DISTANCE SETS

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1. Introduction. With each set of points S of a distance space there is associated a set of non-negative real munbers $D(S)$ called the distance set of S. The number x is an element of $D(S)$ if and only if x is a distance between some pair of points of S. The number zero is always an element of any distance set and no two distinct elements are equal.

Sierpinski [5], Steinhaus [6], Piccard [4], and many others have considered the relationships existing between S and $D(S)$ for subsets of various spaces, particularly the E_n. Most of these investigations have been concerned with the influence of measure and related properties of S on the associated distance set. For example, it has been shown [6] that the distance set of a set of positive Lebesgue measure must contain an interval with one end point zero. Miss Piccard, on the other hand, has considered the converse problem of investigating the nature of spaces with prescribed distance sets. It is with the sharpening and extending of some of her results and substantial simplification of some of her proofs that this paper is concerned. Theorem 4.2 may be regarded as the principal contribution of this paper, but for the sake of completeness and because of the relative inaccessibility of [4] we have included the simplified proofs of certain basic theorems.

2. Preliminary remarks. If p and q are points of a distance space, the distance between the two elements will be denoted by pq. If P and Q are two subsets, $D(P, Q)$ will represent the set of all distances pq, with $p \in P$ and $q \in Q$. The concept of distance set gives rise to a mapping of the subsets S of a given space onto subsets N of real non-negative numbers, ($D(S) = N$) as well as an inverse mapping of certain subsets of the non-negative numbers on the subsets of the space ($D^{-1}(N) = S$). Of course the inverse mapping need not be, and indeed rarely is, single valued. A subscript will serve to distinguish sets having the same distance set, i.e., $D_x^{-1}(N)$ is a particular set of the space with distance set N.

A sequence of non-negative numbers particularly suited to our purposes is one in which $a_{i+1} > 2a_i$. A finite set of numbers which may be so ordered is called an isosceles set, and an infinite set an isosceles sequence. It is readily seen that any metric space whose distance set is isosceles has all of its triangles isosceles with the base as the shortest side. It is also apparent that any subset of an isosceles sequence or set has the isosceles property.

We proceed now to a consideration of the following questions which seem fundamental in any systematic investigation of distance sets. In what spaces,

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if any, can an arbitrary set of non-negative numbers including zero be realized as a distance set? What are the corresponding results for denumerable and finite sets? Are there sets of non-negative numbers including zero which cannot be realized in specified spaces, in particular, the Euclidean and Hilbert spaces?

3. Infinite distance sets.

THEOREM 3.1. *An arbitrary set of non-negative numbers including zero is the distance set for some metric space.*

Proof. Let N be such a set and construct a space whose elements are the numbers of N and with distance defined as follows: $pq = \max(p, q)$ if $p \neq q$, and $pq = 0$ if $p = q$. The space is easily seen to be metric with distance set N.

That this result cannot be substantially improved is shown by the following theorem.

THEOREM 3.2. *There exist sets of non-negative real numbers including zero which are not distance sets for any separable metric space.*

Proof. Consider an uncountable set of non-negative numbers, including zero, whose positive numbers are bounded away from zero. Any space with this as distance set is uncountable and discrete, hence not separable.

In order to establish that any countable set of non-negative numbers including zero is a distance set for some subset of Hilbert space, we need the following lemma.

LEMMA 3.1. *If* $0 < a_1 < a_2 \ldots < a_k$, *and*

$$D(k) \equiv \begin{vmatrix} 0 & 1 & 1 & 1 & . & . & 1 \\ 1 & 0 & a_1 & a_2 & . & . & a_k \\ 1 & a_1 & 0 & a_2 & . & . & a_k \\ 1 & a_2 & a_2 & 0 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & 0 & a_k \\ 1 & a_k & a_k & . & . & a_k & 0 \end{vmatrix}$$

then $\operatorname{sgn} D(k) = (-1)^{k+1}$.

Proof. Let $a_1 \cdot a_2 \cdot a_3 \ldots \cdot a_k = A$. By subtracting appropriate rows and columns, the determinant can be brought into the following form:

$$D(k) = \begin{vmatrix} -2a_1 & a_1 & 0 & . & 0 \\ a_1 & -2a_2 & a_2 & . & 0 \\ 0 & a_2 & -2a_3 & . & 0 \\ . & . & . & . & a_{k-1} \\ 0 & . & . & a_{k-1} & -2a_k \end{vmatrix}$$

$$= -A \cdot \begin{vmatrix} -2 & 1 & 0 & . & 0 \\ a_1/a_2 & -2 & 1 & . & 0 \\ 0 & a_2/a_3 & -2 & . & . \\ . & . & . & -2 & 1 \\ . & . & . & a_{k-1}/a_k & -2 \end{vmatrix} = -A \cdot Q(k)$$

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The problem then is one of showing that the sign of the kth order determinant $Q(k)$ is $(-1)^k$. To do this we first establish the fact that $|Q(r)| > |Q(r-1)|$. Noting that the relation is valid for $r = 2$, we assume it true for all $n < r$. The following recursion is easily verified:

$$Q(r) = -2Q(r-1) - \frac{a_{r-1}}{a_r} Q(r-2).$$

Thus $|Q(r)| - |Q(r-1)| = |2Q(r-1) - \frac{a_{r-1}}{a_r} Q(r-2)| - |Q(r-1)|$. If $Q(r-1)$ and $Q(r-2)$ have the same sign, it follows at once that $|Q(r)| > |Q(r-1)|$. If the signs are opposite, we have

$$\begin{aligned}|Q(r)|-|Q(r-1)| &= 2|Q(r-1) - \frac{a_{r-1}}{a_r} Q(r-2)|-|Q(r-1)| \\ &= |Q(r-1)|-\frac{a_{r-1}}{a_r} |Q(r-2)|,\end{aligned}$$

which is greater than zero, since $a_r > a_{r-1}$ and by the inductive hypothesis $|Q(r-1)| > |Q(r-2)|$.

We return now to the problem of establishing the sign of $Q(k)$, assuming that the sign of $Q(r)$ is $(-1)^r$ for $r < k$. Since $Q(k) = -2\,Q(k-1) - \frac{a_{k-1}}{a_k} \cdot Q(k-2)$, it follows that $\operatorname{sgn} Q(k) = -\operatorname{sgn}\,[2Q(k-1) + \frac{a_{k-1}}{a_k} Q(k-2)]$, and in view of the fact that $|Q(k-1)| > |Q(k-2)|$, we have $\operatorname{sgn} Q(k) = -\operatorname{sgn} Q(k-1) = (-1)^k$. With the observation that the relation is valid for $k = 1, 2$, the induction proof is complete and the lemma is proved.

Remark. It is interesting to note that $Q(k)$ is essentially a continuant. (See any treatise on determinants.)

THEOREM 3.3. *Any countable set of non-negative real numbers including zero is a distance set for some subset of Hilbert space.*

Proof. Let N be such a set and metrize it as in Theorem 3.1. Consider any finite subset of this space consisting of the numbers $0 < a_1 < a_2 < a_3 \ldots < a_k$. The space will be imbeddable congruently in Hilbert space [2, p. 68] if and only if for each k, Δ has the sign $(-1)^{k+1}$ or is zero, where

$$\Delta(k) = \begin{vmatrix} 0 & 1 & 1 & 1 & . & . & 1 \\ 1 & 0 & a_1^2 & a_2^2 & . & . & a_k^2 \\ 1 & a_2^2 & 0 & a_2^2 & . & . & a_k^2 \\ 1 & a_3^2 & a_2^2 & 0 & . & . & . \\ . & . & . & . & . & . & . \\ 1 & . & . & . & . & 0 & a_k^2 \\ 1 & a_k^2 & a_k^2 & . & . & a_k^2 & 0 \end{vmatrix}$$

That Δ has the appropriate sign follows from the lemma.

THEOREM 3.4. *There exist countable sets of non-negative real numbers, including zero, which are not distance sets in any* E_n.

Proof. We will prove that no isosceles sequences $\{a_i\}$ is realizable as a distance set in any E_n. Assume this to be the case for $k < n$ and proceed by induction. Suppose S a subset of E_n with an isosceles sequence as distance set. Let p and q be elements of S such that pq is a minimum. The points of $S - p - q$ are equidistant from p and q and hence are in an E_{n-1}. Furthermore, since S is denumerably infinite, $S - p - q$ is also. If $D(S - p - q)$ were finite, $S - p - q$ would be bounded and hence have an accumulation point. This is impossible and $D(S - p - q)$ is infinite. But $D(S - p - q)$ is a subsequence of an isosceles sequence and is thus itself an isosceles sequence. This contradicts the inductive assumption. To complete the induction, we note that no isosceles sequence is realizable as a distance set in E_1.

Remark. While many other examples might be given of countable sets not realizable as distance sets in any E_n, the following seem particularly worthy of note. The odd integers and zero is not the distance set in any E_n. For Erdös and Anning [1] have shown that any infinite set in E_n all of whose distances are integers is a subset of the line. But it is seen at once that the odd integers and zero cannot be realized as a distance set on the line. As a second example, consider the sequence $1 + 2^{-n}\epsilon$. For "small" ϵ these numbers are "almost equal". But there is no "almost equilateral" infinite set in E_n.

4. Finite distance sets.

THEOREM 4.1. *Any set of n positive numbers and zero is the distance set for a subset of* E_n.

Proof. Let $0 < a_1 < a_2 \ldots < a_n$ be the set of numbers and metrize it as in Theorem 3.1. This space is congruently contained in E_n if and only if the kth ordered bordered principal minors of the determinant $\Delta(n)$ have the sign $(-1)^{k-1}$ or are zero [2, p. 64]. That this is the case follows as in Theorem 3.3.

We are thus assured that any k distinct positive numbers and zero, $k \leqslant n$, can serve as the distance set for some subset of E_n. On the other hand, a set of $\frac{1}{2}n(n-1) + 1$ "almost equal" positive numbers and zero cannot serve as the distance set for a subset of the E_n since any almost equilateral subset of the E_n consists of at most $n + 1$ points. The question as to whether there exists a set of $n + 1$ positive numbers and zero which is not realizable as a distance set in the E_n naturally arises. In order to establish an affirmative answer to this question we need the following lemmas.

LEMMA 4.1. *If the distance set of a finite metric space is isosceles, the space may be decomposed into two non-null sets P and Q with $D(P, Q) = d$ where d is the diameter of M.*

Proof. Let Q be a maximum set such that $D(p, Q) = d$ where p is an element of $M - Q$, and suppose p' any other element of $M - Q$. From the

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triangle inequality it follows that $pq = p'q = d$. Thus Q and $M - Q \equiv P$ is the desired decomposition.

LEMMA 4.2. *If the distance set of a metric space of n points is an isosceles set, it consists of at most n numbers.*

Proof. Proceeding by induction we note that the theorem is true for $n = 1, 2, 3$. Assume it true for all $k < n$. Let P and Q be the two sets of the decomposition of M assured by Lemma 4.1. Suppose P consists of k_1 and Q of k_2 points. Then by the inductive hypothesis, $D(P)$ contains at most k_1 and $D(Q)$ at most k_2 numbers including zero. Thus $D(P)$ and $D(Q)$ together contain at most $k_1 + k_2 - 2 = n - 2$ distinct positive numbers and M has at most $n - 1$ positive numbers.

COROLLARY. *If the distance set of a metric space is an isosceles set of n numbers, the space consists of at least n points.*

DEFINITION. $S_{n,r}$ will denote the surface of the sphere of radius r in E_n.

LEMMA 4.3. *If a set M is a subset of an $S_{n,r}$, but not of any $S_{n-1,r}$, and if the center of $S_{n,r}$ is interior to the convex cover of M, then $r^2 \leqslant \dfrac{n}{2(n+1)} d^2$, where d is the diameter of M.*

Proof. Clearly there are in M vertices of a non-degenerate n-dimensional simplex which contains the centre O of $S_{n,r}$. Using this as a reference simplex, we introduce a barycentric coordinate system and employ a formula of Lagrange. If $m_1, m_2, \ldots, m_{n+1}$ are coordinates of a point Q with $\sum m_i = 1$; $A_1, A_2, \ldots, A_{n+1}$ are vertices of the reference simplex; $a_{ij} = \overline{A_iA_j}$ and P is any other point of E_n, then $\overline{PQ}^2 = \sum \overline{PA}_i^2 m_i - \sum_{i<j} a_{ij}^2 m_i m_j$ (indices from 1 to $n + 1$ in all cases). Let $P = Q = O$. Then $0 = r^2 - \sum_{i<j} a_{ij}^2 m_i m_j$, and $r^2 < a^2 \sum_{i<j} m_i m_j$, a the maximum edge. The numbers m_i are all positive since O is interior to the simplex, and it is easily shown that the maximum value of $\sum_{i<j} m_i m_j$ is $n/2(n + 1)$, from which it follows that $r^2 \leqslant [n/2(n+1)]a^2 \leqslant [n/2(n+1)]d^2$, and the lemma is proved.

LEMMA 4.4. *If the distance set of a non degenerate n-dimensional simplex is isosceles, the circumcenter is a point of the simplex.*

Proof. Proceeding by induction and noting that the theorem is true for $n = 1, 2$, we assume it true for all $k < n$. Suppose P and Q the sets of the decomposition of the simplex (vertices) assured by Lemma 4.1. The points of P and Q form non degenerate simplices each of dimension less than n and hence the circumcenters O_p and O_q of these "faces" are, by the inductive hypothesis, points of the respective faces. Furthermore, the feet of the perpendiculars from the points of P onto the face determined by Q coincide in O_q

since all the distances pq are equal. Thus O_q is equidistant from the points of P and also from the points of Q, and similarly for O_p. Since O_p and O_q are each in the equidistant locus of the points of P as well as that of the points of Q, the line joining them is also.

Consider now the function px/qx where p is a fixed point of P, q a fixed point of Q, and x a variable point on the closed segment O_pO_q. When $x = O_p$, it follows from Lemma 4.3 and the Pythagorean theorem applied to the triangle pO_pq that $pO_p/qO_p < 1$, while from similar considerations $pO_q/qO_q > 1$ when $x = O_q$. Thus for some point O of the segment O_pO_q, $pO = qO$ and O is the center of the circumsphere of the simplex.

COROLLARY. *If r is the circumradius of a non degenerate n-dimensional simplex with isosceles distance set, then $r^2 < [n/2(n-1)]a^2$ where a is the maximum edge of the simplex.*

THEOREM 4.2. *There exist sets of $n + 1$ positive numbers and zero which are not distance sets for any subset of E_n.*

Proof. Let $\{a_i\} \equiv 0 < a_1 < a_2 \ldots < a_{n+1}$ be an isosceles set of numbers. Proceeding by induction, we will show that such a set is not realizable in E_n. We note that the theorem holds for $n = 1$ and assume then that no set in E_k, $k < n$, can have an isosceles distance set of $k + 2$ numbers.

Suppose S is a set in E_n with $\{a_i\}$ as distance set, and let P and Q be the subsets of the decomposition assured by Lemma 4.1. By the Corollary to Lemma 4.2, S contains at least $n + 2$ points. Suppose the points of P lie irreducibly in an m dimensional subspace. Since the points of Q are equidistant from those of P, they are in an E_{n-m}. By the inductive hypothesis, then $D(P)$ consists of at most m positive numbers and $D(Q)$ at most $n - m$. It follows, since $D(S)$ has n + 1 positive numbers, that $D(P)$ and $D(Q)$ contain exactly m and $n - m$ positive numbers respectively, and that neither contains $D(P, Q) = a$.

The feet of the perpendiculars from the points of Q onto the E_m containing P coincide in a point, say O, equidistant from the points of P. Furthermore, O is equidistant from the points of Q and is in the E_{n-m} containing Q (irreducibly). Thus O is the centre of an m-dimensional sphere containing the points of P and also the centre of an $n - m$ dimensional sphere containing the points of Q. Among the points of P are the vertices of a proper m-dimensional simplex and, by the Corollary to Lemma 4.4, its circumradius is less than the largest edge. Thus $r_p < \frac{1}{2}a$. Similarly, $r_q < \frac{1}{2}a$. But in the triangle pOq, $p \in P$ and $q \in Q$, $pO + Oq \geqslant pq$, that is, $r_p + r_q > a$, a contradiction.

COROLLARY. *There exist sets of $n + k$ numbers, $k = 2, 3, 4, \ldots$, including zero, which are not distance sets for any subset of E_n.*

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5. Concluding remarks. Theorem 4.2 focuses attention on sets of points realizable as distance sets in E_{n+1}, but not in E_n. This leads naturally to the definition.

DEFINITION. A set N of positive numbers and zero is said to be *irreducibly n-dimensional* relative to Euclidean spaces if it is realizable as a distance set in E_n, but not in E_{n-1}; i.e., $D_x^{-1}(N) \subset E_n$ for some x, but $D_x^{-1}(N) \not\subset E_{n-1}$ for any x.

DEFINITION. A set N is said to be *properly n-dimensional* relative to Euclidean spaces if $D_x^{-1}(N) \subset E_n$ for all x and $D_x^{-1}(N) \not\subset E_{n-1}$ for any x.

DEFINITION. A set N is said to be *rigid* relative to a space S provided $D_x^{-1}(N) \subset S$ for all x and $D_x^{-1}(N) \subset S$ and $D_y^{-1}(N) \subset S$ implies $D_x^{-1}(N)$ is congruent to $D_y^{-1}(N)$.

Thus any isosceles set of $n+1$ numbers including zero is irreducibly n-dimensional relative to Euclidean spaces, while from the Anning-Erdös theorem, it follows that the even integers, for example, are properly one-dimensional. On the other hand, the distance set of zero together with the integral powers of ten is a D set rigid relative to Euclidean spaces, being realizable on the line in "essentially" only one way. These examples give substance to the definitions, but it would be interesting to know if there exist non-degenerate rigid sets, as well as proper sets, in all dimensions. While we have not yet established the existence of such sets, the following theorem is pertinent.

THEOREM 5.1. *No finite set of non-negative numbers is proper relative to Euclidean spaces.*

Proof. Let N consist of $n+1$ numbers including zero and adjoin to N a second zero, forming the set N^*. Metrize N and N^* as in Theorem 3.1 except for the distance between the two zeros which will be the smallest positive number in N. It is a simple matter to verify that the $(n+2)$-tuple is congruently imbeddable in E_{n+1}, but not E_n. Thus the metrized N and N^* have the same distance sets, but lie in different dimensions.

COROLLARY. *No finite set of non-negative numbers is rigid relative to Euclidean spaces.*

A similar argument is employed to establish the following theorem.

THEOREM 5.2. *No countable set of non-negative numbers whose positive numbers have a minimum is rigid relative to Hilbert space.*

It should be observed that while we have operated largely in Euclidean spaces, many of the results, with obvious modifications, are valid in any locally Euclidean space (i.e., Riemannian), in particular, hyperbolic and elliptic spaces. A more complete analysis of sets of distances peculiar to various familiar spaces is in progress.

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