# A COINCIDENCE PROBLEM IN TELEPHONE TRAFFIC WITH NON-RECURRENT ARRIVAL PROCESS

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## 1. Introduction

We consider the following problem. Calls arrive at a telephone exchange at the instants $t_0, t_1, \cdots, t_m, \cdots$. The telephone exchange contains a denumerable infinity of channels. The holding times of calls are non-negative random variables distributed independently of the times at which calls arrive, independently of which channel a call engages and independently of each other with a common distribution function $B(x)$. Takacs [3] has studied the case when the arrival process is a recurrent process, that is, the inter-arrival intervals $\tau_m = t_{m+1} - t_m$, $m \geqq 0$ are identically and independently distributed non-negative random variables with common distribution function, $A(x)$. Finch [1] has studied the transient behaviour in the case of a recurrent arrival process and exponential holding time, that is when the common distribution of holding time is given by

$$B(x) = 1 - e^{-\mu x}, \qquad x \geqq 0 \tag{1.1}$$

In this paper we make no assumption about the arrival process $\{t_m\}$. The underlying principle of this paper is the same as that of Finch [2]. We consider the instants of arrival $t_0, t_1, \cdots, t_m, \cdots$ as given and determine various probabilities of interest conditionally as functions of the inter-arrival intervals $\tau_0, \tau_1, \cdots, \tau_m, \cdots$. When the arrival process is a stochastic process we can then determine the relevant unconditional probabilities by integration.

The assumption of an infinite number of channels is, of course, practically unrealistic, but apart from the mathematical interest of such a system it can be used to obtain approximate results for a system with a finite number of channels when the probability of loss is small.

## 2. Transient behaviour: specified arrival times

Suppose that the instants $t_0, t_1, \cdots, t_m, \cdots$ at which the calls arrive are given and write $\tau_m = t_{m+1} - t_m$, $m \geqq 0$. Suppose that initially no calls are



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in the system. Denote the number of calls in the system at $t_m - 0$ by $\eta_m(\tau_0, \tau_1, \cdots, \tau_{m-1})$ and write

$$P_j^m(\tau_0, \tau_1, \cdots, \tau_{m-1}) = Pr\{\eta_m(\tau_0, \tau_1, \cdots, \tau_{m-1}) = j\} \quad 0 \leqq j \leqq m, \quad 1 \leqq m.$$

Let $h$ be the holding time of the call which arrives at $t_0$ and distinguish the two cases (i) $h \leqq \tau_0 + \cdots + \tau_m$ (ii) $h > \tau_0 + \cdots + \tau_m$. Since an infinite number of channels is available and since the holding times of calls are identically distributed with common distribution function $B(x)$ independently of each other and of the instants at which calls arrive we have

$$\eta_{m+1}(\tau_0, \tau_1, \cdots, \tau_m) = \begin{cases} \eta_m^*(\tau_1, \tau_2, \cdots, \tau_m), & \text{if} \quad h \leqq \tau_0 + \tau_1 + \cdots + \tau_m \\ \eta_m^*(\tau_1, \tau_2, \cdots, \tau_m) + 1, & \text{if} \quad h > \tau_0 + \tau_1 + \cdots + \tau_m \end{cases} \tag{2.1}$$

where $\eta_m^*(\tau_1, \tau_2, \cdots, \tau_m)$ is distributed as $\eta_m(\tau_1, \tau_2, \cdots, \tau_m)$ independently of $\eta_{m+1}(\tau_0, \tau_1, \cdots, \tau_m)$. Introduce the generating function

$$P^k(z, \tau_0, \tau_1, \cdots, \tau_{k-1}) = \sum_{j=0}^{k} P_j^k(\tau_0, \tau_1, \cdots, \tau_{k-1}) z^j, \qquad |z| \leqq 1 \tag{2.2}$$

Then from (2.1) we obtain

$$P^{m+1}(z, \tau_0, \tau_1, \cdots, \tau_m) = [B(\theta_{m,m}) + z(1 - B(\theta_{m,m})] P^m(z, \tau_1, \tau_2, \cdots, \tau_m) \tag{2.3}$$

where

$$\theta_{m,j} = \tau_m + \tau_{m-1} + \cdots + \tau_{m-j}, \qquad 0 \leqq j \leqq m. \tag{2.4}$$

Since the initial condition is $\eta_0 = 0$ by hypothesis we have $P^0(z) = 1$ and from (2.3) we obtain

$$P^{m+1}(z, \tau_0, \tau_1, \cdots, \tau_m) = \prod_{s=0}^{m} [B(\theta_{m,s}) + z(1 - B(\theta_{m,s}))] \tag{2.5}$$

From (2.5) we obtain explicit expressions for the moments of the distribution $\{P_j^{m+1}(\tau_0, \tau_1, \cdots, \tau_m)\}$, in particular we find for the mean and variance respectively.

$$E\{\eta_{m+1}(\tau_0, \tau_1, \cdots, \tau_m)\} = \sum_{s=0}^{m} \{1 - B(\theta_{m,s})\} \tag{2.6}$$

$$\operatorname{Var}\{\eta_{m+1}(\tau_0, \tau_1, \cdots, \tau_m)\} = \sum_{s=0}^{m} \{1 - B(\theta_{m,s})\} B(\theta_{m,s}) \tag{2.7}$$

## 3. Transient behaviour: stochastic arrival times

In this section we suppose, in addition to assumption of initial emptiness, that the sequence of arrival times $\{t_m\}$ forms a stochastic process and that

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the joint distribution of the non-negative random variables $\theta_{m,j}$, $0 \leqq j \leqq m$, given by (2.4) is

$$F_m(x_0, x_1, \cdots, x_m) = P(\theta_{m,j} \leqq x_j, 0 \leqq j \leqq m) \tag{3.1}$$

Let

$$P_j^{m+1} = E \cdot P_j^{m+1}(\tau_0, \tau_1, \cdots, \tau_m) \tag{3.2}$$

be the unconditional probability that the $(m+2)$th call to arrive find $j$ calls in the system, $j = 0, 1, \cdots, m+1$. Write $P^{m+1}(z) = \sum_{j=0}^{m+1} P_j^{m+1} z^j$, from (2.5) we have

$$P^{m+1}(z) = \int \prod_{s=0}^{m} [B(\theta_{m,s}) + z(1 - B(\theta_{m,s}))] dF_m(\theta_{m,0}, \cdots, \theta_{m,m}). \tag{3.3}$$

If $\eta_m$ is the random variable specifying the number of calls in the system on the $m$th arrival we can obtain the moments of $\eta_{m+1}$ from (3.3), in particular we find for the mean and variance respectively

$$E(\eta_{m+1}) = \int_{0-}^{\infty} \{1 - B(x)\} dG_{m+1}(x) \tag{3.4}$$

$$\begin{aligned} \text{Var}(\eta_{m+1}) &= E(\eta_{m+1})\{1 - E(\eta_{m+1})\} \\ &\quad + \int_{0-}^{\infty} \int_{0-}^{\infty} \{1 - B(x)\}\{1 - B(y)\} dG_{m+1}(x, y) \end{aligned} \tag{3.5}$$

where

$$G_{m+1}(x) = \sum_{s=0}^{m} F_{m,s}(x) \tag{3.6}$$

$$G_{m+1}(x, y) = \sum_{s,t=0}^{m} F_{m,s,t}(x, y) \tag{3.7}$$

$$F_{m,s}(x) = P\{\theta_{m,s} \leqq x\} \tag{3.8}$$

$$F_{m,s,t}(x) = P\{\theta_{m,s} \leqq x, \theta_{m,t} \leqq y\}, \quad s \neq t. \tag{3.9}$$

## 4. An Example

In this section we consider an example of the method introduced above. We consider the case of an exponential distribution of holding time $B(x)$, given by (1.1) with $\mu = 1$. We have from (3.4) and (3.5).

$$E(\eta_{m+1}) = {}^*G_{m+1}(1) \tag{4.1}$$

$$\text{Var}(\eta_{m+1}) = {}^*G_{m+1}(1, 1) \tag{4.2}$$

where

(4.3) $$^*G_{m+1}(s) = \int_{0-}^{\infty} e^{-sx}\, dG_{m+1}(x),$$

and

(4.4) $$^*G_{m+1}(s, u) = \int_{0-}^{\infty}\int_{0-}^{\infty} e^{-(sx+uy)}\, dG_{m+1}(x, y).$$

For example if the inter-arrival intervals $\tau_m$ are generated by a simple moving average of order two, namely

(4.5) $$\tau_m = u_m + bu_{m-1}, \quad m \geqq 0$$

where $b > 0$ and the $u_m$ are identically distributed, uncorrelated non-negative random variables with $\psi(s) = E(e^{-su_m})$ we find that

(4.6) $$E(\eta_{m+1}) = \psi(1)\psi(b)[1-\{\psi(1+b)\}^{m+1}][1-\psi(1+b)]^{-1}$$

A slightly more complicated expression can be written down for the variance of $\eta_{m+1}$, we shall give here only the limiting value of this expression, namely

(4.7) $$\lim_{m\to\infty} \mathrm{Var}\,(\eta_{m+1}) = \psi(1)\psi(b)[1-\psi(1)\psi(b)][1-\psi(1+b)]^{-1} + 2\psi(2)\psi(b)\psi(1+2b)[1-\psi(1+b)]^{-1}[1-\psi(2+2b)]^{-1}.$$

It is clear that similar results can be obtained for a moving average of any finite order, that is when

$$\tau_m = u_{m+p} + b_1 u_{m+p-1} + \cdots + b_p u_m; \quad b_i > 0$$

where the $u_m$ are identically distributed, uncorrelated non-negative random variables. With only an additional algebraic complexity the procedure above is easily generalised to an Erlang distribution of holding time given by

$$dB(x) = e^{-\mu x}\mu^{+1}x^k dx/k!$$

## References

[1] Finch, P. D., On the transient behaviour of a coincidence variate in telephone traffic, *Annals of Math. Stat.*, 32 (1961), 230–234.

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[3] Takacs, L., On a coincidence problem concerning telephone traffic, *Acta Math. Sci. Hung.* 9 (1958), 45–81.

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