a \cdot (b \cdot c_1 + b \cdot c_2) = a \cdot (b \cdot c).\end{aligned}$$

Hence $c \in F'$ and F' coincides with F, by Theorem 1.

Degree, $\delta(a)$, and altitude, $\alpha(a)$, are defined by induction as real functions of all $a \in F$, thus:

$$\begin{aligned}\delta(e) &= 1, & \delta(a + b) &= \delta(a) + \delta(b) \\ \alpha(e) &= 0, & \alpha(a + b) &= 1 + \operatorname{Max}(\alpha(a), \alpha(b)).\end{aligned}$$

As an example of this procedure of induction we consider the definition of degree in more detail. The first step is to define $\delta(e) = 1$. The n^{th} step is to consider all elements $c = a + b$, for which $\delta(c)$ has not been defined in the first $(n-1)$ steps, but $\delta(a)$ and $\delta(b)$ have both been defined in the first $(n-1)$ steps. We then define $\delta(a+b) = \delta(a) + \delta(b)$.

Two simple forests F, F' will be called *isomorphic* if there is a one-to-one correspondence $\longleftrightarrow$ between their elements, $a, b, \ldots$ and $a', b', \ldots$ respectively such that if $a \longleftrightarrow a'$, $b \longleftrightarrow b'$, then $(a+b) \longleftrightarrow (a'+b')$. The correspondence will be called an *isomorphism.*

THEOREM 4. *Given an isomorphism* $\longleftrightarrow$ *between two simple forests* F *and* F', *we have* $e \longleftrightarrow e'$, *and if* $a \longleftrightarrow a'$, $b \longleftrightarrow b'$, *then* $a{\cdot}b \longleftrightarrow a'{\cdot}b'$.

To prove the first part of the theorem, assume contrary to assertion that $e \longleftrightarrow c'$, $c \neq e'$, $c' = a' + b'$. Let a and b in F correspond to a' and b' respectively, then $c = a + b$ corresponds to $c' = a' + b'$. This implies $e = c = a + b$, contrary to the fact that e has no components.

To prove the second part, we keep a, a' fixed, $a \longleftrightarrow a'$, and prove that the set of elements b (corresponding to b' in F') for which $a{\cdot}b \longleftrightarrow a'{\cdot}b'$ coincides with F.

The proof is by induction. In fact, if $b = e$, then $b' = e'$, by the first part of this theorem, and so $a{\cdot}b = a$ corresponds to $a'{\cdot}b' = a'$. Also if $b = b_1 + b_2$, $b' = b'_1 + b'_2$, $b_1 \longleftrightarrow b'_1$, $b_2 \longleftrightarrow b'_2$, $a{\cdot}b_1 \longleftrightarrow a'{\cdot}b'_1$, $a{\cdot}b_2 \longleftrightarrow a'{\cdot}b'_2$, then $(a{\cdot}b_1 + a{\cdot}b_2) \longleftrightarrow (a'{\cdot}b'_1 + a'{\cdot}b'_2)$ by the definition of an isomorphism. Hence $ab \longleftrightarrow a'b'$, and so, by Theorem 1, the set of elements b, for which $b \longleftrightarrow b'$ implies $a{\cdot}b \longleftrightarrow a'{\cdot}b'$, coincides with F.

THEOREM 5. *Any two simple forests (realisations of Axioms I-IV) are isomorphic.*

This theorem is a special case of Theorem 7 which is proved in detail in §3 below. It can readily be shown that the arithmetic of binary non-commutative shapes as defined in ref. (1) (see §1 above) satisfies Axioms I-IV. It follows that this set of axioms does in fact supply a satisfactory axiomatic foundation for the arithmetic of these shapes (in the same way as the axioms for an infinite field with minimum condition supply a satisfactory foundation for the arithmetic of rational numbers).

THEOREM 6. *There is one and only one isomorphism between any two simple forests.*

The proof is left to the reader.

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3. *Forests.*

We now drop Axiom III and consider systems which satisfy only Axioms I, II and IV. Any such system will be called a "forest." An element of a forest will be said to be *irreducible* if it has no components. According to IV, a non-empty forest contains at least one irreducible element. The (cardinal) number of irreducible elements in a given forest will be called the *order* of the forest. It may be finite or infinite. A simple forest is a forest of order 1. The set of irreducible elements of a forest will be called the *base* of the forest.

THEOREM 7. *Two forests of equal order are isomorphic.*

Proof: Let $e_1, e_2, \ldots$, and $e'_1, e'_2, \ldots$ be the bases of the two forests, F and F', respectively. By assumption, there is a one-to-one correspondence, say $e_1 \longleftrightarrow e'_1$, $e_2 \longleftrightarrow e'_2, \ldots$ between the elements of the two bases. We now define a many-to-one correspondence between the elements of F and some of the elements of F', and then show (i) that the correspondence is in fact one-to-one, (ii) that it comprises all the elements of F', and (iii) that $a \longleftrightarrow a'$, $b \longleftrightarrow b'$ implies $(a+b) \longleftrightarrow (a'+b')$. The function f which yields this correspondence will be defined as follows. As a first step, we define $f(e_1)=e'_1$, $f(e_2)=e'_2, \ldots$ as given by the correspondence between the bases of F and F'. As the n^{th} step, we define $f(a+b)=f(a)+f(b)$, if $f(a+b)$ has not been defined at one of the first $(n-1)$ steps, but $f(a)$ and $f(b)$ have been so defined.

The function $f(a)$ is defined for all $a \in F$; assume on the contrary that the set F_1, of elements of F for which $f(a)$ is not defined, is non-empty. This implies, by Axiom IV, that F_1 contains an element c without components in F_1. c cannot be irreducible in F, for in that case c would belong to the base so that $f(c)$ would have been defined at the first step. Hence c has two components, $c=a+b$ say, so that neither a nor b is in F_1. That is to say that $f(a)$ and $f(b)$ have both been defined, at the $n_1{}^{th}$ and $n_2{}^{th}$ steps respectively, and we may assume without any essential limitation of the generality of our argument that $n_1 \geqq n_2$. But then, according to the construction of f, $f(c)$ has actually been defined by $f(a)+f(b)$ at the $(n_1+1)^{th}$ step, contrary to assumption.

The function f takes values in F'. We are going to show that every element of F' is taken at least once. Assume on the contrary that some of the elements of F' never occur as values of the function f, and let F'_1 be the set of these elements. Then there is at least one element of F'_1, say c', without components in F'_1, but as before, it cannot be irreducible in F'. Hence $c' = a' + b'$ where a is taken at least once by f, for an argument a, say, $a' = f(a)$, and b' is taken at least once by f; for an argument b, say, $b' = f(b)$. But in this case, c' must be the value of f, for $c = a + b$, $c' = f(c)$, contrary to assumption.

Next, we show that f does not take any element of F' more than once. Let F'_1 be the set of elements of F' which are taken more than once. If F'_1 is not empty, then by Axiom IV it contains at least one element c' say, without components in F'_1. Assume first that c' is irreducible. In this case c' is in fact taken once by f, at the first step, but it cannot possibly be taken as a value of f at the n^{th} step, $n > 1$, since at these steps f takes only values with components in F'. On the other hand if c' is not irreducible, then it can be written $c' = a' + b'$, where a' and b$'$ are employed just once as values of f, $a' = f(a)$, $b' = f(b)$ say. Hence $c' = f(c)$, $c = a + b$. Assume that at the same time $c' = f(c_1)$, $c_1 = a_1 + b_1$. Then $a' = f(a_1)$, $b' = f(b_1)$, by the construction of f, and so $a = a_1$, $b = b_1$, since a and b are employed only once. Hence $c_1 = c$, so that c' is in fact taken only once, contrary to assumption.

Finally, we have to show that the one-to-one correspondence which, as we have seen, is set up by f between F and F', does in fact establish an isomorphism. That is to say, we have to show that $a' = f(a)$, $b' = f(b)$ implies $c' = f(c)$, $c' = a' + b'$, $c = a + b$, and this follows immediately from the manner of construction of f.

The proof of Theorem 7 is now complete.

Given a forest F, any subset F' of F which is closed with respect to addition (*i.e.* $a \in F'$, $b \in F'$ implies $(a + b) \in F'$) is itself a forest. The meet (set-product) of any number of forests which are subsystems of a given forest is itself a forest.

Given a forest F and a set S of elements in F, the smallest forest $F' \leqq F$ which contains S will be said to be *generated* by S. F' is the meet of all forests in F which contain S. It can be shown that the base of F' is a subset of S. Also if F' is a forest in F, and F' contains the base of F, then $F' = F$.

Given a forest F, a set S of elements of F will be said to be *independent*, if no element $s \in S$ is contained in the forest generated by the remainder of the elements of S. In that case, S coincides with the base of the forest generated by it.

Thus, taking into account Theorem 7, in order to gain an impression of the possible types of forests which are subsystems of a given forest F, we only want to know the magnitude (cardinal number) of the various independent sets in F.

It is demonstrated below that a simple forest F contains an infinite independent subset S. Also it is easily shown that a forest is denumerable (of cardinal number $\aleph_0$). It follows that F contains simple forests of all finite orders and of order $\aleph_0$.

THEOREM 8. *A simple forest F contains at least one infinite independent subset S.*

Proof: Put $e_1 = (e+e)+e$ and $e' = e + (e + e)$. Then e' generates a simple forest of order 1, which is therefore a simple forest, F_1, say. By Theorem 6 there is just one isomorphism between F and F_1, $F \longleftrightarrow F_1$, say. If we take F_1 as a subsystem of F, F_1 corresponds to a subsystem of F_1, F_2, say, itself a simple forest. Again, if we take F_2 as a subsystem of F, F_2 corresponds to a subsystem of F_1, F_3, say, itself a simple forest. In this way we obtain a chain of simple forests,

$$\begin{array}{ccccccccc} F & > & F_1 & > & F_2 & > & F_3 & > & \ldots\ldots \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\ F_1 & > & F_2 & > & F_3 & > & F_4 & > & \ldots\ldots \end{array}$$

Also, under the stated isomorphism, e_1 as an element of F corresponds to an element of F_1, e_2, say; e_2 as an element of F corresponds to an element of F_2, e_3, say; and so on. In this way we obtain a set S of elements $e_1, e_2, e_3, e_4, \ldots$ We are going to show that $e_n \neq e_m$ for $n > m$.

In fact, by construction, e_m is not in F_m, while $e_n \in F_\kappa$ for all $\kappa < n$, and so $e_n \in F_m$. Hence $e_n \neq e_m$, so that S is infinite.

Let S_κ, $\kappa = 1, 2, 3, \ldots$, be the subset of S obtained by omitting e_κ from S. In order to prove that S is an independent set, we have to show that e_κ is not contained in the forest generated by S_κ.

Let $\delta(e_\kappa)$ be the degree of e_κ regarded as an element of F, $\kappa = 1, 2, 3, \ldots$ It is easy to deduce from the definition of the degree given in §2 above, that $\delta(e_\kappa) = 3^\kappa$, $\kappa = 1, 2, 3, \ldots$ Again the degree of any element of a forest cannot possibly be smaller than the smallest of the degrees of the elements of a set generating the forest. It follows that e_1 cannot be contained in the forest generated by S_1. Similarly, if an element a is contained in the forest generated by $S_{\kappa+1}$, $\kappa \geqq 1$, but not in the forest generated by the set $(e_1, \ldots, e_\kappa)$, then its degree is necessarily $\geqq 3^{\kappa+2}$. It is therefore sufficient to show that $e_{\kappa+1}$ is not contained in the forest generated by $(e_1, \ldots, e_\kappa)$, $\kappa \geqq 1$.

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Let e'_κ be the irreducible element in F_κ, so that $e'_1 = e'$. Then it will be observed that $\delta(e'_\kappa) = \delta(e_\kappa)$, and $e_{\kappa+1} = (e'_\kappa + e'_\kappa) + e'_\kappa$. Hence if $e_{\kappa+1}$ is contained in the forest generated by $(e_1, \ldots, e_\kappa)$ $\kappa = 2, 3, 4, \ldots$, so are $(e'_\kappa + e'_\kappa)$ and e'_κ. Also, as before, since $\delta(e'_\kappa) = \delta(e_\kappa)$, it follows that if e'_κ belongs to the forest generated by $(e_1, \ldots, e_\kappa)$, it also belongs to the forest generated by $(e_1, \ldots, e_{\kappa-1})$. Now $e'_\kappa = e'_{\kappa-1} + (e'_{\kappa-1} + e'_{\kappa-1})$, for all $\kappa \geqq 2$. Hence, if e_κ belongs to the forest generated by $(e_1, \ldots, e_{\kappa-1})$, so do $e'_{\kappa-1}$ and $(e'_{\kappa-1} + e'_{\kappa-1})$, and therefore so does $e_\kappa = (e'_{\kappa-1} + e'_{\kappa-1}) + e'_{\kappa-1}$.

We have shown that if $e_{\kappa+1}$ belongs to the forest generated by $(e_1, \ldots, e_\kappa)$, $\kappa = 2, 3, \ldots$, then e_κ belongs to the forest generated by $(e_1, \ldots, e_{\kappa-1})$. Hence if there is an element e_κ, $\kappa \geqq 2$, which belongs to the forest generated by S_κ, then e_2 belongs to the forest generated by e_1. And exactly as before this implies that e'_1 belongs to the forest generated by e_1, which is impossible since $\delta(e'_1) = \delta(e_1)$. It follows that S is an independent set, as asserted.

THEOREM 9. (Principle of mathematical induction for general forests.) *If the base of a forest F belongs to a subset F' of F, and if $a \in F'$, $b \in F'$ implies $(a+b) \in F'$, then $F' = F$.*

The proof of this theorem is similar to that of Theorem 1.

There is no "natural" definition of multiplication in terms of addition for forests in general, as there is for simple forests.

4. *Commutative Forests.*

If the original operation, from which the "logarithmetic" is derived after the manner of ref. 1, is commutative, then addition in the logarithmetic is commutative. In the axiomatic approach this requires the following modifications:

Axiom II is replaced by Axiom II':—

If $a + b = c + d$, then $a = c$, $b = d$, or $a = d$, $b = c$.

A new axiom is added

$$V': \; a + b = b + a$$

Axioms I, III, and IV remain unaltered.

Then all the theorems and definitions of §2 remain valid.

A system satisfying the modified set of axioms will be called a *simple commutative forest*, and a system satisfying all the axioms of the set except (possibly) III will be called a *commutative forest*. The definitions and theorems of §3 are again valid for commutative forests, although the proof of Theorem 8 requires some modification.

5. *N-ary Forests.*

In an associative system, the product of three terms, A, B, C, can be defined in terms of a binary operation, by $A\ (B\ C)$ or by $(A\ B)\ C$, and these two expressions are equal. On the other hand, in a non-associative system, $A\ (B\ C) \neq (A\ B)\ C$, but in addition $A\ B\ C$ may be specified as a ternary operation again leading to a different result. From these and similar considerations we see that it may be necessary to investigate logarithmetics in which the "sum" is defined as an n-ary operation, $n>2$ ("n-ary forests"), or alternatively, logarithmetics in which "sums" are defined for all κ-ary operations, $2 \leqq \kappa \leqq n$, for specified $n > 2$ ("mixed n-ary forests"). Again, the specified operations may be either commutative or non-commutative. It will be sufficient here to formulate the axioms for one of these cases, say that of the simple mixed n-ary forest.

A system F is called a *simple mixed n-ary forest* if it satisfies the following axioms:

I. For every $a_1 \epsilon F$, $a_2 \epsilon F$, $\ldots$, $a_\kappa \epsilon F$, $2 \leqq \kappa \leqq n$, there is defined a unique sum $c \epsilon F$, $c = a_1 + a_2 + \ldots + a_\kappa$.
Each one of the elements $a_1, a_2, \ldots, a_\kappa$ is called a *component* of c.

II. If $a_1 + a_2 + \ldots + a_\kappa = b_1 + b_2 + \ldots + b_m$, then $\kappa = m$ and $a_1 = b_1$, $a_2 = b_2$, $\ldots$, $a_\kappa = b_\kappa$.

III. There is not more than one element in F without component.

IV. In every non-empty subset F' of F there is at least one element without component in F'.

With suitable modifications, the theory of simple mixed n-ary forests, and of mixed n-ary forests, can be developed in parallel with §§2 and 3 above.

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COLLEGE OF AERONAUTICS,
CRANFIELD,
BLETCHLEY, BUCKS.