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GOLDIE M-GROUPS

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Abstract

If $(G+)$ is a group and M is a nonempty set of endomorphisms of G operating on the left then G is said to be M-Goldie when

(i) G has no infinite independent family of nonzero M-subgroups, and
(ii) annihilators in M of subsets of G satisfy the a.c.c. (under set inclusion).

Here we prove some results, analogous to those of a Noetherian module in some special cases, even when the set M of operators has no other algebraic structure than the existence of a zero element or in some cases M is at most a finite dimensional commutative near-ring. Precisely speaking, we prove that the collection of associated operating sets of G is finite and there exists a primary decomposition of 0 of such a Goldie M-group, and then if M is a finite dimensional commutative near-ring with unity, for any x belonging to each associated operating set of G, a power of it belongs to the annihilator of G.

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1. Introduction

In this paper we introduce the notion of a Goldie operator group and establish some interesting properties of such a system.

If $(G+)$ is a group and M is a nonempty set of endomorphisms of G operating on the left then G is said to be *M-Goldie* when

(i) G has no infinite independent family of nonzero M-subgroups, and

(ii) annihilators of subsets of G in M satisfy the ascending chain condition (under set inclusion).

A Goldie ring is clearly a Goldie M-group. Every finite dimensional left module over a left Noetherian ring is a Goldie M-group. An Artinian left module over a left Artinian ring is a Goldie M-group.

It can easily be seen that a direct sum of two Goldie M-groups is again a Goldie M-group. An M-subgroup of a Goldie M-group is a Goldie M-group. But the homomorphic image of a Goldie M-group need not be a Goldie M-group. For in the case of a Goldie ring, a homomorphic image of a Goldie ring need not be a Goldie ring [2]. A Goldie M-group is called *fully Goldie* if every homomorphic image of it is a Goldie M-group.

An M-subgroup H of G is called an *essential M-subgroup* of G if for each nonzero M-subgroup K of G, $H \cap K \neq 0$. We denote this by $H \leq_e G$. Clearly $G \leq_e G$ and $0 \leq_e G$ if and only if $0 = G$. Moreover if H, K are M-subgroups of G, $H \subseteq K \subseteq G$, then $H \leq_e G$ if and only if $H \leq_e K \leq_e G$.

If an M-subgroup H of G has no proper essential extension inside G (that is, if H and K are M-subgroups of G then $H \leq_e K < G$ implies $H = K$) then H is called a *closed M-subgroup* of G and we write $H \leq_c G$. Thus 0 and G are always closed M-subgroups of G.

An ordered family $\{G_1, G_2, \dots, G_n\}$ of M-subgroups of G is called an *independent family* if $(G_1 + \cdots + \widehat{G}_t + \cdots + G_n) \cap G_t = 0$, for $1 \leq t \leq n$. (The symbol $\widehat{\ }$ denotes omission of G_t.)

An M-group G is called finite dimensional provided G has no infinite direct sum of nonzero normal M-subgroups. To prove G is finite dimensional, it suffices to show that G has no infinite independent sequence of nonzero normal M-subgroups.

The *annihilator* $A(S)$ of a subset S of G is defined as

$$A(S) = \{m \in M \mid ms = 0 \text{ for all } s \in S\}.$$

In our discussion, M will always contain a zero element 0 such that $0g = 0$ for all $g \in G$. Thus $A(S) \neq \varnothing$ for all S. A nonzero M-subgroup H of G is called a *prime M-subgroup* of G if for every nonzero M-subgroup K of H, $A(K) = A(H)$. If, for each M-subgroup H of the M-group G, $A(G) = A(H)$, then G is called a *prime M-group.*

The collection

$$\mathscr{A}(G) = \{P \subseteq M \mid P = A(H) \text{ for some prime } M\text{-subgroup } H \text{ of } G\}$$

is the family of *associated operating subsets* of G. An M-group G is M-primary if $\mathscr{A}(G)$ is a singleton.

Let G be a Goldie M-group with closed normal M-subgroups $G_1, \dots, G_t$ such that

(1) $G_1 \cap \cdots \cap G_t = 0$ and $G_1 \cap \cdots \cap \widehat{G}_i \cap \cdots \cap G_t \neq 0$, for $i = 1, \dots, t$ and

(2) each quotient M-group G/G_i is an M-primary group with $\mathscr{A}(G/G_i) \neq \mathscr{A}(G/G_j)$ for $i \neq j$.

Then $G_1 \cap \cdots \cap G_t$ is called an *M-primary decomposition* of 0 of G.

In a unique factorisation domain one can express a non-unit as a finite product $p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ of positive powers of distinct primes. This result can be expressed in terms of ideals as $(a) = (p_1^{\alpha_1}) \cap \cdots \cap (p_t^{\alpha_t})$.

A similar decomposition of ideals of a commutative Noetherian ring is known. We extend some portions of this theory to Goldie M-groups.

Here we prove that if G is a fully Goldie M-group and if $\mathscr{A}(G) = X \cup Y$, $X \cap Y = \varnothing$, then in some cases there exists a closed normal M-subgroup G' of G such that $\mathscr{A}(G) = \mathscr{A}(G') \cup \mathscr{A}(G/G')$ where $\mathscr{A}(G') = X$ and $\mathscr{A}(G/G') = Y$. Another interesting result is that in some special cases $\mathscr{A}(G)$ is a finite collection. Moreover the very interesting and important result we prove here is the existence of an M-primary decomposition of 0 of such a Goldie M-group. If $G_1 \cap \cdots \cap G_t$ is such a decomposition of 0 then $\mathscr{A}(G) = \mathscr{A}(G/G_1) \cup \cdots \cup \mathscr{A}(G/G_t)$. Next, if G is a Goldie M-group where M is a right near-ring having no infinite direct sum of ideals and is such that $Z_1(G) = 0$ (where $Z_1(G) = \{g \in G \mid Ig = 0$ for some essential M-subgroup I of $G\}$) then the annihilators of subsets of G in M satisfy the d.c.c. and if M is a commutative near-ring then for any $x \in \bigcap_{P \in \mathscr{A}(G)} P$ there exists a $t \in \mathbb{Z}^+$ such that $x^t \in A(G)$.

2. Preliminaries

Following are some preliminary lemmas for use in the proofs of the main results. First we prove the following important lemma which will play a key role in our theory.

LEMMA 2.1. *If an M-group G has no infinite independent family of M-subgroups then it satisfies the a.c.c. on closed normal M-subgroups.*

PROOF. Suppose G does not satisfy the a.c.c. on closed normal M-subgroups. Then G has a chain $G_1 < G_2 < \cdots$ of closed normal M-subgroups of G. Since $G_n <_c G$, we have $G_n \not<_e G_{n+1}$. Therefore G_{n+1} must have a nonzero M-subgroup C_n such that $G_n \cap C_n = 0$. And this is true for each n. We claim for any $t \in \mathbb{Z}^+$, $i \leq t$, that

$$(C_1 + \cdots + \widehat{C}_i + \cdots + C_t) \cap C_i = 0.$$

Here

$$\begin{aligned}(C_1 + \cdots + \widehat{C}_i + \cdots + C_t) \cap C_i \\ \subseteq (G_2 + \cdots + G_i + C_{i+1} + \cdots + C_t) \cap C_i.\end{aligned}$$

Now if $c_i = g_2 + \cdots + g_i + c_{i+1} + \cdots + c_t$ (where $g_k \in G_k$, $k = 2, \ldots, i$, and $c_l \in C_l$, $l = i, i+1, \ldots, t$) is an element of $(G_2 + \cdots + G_i + C_{i+1} + \cdots + C_t) \cap C_i$ then $-c_i + g_2 + \cdots + g_i + c_{i+1} + \cdots + c_t = 0$ implies $g_2' + \cdots + g_i' - c_i + c_{i+1} + \cdots + c_{t-1} = -c_t$ (since $G_2, \ldots, G_i$ are normal subgroups of G), where $g_k' \in G_k$, $k = 2, \ldots, i$. Thus

$$c_t \in (G_2 + \cdots + G_i + G_{i+1} + \cdots + G_t) \cap C_t \subseteq G_t \cap C_t = 0.$$

Similarly $c_{t-1}, \ldots, c_i$ are all zeros.

So $(G_2 + \cdots + G_i + C_{i+1} + \cdots + C_t) \cap C_i = 0$ and therefore

$$(C_1 + \cdots + \widehat{C}_i + \cdots + C_t) \cap C_i = 0.$$

Hence $\{C_1, C_2, \ldots\}$ is an independent family of nonzero M-subgroups of G. Since G has no infinite independent family of M-subgroups we can not have a strictly ascending infinite sequence of closed normal M-subgroups of G. Thus G satisfies the a.c.c. on closed normal M-subgroups.

LEMMA 2.2. *Let G be an M-group satisfying the a.c.c. for annihilators of subsets of G in M. Then $A(G) \neq \varnothing$ if and only if $G = 0$.*

PROOF. Suppose $G = 0$. Then G has no prime M-subgroup. Hence $\mathscr{A}(G) = \varnothing$. Again if $G \neq 0$ consider $\mathscr{H} = \{A(G^*) \mid G^*$ is an M-subgroup of $G\}$. Since G is Goldie, $\mathscr{H}$ has a maximal element, say $A(N)$. Now let N' $(\neq 0)$ be an M-subgroup of G such that $N' \subseteq N$. Then $A(N') \supseteq A(N)$. So by maximality of $A(N)$ it follows that $A(N) = A(N')$. Thus N is a prime M-subgroup of G. Therefore $A(N) \in \mathscr{A}(G)$, that is, $\mathscr{A}(G) \neq \varnothing$.

LEMMA 2.3. *Let G be an M-group as above with an exact sequence*

$$0 \to G' \xrightarrow{g} G \xrightarrow{f} G'' \to 0.$$

Then $\mathscr{A}(G') \subseteq \mathscr{A}(G) \subseteq \mathscr{A}(G') \cup \mathscr{A}(G'')$.

PROOF. If $G = 0$ then $G' = 0$ and $G'' = 0$ and thus the result is true in this case. Assume $G \neq 0$. Since g is injective, G' is an M-subgroup of G. Therefore $\mathscr{A}(G') \subseteq \mathscr{A}(G)$. Let $A(N) \in \mathscr{A}(G)$ for some prime M-subgroup N of G. If $N \cap G' \neq 0$, then $A(N \cap G') = A(N)$ since N is a prime M-subgroup of G and $N \cap G'$, being an M-subgroup of the prime M-subgroup N, is also prime. Therefore $A(N \cap G') \in \mathscr{A}(G')$. Thus $A(N) \in \mathscr{A}(G')$. Now suppose $N \cap G' = 0$ and h is the restriction of f to N. Then h is injective, so $h(N) \cong N \subseteq G''$. Thus $A(N) \in \mathscr{A}(G'')$. Hence $\mathscr{A}(G) \subseteq \mathscr{A}(G') \cup \mathscr{A}(G'')$.

LEMMA 2.4. *Let N be a normal M-subgroup of an M-group G such that A is a closed M-subgroup of G with $N \leq A \leq_c G$. Then $A/N \leq_c G/N$.*

PROOF. If not, let $A/N \lneqq_e L/N \leq G/N$. Then $N \leq A \lneqq L \leq G$ and there is an M-epimorphism $f: L \to L/N$.

Here $f^{-1}(A/N) = A$. Since $A/N <_e L/N$, it follows that $A <_e L$ and this is not possible for $A \leq_c G$. Hence $A/N \leq_c G/N$. The following two lemmas are easy to prove.

LEMMA 2.5. *If G_1 and G_2 are two Goldie M-groups then*

$$\mathscr{A}(G_1 \oplus G_2) = \mathscr{A}(G_1) \cup \mathscr{A}(G_2).$$

LEMMA 2.6. *If G is a Goldie M-group and P, Q, N are M-subgroups of G, $N \triangleleft G$ such that $N \leq P, Q$ then $\mathscr{A}(P \cap Q/N) = \mathscr{A}(P/N \cap Q/N) \subseteq \mathscr{A}(P/N) \cap \mathscr{A}(Q/N)$.*

Let H and K be two M-subgroups of an M-group G such that $H \leq K \leq G$. Then H is M-essential in K if for any M-subgroup L $(\subseteq K)$, $H \cap L \neq 0$.

We now consider the set M of operators as a right near-ring with 1 such that $1g = g$, $(m_1 + m_2)g = m_1 g + m_2 g$, $(m_1 m_2)g = m_1(m_2 g)$ for $g \in G$, $m_1, m_2 \in M$ (in other words, G is a left near module over the right near-ring M).

LEMMA 2.7. *Let N and H be M-subgroups of an M-group G such that H is M-essential in N. If $a \in N$, $a \neq 0$, then there is an essential left M-subgroup L of M such that $La \neq 0$, $La \subseteq H$.*

PROOF. Let $L = \{m \in M \mid ma \in H\}$. Then L is left M-subgroup of M and $Ma \subseteq N$ (since N is an M-subgroup of G and $a \in N$). Also $Ma \neq 0$ (for $1 \in M$ Implies $a \in Ma$). Since H is M essential in N, we get $Ma \cap H \neq 0$. Let $h = ma$ $(\neq 0) \in H$. So $La \neq 0$. We now show that L is an essential left M-subgroup of M. Let I $(\neq 0)$ be a left M-subgroup of M. We claim that $I \cap L \neq 0$. Suppose $Ia = 0$. Then $Ia \subseteq H$. So $I \subseteq L$. Hence $I \cap L \neq 0$. And if $Ia \neq 0$ then Ia is an M-subgroup of G and $Ia \subseteq N$. Since H is M-essential in N, $Ia \cap H \neq 0$. Hence for some $x(\neq 0) \in I$, $xa \in H$. Thus $x \in L$. Therefore $I \cap L \neq 0$ which implies that L is an essential left M-subgroup of M.

We define

$$Z_1(G) = \{x \in G \mid Ax = 0 \text{ for some essential left } M\text{-subgroup } A \text{ of } M\}$$

and for any S $(\subseteq M)$,

$$\mathrm{r}_G(S) = \{g \in G \mid sg = 0 \text{ for all } s \in S\}.$$

LEMMA 2.8. *Let P, Q be annihilators of subsets of G in M such that $P \subseteq Q$ and P is M-essential in Q. If $Z_1(G) = 0$ then $P = Q$.*

PROOF. Let $q \in Q$, $q \neq 0$. Since $P \subseteq Q$ and P is M-essential in Q there exists an essential left M-subgroup L of M such that $Lq \in P$. $Lq \neq 0$ (Lemma 2.7). thus $Lq\mathrm{r}_G(P) = 0$. So $q\mathrm{r}_G(P) = 0$ implies $q \in A$ $(\mathrm{r}_G(P)) = P$ (since P is an annihilator of a subset of G in M). Hence $P = Q$.

We see that the $\mathbb{Z}$ groups $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathbb{Z}_{15}$, etc. are such $\mathbb{Z}$ Goldie groups that their proper quotients are all prime $\mathbb{Z}$-groups. And in a prime M-group all of its M-subgroups are prime and at least the M-group itself is a prime closed extension of each of its prime M-subgroups. Again $G = \mathbb{Z}_{30}$ is such a Goldie $\mathbb{Z}$-group that its $\mathbb{Z}$-subgroups are

$$A_1 = \{0, 2, 4, \dots, 28\}, \quad A_2 = \{0, 3, 5, \dots, 27\},$$
$$A_3 = \{0, 5, 10, \dots, 25\}, \quad A_4 = \{0, 6, 12, \dots, 24\},$$
$$A_5 = \{0, 10, 20\}, \quad \text{and} \quad A_6 = \{0, 15\}$$

of which $A_4 < A_2 <_c G$, $A_5 < A_3 <_c G$ and $A_6 < A_3 <_c G$. So by Lemma 2.4, $A_2/A_4 <_c G/A_4$, $A_3/A_5 <_c G/A_5$ and $A_3/A_6 <_c G/A_6$.

Here each of A_2/A_4, A_3/A_5 and A_3/A_6 is a prime M-subgroup. Thus each of a closed extension of itself which is prime. And the remaining quotients G/A_1, G/A_2 and G/A_3 are all primes.

These are such Goldie M-groups that any prime M-subgroups N/G' of G/G' has a prime closed normal extension T/G' such that $G' \leq N \leq T \leq_c G$.

In what follows our Goldie M-group G will be of this type.

3. Main results

THEOREM 3.1. *Let G be a Goldie M-group described as above. If the set $\mathscr{A}(G)$ is a union of two disjoint sets X and Y, then there exists a normal closed M-subgroup G' such that $\mathscr{A}(G') = X$ and $\mathscr{A}(G/G') = Y$.*

PROOF. Let $\mathscr{H} = \{N \trianglelefteq_c G \mid \mathscr{A}(N) \subseteq X\}$. As 0 is a closed normal M-subgroup of G and $\mathscr{A}(0) = \varnothing$, we have $\mathscr{H} \neq \varnothing$ (since $\varnothing \subseteq X$).

Since G is M-Goldie, by Lemma 2.1, $\mathscr{H}$ has a maximal element, (say) G'. Also, $X \cup \mathscr{A}(G/G') \supseteq X \cup Y$ (Lemma 2.3). Since $X \cap Y = \varnothing$, we have

$Y \subseteq \mathscr{A}(G/G')$. Suppose $\mathscr{A}(G/G') \nsubseteq Y$. Then there exists a prime M-subgroup N/G' of G/G' such that $A(N/G') \in \mathscr{A}(G/G')$ and $A(N/G') \notin Y$. Moreover by hypothesis there is a prime closed M-normal extension T/G' such that $N/G' \leq T/G' \trianglelefteq_c G/G'$ and $G' \leq N \leq T \trianglelefteq_c G$. Thus T is a closed normal M-subgroup of G. Since T/G' is nonzero, $G' \subseteq T$ and $A(T/G') = A(N/G')$. Since T/G' is prime, $\mathscr{A}(T/G')$ is a singleton set, say $\{P\}$. We write simply P. Thus $\mathscr{A}(T/G') = P$ and $P \notin Y$. Again by Lemma 2.4, $\mathscr{A}(T) \subseteq \mathscr{A}(G') \cup \mathscr{A}(T/G')$. Since $\mathscr{A}(G') \subseteq X$ and $\mathscr{A}(T/G') = P$, we get $\mathscr{A}(T) \subseteq X \cup P$. Also $T \subseteq G$ and $\mathscr{A}(G) = X \cup Y$ give $\mathscr{A}(T) \subseteq X \cup Y$.

So $P \notin Y$ gives $\mathscr{A}(T) \subseteq X$. Thus $T \in \mathscr{H}$ and this contradicts the maximality of G. Therefore $\mathscr{A}(G/G') \subseteq Y$. Thus $X \cup Y \subseteq \mathscr{A}(G') \cup Y$ and $X \cap Y = \varnothing$ gives $X \subseteq \mathscr{A}(G')$.

Theorem 3.2. *Let G be a Goldie M-group as above. Then $\mathscr{A}(G)$ is finite.*

Proof. We assume the opposite, that is, that $\mathscr{A}(G) = \{P, Q, R, \ldots\}$ is infinite.

If $\mathscr{A}(G) = P \cup Y$ and $P \notin Y$ (we write P for $\{P\}$) then by Theorem 3.1 there exists a closed normal M-subgroup G' of G such that $\mathscr{A}(G') = P$, $\mathscr{A}(G/G') = Y$. Thus

$$\mathscr{A}(G) = \mathscr{A}(G') \cup \mathscr{A}(G/G').$$

Since $Q \in \mathscr{A}(G)$ we have $Q \in \mathscr{A}(G/G')$, so for some prime M-subgroup B'/G' of G/G', $A(B'/G') = Q$. Thus $\mathscr{A}(B'/G') = Q$. By hypothesis there is a prime extension G''/G' such that $B'/G' < G''/G' \trianglelefteq_c G/G'$ and $G' \leq B' \leq G'' \trianglelefteq_c G$. Hence $A(G''/G') = A(B'/G') = Q$. Therefore $\mathscr{A}(G''/G') = \mathscr{A}(B'/G') = Q$. And by Lemma 2.3, $\mathscr{A}(G'') \subseteq \mathscr{A}(G') \cup \mathscr{A}(G''/G')$. It follows that $\mathscr{A}(G'') \subseteq \{P, Q\}$. Also by Lemma 2.3, $\mathscr{A}(G) \subseteq \mathscr{A}(G'') \cup \mathscr{A}(G/G'')$. Therefore $\mathscr{A}(G) \subseteq \{P, Q\} \cup \mathscr{A}(G/G'')$, that is, $R \in (G/G'')$.

In a like manner we get another closed normal M-subgroup G''' of G such that $G' < G'' < G'''$ and for $S \in \mathscr{A}(G)$, $S \in \mathscr{A}(G/G''')$. Since $\mathscr{A}(G)$ is infinite, we get a strictly ascending infinite sequence of closed normal M-subgroups, which contradicts the Goldie character of G because of Lemma 2.1. Hence $\mathscr{A}(G)$ is finite.

Theorem 3.3. *Let the M-group G be fully Goldie as above.*

(I) *There exists an M-primary decomposition of* 0 *in* G.

(II) *If $G_1 \cap \cdots \cap G_t$ is an M-primary decomposition of* 0 *in G then* $\mathscr{A}(G) = \mathscr{A}(G/G_1) \cup \cdots \cup \mathscr{A}(G/G_t)$.

Proof. (I) By the above theorem, $\mathscr{A}(G)$ is finite.

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Let $\mathscr{A}(G) = \{P_1, \ldots, P_t\}$. Since $\mathscr{A}(G)$ is expressible as a union of two disjoint sets $\{P_1, \ldots, \widehat{P}_i, \ldots, P_t\}$ and $\{P_i\}$, by Theorem 3.1, we get closed normal M-subgroups $G_1, \ldots, G_t$ of G such that for each i,

$$\mathscr{A}(G_i) = \{P_1, \ldots, \widehat{P}_i, \ldots, P_t\} \text{ and } \mathscr{A}(G/G_i) = \{P_i\}.$$

Also, for each i, G/G_i is M-primary and $\mathscr{A}(G/G_i) \neq \mathscr{A}(G/G_j)$ for $i \neq j$ and $\mathscr{A}(G_1 \cap \cdots \cap G_i) \subseteq \mathscr{A}(G_1) \cap \cdots \cap \mathscr{A}(G_t)$. Since clearly $\mathscr{A}(G_1) \cap \cdots \cap \mathscr{A}(G_t) = \varnothing$, we then have $\mathscr{A}(G_1 \cap \cdots \cap G_t) = \varnothing$ and therefore by Lemma 2.2, $G_1 \cap \cdots \cap G_t = 0$. If possible let $G_1 \cap \cdots \cap \widehat{G}_i \cap \cdots \cap G_t = 0$, that is, $\bigcap_{j \neq i} G_j = 0$, for some i, $1 \leq i \leq t$. Then we get an M-homomorphism.

$$\alpha : G \to \bigoplus_{j \neq i} G_j, \quad g \mapsto (g + G_1, \ldots, \widehat{g + G_i}, \ldots, g + G_t).$$

We note that $\operatorname{Ker} \alpha = \{g \mid g \in \bigcap_{j \neq i} G_j = 0\} = 0$. Thus α is an embedding and hence $\mathscr{A}(G) \subseteq \mathscr{A}(\bigoplus_{j \neq i} G/G_j)$. Since G is fully Goldie, each G/G_i is Goldie. so it follows from Lemma 2.5 that for each i, $\mathscr{A}(G) \subseteq \bigcup_{j \neq i} \mathscr{A}(G/G_j)$, that is, $\mathscr{A}(G) \subseteq \{P_1, \ldots, \widehat{P}_i, \ldots, P_t\}$ which is absurd. Hence $\bigcap_{j \neq i} G_j \neq 0$.

(II) Next suppose that $\bigcap_{j=1}^{t} G_j$ is an M-primary decomposition of 0 in G. Then the map

$$\alpha : G \to \bigoplus_{j=1}^{t} G/G_j, \quad g \mapsto (g + G_1, \ldots, g + G_t)$$

is an embedding, which means that $\mathscr{A}(G) \subseteq \mathscr{A}(\bigoplus G/G_j)$ and hence $\mathscr{A}(G) \subseteq \bigcup \mathscr{A}(G/G_j)$. To see the opposite inclusion consider the M-homomorphism

$$\beta : \bigcap_{j \neq i} G_j \to G/G_i, \quad g \mapsto g + G_i.$$

Now $\operatorname{Ker} \beta = \{g \mid g \in \bigcap G_j\} = 0$. Thus $\mathscr{A}(\bigcap_{j \neq i} G_j) \subseteq \mathscr{A}(G/G_i)$ and by Lemma 2.2, $\mathscr{A}(\bigcap_{j \neq i} G_j) \neq \varnothing$. Since $\mathscr{A}(G/G_i)$ is a singleton, we get $\mathscr{A}(\bigcap_{j \neq i} G_j) = \mathscr{A}(G/G_i)$ for each i. Hence

$$\bigcup_{j=1}^{t} \mathscr{A}(G/G_j) = \bigcup_{i=1}^{t} (\mathscr{A}(\bigcap_{j \neq i} G_j)$$

and since $\mathscr{A}(\bigcap_{j \neq i} G_j) \subseteq \mathscr{A}(G)$ for each i, we finally get $\bigcup_{j=1}^{t} \mathscr{A}(G/G_j) \subseteq \mathscr{A}(G)$. Thus $\mathscr{A}(G) = \bigcup_{j=1}^{t} \mathscr{A}(G/G_j)$.

We now give two results on a Goldie M-group when the operating set M is a right near-ring with no infinite direct sum of left ideals and $Z_1(G) = 0$. Theorem 7 of Oswald [5] follows as a corollary to the following result in the case of a regular left Goldie near-ring [3].

THEOREM 3.4. *Let G be a Goldie M-group with $Z_1(G) = 0$ as above and such that an essential left ideal of M is essential as a left M-subgroup also. Then the annihilators of subsets of G in M satisfy the d.c.c.*

PROOF. Let $B = A(Y)$, $C = A(X)$, $X, Y \subseteq G$. Then if $X \subseteq Y$ we have $B \subseteq C$. Suppose $B \subset C$. Then by Lemma 2.8, there exists a left M-subgroup D of M such that $D \subseteq C$, $B \cap D = 0$. Thus if in the descending chain $A(S_1) \supseteq A(S_2) \supseteq \cdots$ we have $A(S_k) \supsetneq A(S_{k+1})$, then there exists left M-subgroups P_k such that $P_k \subseteq A(S_k)$ and $A(S_{k+1}) \cap P_k = 0$. Again we choose a left ideal X_k such that $A(S_{k+1}) \cap X_k = 0$ and X_k is maximal for this.

Being the left annihilator of S_{k+1} in M, $A(S_{k+1})$ is a left ideal of M. So $A(S_{k+1}) + X_k$ is a left ideal of M. So it is essential as a left M-subgroup. Therefore $P_k \cap (A(S_{k+1}) + X_k) \neq 0$ (we write A_k for $A(S_k)$). Now let $(0 \neq)$ b_k $(\in P_k) = a_{k+1} + x_k$, $a_{k+1} \in A_{k+1}$, $x_k \in X_k$. This implies $x_k = -a_{k+1} + b_k \in A_{k+1} + P_k \subseteq A_k + P_k \subseteq A_k \cap X_k$ $(= C_k$, say). Now if x_k were 0, we would have $a_{k+1} = b_k \in P_k \cap A_{k+1} = 0$. So $x_k \neq 0$. Therefore we get a nonzero left ideal C_k and $C_k \cap A_{k+1} = 0$. An infinite descending chain of left annihilators of subsets of G in M gives an infinite direct sum of left ideals of M. Since M has no infinite direct sum of left ideals, the descending chain $A_1 \supseteq A_2 \supseteq \cdots$ is a finite one. Now we prove our last result of this paper, in the case of a finite dimensional commutative near-ring with 1.

THEOREM 3.5. *Let G be a Goldie M-group where M is a commutative near-ring with 1 having no infinite direct sum of ideals and is such that $Z_1(G) = 0$. Then for any $x \in \bigcap_{p\in\mathscr{A}(G)} P$, there exists $t \in \mathbb{Z}^+$ such that $x^t \in A(G)$.*

PROOF. Let $x \in \bigcap_{p\in\mathscr{A}(G)} P$. Then for every positive integer i, we get M-homomorphisms $\varphi_i : G \to G$, $g \mapsto x^i g$, $i = 1, 2, \ldots$. Clearly $\operatorname{Ker}\varphi_i \subseteq \operatorname{Ker}\varphi_{i+1}$. In other words $\mathrm{r}_G(x^i) \subseteq \mathrm{r}_G(x^{i+1})$ which gives

$$A(\mathrm{r}_G(x^i)) \supseteq A(\mathrm{r}_G(x^{i+1})).$$

By Theorem 3.4, we get $A(\mathrm{r}_G(x^r)) = A(\mathrm{r}_G(x^{t+1}))$ for some $t \in \mathbb{Z}^+$. Then $\mathrm{r}_G(A(\mathrm{r}_G(x^t))) = \mathrm{r}_G(A(\mathrm{r}_G(x^{t+1})))$, that is, $\mathrm{r}_G(x^t) = \mathrm{r}_G(x^{t+1})$ on $\operatorname{Ker}\varphi_t = \operatorname{Ker}\varphi_{t+1}$. Now we consider the M-homomorphism

$$f : x^t G \to x^t G, \quad x^t g \mapsto x^{t+1} g.$$

If $x^{t+1} g = x^{t+1} g'$ then $x^{t+1}(g - g') = 0$ so $g - g' \in \operatorname{Ker}\varphi_{t+1} = \operatorname{Ker}\varphi_t$ and thus $x^t g = x^t g'$. Hence f is injective. Now $x^t G \leq G$ so $\mathscr{A}(x^t G) \subseteq \mathscr{A}(G)$.

If $x^tG \neq 0$ then $\mathscr{A}(x^tG) \neq \varnothing$. Then there exists a nonzero M-subgroup G' of x^tG such that $A(G') \in \mathscr{A}(x^tG)$. Since $x \in P$ for each $P \in \mathscr{A}(G)$, we get $x \in P$ for each $P \in \mathscr{A}(x^tG)$. So $x \in A(G')$. And this gives that $xG' = 0$, that is, $f(G') = 0$. Since f is injective, it follows that $G' = 0$, a contradiction. Hence $x^tG = 0$, that is, $x^t \in A(G)$.

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