A REMARK ON DECOMPOSITIONS OF THE PERMUTATION REPRESENTATION OF A PERMUTATION GROUP

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To RICHARD BRAUER on the occasion of his 60th birthday

Let $\mathfrak{G}$ be a permutation group on n-letters $\underset{\sim}{1}, \underset{\sim}{2}, \ldots, \underset{\sim}{n}$. Let $\mathfrak{G}_1$ be the subgroup of $\mathfrak{G}$ fixing suitable one letter, say $\underset{\sim}{1}$. For any element G of $\mathfrak{G}$, a non-singular matrix $G^* = (g_{ij})$ of degree n is defined by the equation

$$(1) \qquad \begin{pmatrix} \underset{\sim}{1}^G \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{n}^G \end{pmatrix} = G^* \begin{pmatrix} \underset{\sim}{1} \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{n} \end{pmatrix}.$$

Since $g_{ij}'s$ are 0 or 1, we may assume that G^* is a matrix whose coefficients are in an arbitrary unitary ring K. Then if for any element G of $\mathfrak{G}$ we take the mapping $G \to G^*$, this mapping will be a representation P_K of $\mathfrak{G}$ by the non-singular $n \times n$ matrices over K. By the formula (1) the representation P_K is also the representation of $\mathfrak{G}$ induced by the identity representation of $\mathfrak{G}_1$ over K. We call P_K the permutation representation of $\mathfrak{G}$ over K. If K is a field of characteristic 0 (more generally, if the characteristic of K does not divide the order of $\mathfrak{G}$), then it is well known that $\mathfrak{G}$ is a doubly transitive group when and only when P_K is directly decomposed into two irreducible constituents (see [2]). Now in the present note we consider decompositions of the permutation representation P_K of $\mathfrak{G}$ over an arbitrary unitary ring K, instead of such a field of characteristic 0.

THEOREM 1. *Assume that $\mathfrak{G}$ is a doubly transitive group.*

i) If n is an inversible element of K (e.g. K is a field whose characteristic does not divide n), then P_K is directly decomposed into two indecomposable

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constituents and one of them is the identity representation.

ii) If n is not an inversible element of K (e.g. K is rational integer ring or a field whose characteristic divides n), then P_K is a indecomposable representation.

Proof. Let M_K be the representation module of $\mathfrak{G}$ corresponding to P_K. Then we may suppose that M_K has a basis $\underline{1}, \underline{2}, \ldots, \underline{n}$ over K and, for any element G of $\mathfrak{G}$, G^* operates on this basis such that $\underline{i} \to \underline{i}^G$. If M_K is directly decomposed into a certain number of $\mathfrak{G}$-submodules of M_K, say $M_1, \ldots, M_r$, then we have, for $u \in M$ uniquely, $u = \sum_{i=1}^{r} u_i$, $u_i \in M_i$ and the mappings δ_i: $u \to u_i$, $i = 1, \ldots, n$, are idempotent $\mathfrak{G}$-endomorphisms of M_K such that

$$\delta_i \delta_j = 0 \qquad \text{for } i \neq j \text{ and } \sum_{i=1}^{r} \delta_i = \text{identity}. \tag{2}$$

Conversely, if there exist idempotent $\mathfrak{G}$-endomorphisms $\delta_1, \ldots, \delta_r$ of M_K satisfying the relations (2), then it is easy to see that M_K is directly decomposed into r $\mathfrak{G}$-submodules of M_K. Therefore, in order to determine the direct decomposition of M_K, we need only to look for idempotent $\mathfrak{G}$-endomorphisms of M_K satisfying the relations (2). Let δ be a $\mathfrak{G}$-endomorphism of M_K and put $\underline{i}^\delta = \sum_{j=1}^{n} \lambda_{ij} \underline{j}$. Since $\underline{i}^{\delta G} = \underline{i}^{G\delta}$ for any element G of $\mathfrak{G}$, we have $\sum_{j=1}^{r} \lambda_{ij} \underline{j}^G = \sum \lambda_{i_G j_G} \underline{j}_G$, hence $\lambda_{ij} = \lambda_{i_G j_G}$ for any element G of $\mathfrak{G}$ and for any integers $1 \leqq i, j \leqq n$. Since $\mathfrak{G}$ is doubly transitive it is easy to see that $\lambda_{11} = \cdots = \lambda_{nn} (= \lambda)$ and $\lambda_{ij} = \lambda_{k1} (= \mu)$ if $i \neq j$ and $k \neq 1$. Hence we have

$$\begin{pmatrix} 1^\delta \\ \cdot \\ \cdot \\ \cdot \\ n^\delta \end{pmatrix} = \Delta(\delta) \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ n \end{pmatrix} \quad \text{where} \quad \Delta(\delta) = \begin{pmatrix} \lambda & \mu & \cdot & \cdot & \cdot & \mu \\ \mu & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \mu \\ \mu & \cdot & \cdot & \cdot & \mu & \lambda \end{pmatrix}$$

If δ is a idempotent $\mathfrak{G}$-endomorphism of M_K, i.e. $\delta^2 = \delta$, then

$$\Delta(\delta) = \Delta(\delta)^2 = \begin{pmatrix} \lambda^2 + (n-1)\mu^2 & & 2\lambda\mu + (n-2)\mu^2 \\ & \ddots & \\ 2\lambda\mu + (n-2)\mu^2 & & \lambda^2 + (n-1)\mu^2 \end{pmatrix},$$

therefore we have the equations $\lambda = \lambda^2 + (n-1)\mu^2$, $\mu = 2\lambda\mu + (n-2)\mu^2$. From these equations we see that $\lambda = \mu = 0$, $\lambda = \mu$ and $\lambda n = 1$, $\lambda = 1$ and $\mu = 0$, or $\lambda = \mu + 1$ and $n\lambda = n - 1$. Hence if n is not a inversible element of K then we

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have no non trivial idempotent $\mathfrak{G}$-endomorphisms of M_K and if n is a inversible element of K then there exist exactly two non trivial $\mathfrak{G}$-endomorphisms δ_1, δ_2 of M_K where

$$\Delta(\delta_1) = \begin{pmatrix} \frac{1}{n} & & \frac{1}{n} \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ \frac{1}{n} & & \frac{1}{n} \end{pmatrix} \text{ and } \Delta(\delta_2) = \begin{pmatrix} \frac{n-1}{n} & & -\frac{1}{n} \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ -\frac{1}{n} & & \frac{n-1}{n} \end{pmatrix}.$$

It is easy to see that $\delta_1\delta_2 = \delta_2\delta_1 = 0$ $\delta_1 + \delta_2 =$ identity and $M_K^{\delta} = k\sum_{i=1}^{n} \underline{i}$. The proof is complete.

It seems to us of interest to determine irreducible constituents of M_K. When $\mathfrak{G}$ is the symmetic groups, H. K. Farahat determined irreducible constituents of M_K (see [1]). Using Farahat's method we can prove a following theorem.

THEOREM 2. *Let $\mathfrak{G}$ be a triply transitive group. If K is a field whose characteristic p divides n and does not divide the order of $\mathfrak{G}_1$ then, for the $\mathfrak{G}$-module M_K, we have a composition series $M_K \supset M_1 \supset M_2 \supset 0$ where* $\dim_K M_2 = \dim_K M_K/M_1 = 1$.

Proof. Put $M_1 = \sum_{i=2}^{n} K(\underline{i} - \underline{1})$ and $M_2 = K\sum_{i=1}^{n} \underline{i}$. Then M_1, M_2 are $\mathfrak{G}$-submodules of M_K and, by our assumption $p\,|\,n$, $M_1 \supset M_2$ and $\dim_K M_2 = \dim_K M_K/M_1 = 1$. Put $M_1^* = \sum_{i=2}^{n} K\underline{i}$. Then, since $\underline{1}^G = \underline{1}$ for any element G of $\mathfrak{G}_1$, we see that M_1^* is a $\mathfrak{G}_1$-module and there is a $\mathfrak{G}_1$-isomorphism θ of M_1 onto M_1^*, for which $\theta(\underline{i} - \underline{1}) = \underline{i}$. Furthermore, since $n \cdot 1 = 0$ in K, θ carries M_2 onto $M_2^* = K\sum_{i=2}^{n} \underline{i}$. It follows that θ induces a $\mathfrak{G}_1$-isomorphism of the factor module M_1/M_2 onto the factor module M_1^*/M_2^*. Since p does not divide the order of $\mathfrak{G}_1$ and $\mathfrak{G}_1$ is doubly transitive, M_1^*/M_2^* is a irreducible $\mathfrak{G}_1$-module. It follows that M_1/M_2 is a irreducible $\mathfrak{G}_1$-module. Hence M_1/M_2 is a irreducible $\mathfrak{G}$-module. The proof is complete.

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