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Abstract

Let $\mathfrak{F}_n$ be the set of all cuspidal automorphic representations π of GL_n with unitary central character over a number field F. We prove the first unconditional zero density estimate for the set $\mathcal{S} = \{L(s, \pi \times \pi') : \pi \in \mathfrak{F}_n\}$ of Rankin–Selberg L-functions, where $\pi' \in \mathfrak{F}_{n'}$ is fixed. We use this density estimate to establish: (i) a hybrid-aspect subconvexity bound at $s = \frac{1}{2}$ for almost all $L(s, \pi \times \pi') \in \mathcal{S}$; (ii) a strong on-average form of effective multiplicity one for almost all $\pi \in \mathfrak{F}_n$; and (iii) a positive level of distribution for $L(s, \pi \times \widetilde{\pi})$, in the sense of Bombieri–Vinogradov, for each $\pi \in \mathfrak{F}_n$.

1. Introduction and statement of the main result

Let $\mathbb{A}_F$ be the ring of adèles over a number field F with absolute norm $\mathrm{N} = \mathrm{N}_{F/\mathbb{Q}}$ and absolute discriminant D_F. Let $\mathfrak{F}_n$ be the set of cuspidal automorphic representations $\pi = \bigotimes_v \pi_v$ of $\mathrm{GL}_n(\mathbb{A}_F)$, where the (restricted) tensor product runs over all places of F and π is normalized so that its central character is trivial on the diagonally embedded copy of the positive reals. Let $\mathfrak{q}_\pi$ be the arithmetic conductor of π, $C(\pi) \geq 1$ the analytic conductor of π (see (3.4)), and $\mathfrak{F}_n(Q) = \{\pi \in \mathfrak{F}_n : C(\pi) \leq Q\}$. The analytic conductor $C(\pi)$ is a useful measure for the arithmetic and spectral complexity of π. Our normalization for the central characters ensures that $|\mathfrak{F}_n(Q)|$ is finite.

Given $\pi \in \mathfrak{F}_n$ and $\pi' \in \mathfrak{F}_{n'}$, let $L(s, \pi \times \pi')$ be the associated Rankin–Selberg L-function, and let $\widetilde{\pi} \in \mathfrak{F}_n$ and $\widetilde{\pi}' \in \mathfrak{F}_{n'}$ be the contragredient representations. When $\pi' \in \{\widetilde{\pi}, \widetilde{\pi}'\}$, work of Brumley [Hum19, Appendix] and the authors [HT22] shows that there exists an effectively computable constant $c_1 = c_1(n, n') > 0$ such that $L(s, \pi \times \pi')$ has a 'standard' zero-free region of the shape

$$\mathrm{Re}(s) \geq 1 - \frac{c_1}{\log(C(\pi)C(\pi')(|\mathrm{Im}(s)| + 3)^{[F:\mathbb{Q}]})} \tag{1.1}$$

apart from at most one real simple zero. This is comparable to the classical zero-free region for Dirichlet L-functions. Brumley (see [Bru06a] and [Lap13, Appendix]) established a much narrower zero-free region for all choices of π and π'. The generalized Riemann hypothesis (GRH) asserts that $L(s, \pi \times \pi') \neq 0$ for $\mathrm{Re}(s) > \frac{1}{2}$. Zeros near the line $\mathrm{Re}(s) = 1$ are typically most damaging in applications, but even a zero-free region of the shape $\mathrm{Re}(s) \geq 1 - \delta$ for some constant $\delta = \delta(n, n', [F : \mathbb{Q}]) > 0$ would be sufficient to obtain many spectacular arithmetic consequences.

Since such strong zero-free regions for Rankin–Selberg L-functions remain out of reach, it is useful to show that zeros near the line $\mathrm{Re}(s) = 1$ must be 'sparse'. A suitable quantitative formulation can serve as a proxy for a zero-free region of the shape $\mathrm{Re}(s) \geq 1 - \delta$.

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Famous consequences of this philosophy include Hoheisel's proof [Hoh30] that $p_{n+1} - p_n \ll p_n^{1-1/33\,000}$ (where p_n is the nth prime) and Linnik's proof [Lin44] that if $\gcd(a,q) = 1$, then there exists an absolute, effectively computable constant $B > 0$ and a prime $p \leq q^B$ such that $p \equiv a \pmod{q}$.

To quantify our notion of 'sparse', we define for $\sigma \geq 0$ and $T \geq 1$ the quantity

$$N_{\pi\times\pi'}(\sigma,T) = |\{\rho = \beta + i\gamma\colon L(\rho,\pi\times\pi') = 0,\ \beta \geq \sigma,\ |\gamma| \leq T\}|.$$

Note that $N_{\pi\times\pi'}(\frac{1}{2},T)$ is roughly $T\log(C(\pi)C(\pi')T)$ via the argument principle and the functional equation, and GRH can be restated as $N_{\pi\times\pi'}(\sigma,T) = 0$ for all $\sigma > \frac{1}{2}$. The zero density estimate

$$N_{\pi\times\pi'}(\sigma,T) \ll_{n,n',[F:\mathbb{Q}]} (C(\pi)C(\pi')T^{[F:\mathbb{Q}]})^{10^7(n'n)^4(1-\sigma)}, \tag{1.2}$$

follows from work of Soundararajan and Thorner [ST19, Corollary 2.6]. Therefore, while an arbitrary Rankin–Selberg L-function $L(s,\pi\times\pi')$ is not yet known to have the standard zero-free region (1.1), the bound (1.2) ensures that the number of zeros in the region (1.1) is $O_{n,n',[F:\mathbb{Q}]}(1)$.

Let $\mathcal{S} \subseteq \mathfrak{F}_n$, and let $\mathcal{S}(Q) = \{\pi \in \mathcal{S}\colon C(\pi) \leq Q\}$. In this article, we seek a strong averaged form of (1.2), namely

$$\sum_{\pi\in\mathcal{S}(Q)} N_{\pi\times\pi'}(\sigma,T) \ll_{n,n',[F:\mathbb{Q}],\varepsilon} (Q|\mathcal{S}(Q)|C(\pi')T^{[F:\mathbb{Q}]})^{A(1-\sigma)+\varepsilon}, \tag{1.3}$$

where $A = A(n,n',[F:\mathbb{Q}]) > 0$ is a constant and $\varepsilon > 0$. The bound (1.3) follows from the works of Brumley, Thorner, and Zaman under at least one of the following hypotheses:[1]

- $\pi' \in \mathfrak{F}_1$ is trivial [TZ21, Theorem 1.2];
- $\max\{n,n'\} \leq 4$ (see [BTZ22, Theorem 1.3]); or
- π' and each $\pi \in \mathcal{S}(Q)$ satisfy certain unproven partial progress towards the generalized Ramanujan conjecture (GRC) [BTZ22, Hypothesis 1.1 and Theorem 1.3].[2]

Here, we prove the first completely unconditional zero density estimate of the form (1.3).

Theorem 1.1. *Let $n, n' \geq 1$ and $\varepsilon > 0$. Let $\mathcal{S} \subseteq \mathfrak{F}_n$ and $\mathcal{S}(Q) = \{\pi \in \mathcal{S}\colon C(\pi) \leq Q\}$. If $0 \leq \sigma \leq 1$, $\pi' \in \mathfrak{F}_{n'}$, and $Q, T \geq 1$, then*

$$\sum_{\pi\in\mathcal{S}(Q)} N_{\pi\times\pi'}(\sigma,T) \ll_{n,n',[F:\mathbb{Q}],\varepsilon} \big(|\mathcal{S}(Q)|^4\big(C(\pi')QT^{[F:\mathbb{Q}]}\big)^{6.15\max\{n^2,n'n\}}\big)^{1-\sigma+\varepsilon}.$$

Theorem 1.1 is non-trivial when

$$\delta_{\mathcal{S}} = \liminf_{Q\to\infty} \frac{\log|\mathcal{S}(Q)|}{\log Q} > 0. \tag{1.4}$$

This is important because in applications, it is usually convenient to bound Q by a power of $|\mathcal{S}(Q)|$ or vice versa. When $\mathcal{S} = \mathfrak{F}_n$, we have the bounds

$$Q^{n+1} \ll_{n,F} |\mathfrak{F}_n(Q)| \ll_\varepsilon D_F^{-n^2} Q^{2n+\varepsilon}. \tag{1.5}$$

The upper bound in (1.5) was proved by Brumley *et al.* [BTZ22, Theorem A.1]. The lower bound in (1.5) follows from work of Brumley and Milićević [BM18, Theorem 1.1], who computed a constant $c_{n,F} > 0$ such that if $\mathfrak{F}_n^*(Q)$ is the subset of $\pi \in \mathfrak{F}_n(Q)$ that are spherical at the archimedean places of F, then $|\mathfrak{F}_n^*(Q)| \sim c_{n,F}Q^{n+1}$. (The lower bound in (1.5) reflects the conjectured order

[1] In [BTZ22, ST19], it is assumed that $F = \mathbb{Q}$. Uniformity over $F \neq \mathbb{Q}$ requires minor modifications.
[2] If θ_n in (3.3) satisfies $\theta_n \leq \frac{1}{4} - \delta_n$ for some $\delta_n > 0$, then each $\pi \in \mathfrak{F}_n$ satisfies [BTZ22, Hypothesis 1.1].

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of growth; see [BM18].) Together, Theorem 1.1 and (1.5) imply that

$$\sum_{\pi\in\mathfrak{F}_n(Q)} N_{\pi\times\pi'}(\sigma,T) \ll_{n,n',F,\varepsilon} (|\mathfrak{F}_n(Q)|C(\pi')^n T^{n[F:\mathbb{Q}]})^{7.1\max\{n,n'\}(1-\sigma)+\varepsilon}. \tag{1.6}$$

Furthermore, if $n'=n\geq 3$ and $\pi'\in\mathfrak{F}_{n'}$, then Theorem 1.1 and (1.5) together imply that

$$\sum_{\pi\in\mathfrak{F}_n(Q)} N_{\pi\times\pi'}(\sigma,T) \ll_{\varepsilon} (C(\pi')QT^{[F:\mathbb{Q}]})^{9n^2(1-\sigma)+\varepsilon}. \tag{1.7}$$

2. Applications

We now describe some applications of Theorem 1.1. In what follows, we write $f\ll_\nu g$, $f=O_\nu(g)$, and $g\gg_\nu f$ to denote that there exists a constant $c>0$ such that $|f|\leq c|g|$ in the stated range. The implied constant c, which is effectively computable unless otherwise stated, will depend at most on ν, n, n', and $[F:\mathbb{Q}]$. The expression $f\asymp_\nu g$ means that $f\ll_\nu g$ and $g\ll_\nu f$. We use $\varepsilon>0$ to denote an arbitrarily small quantity that depends at most on n, n', and $[F:\mathbb{Q}]$.

2.1 Bounds for Rankin–Selberg L-functions

It is a classical problem for Dirichlet L-functions to find strong bounds on the critical line $\mathrm{Re}(s)=\frac{1}{2}$. The Phragmén–Lindelöf convexity principle shows that if q_χ is the conductor of a primitive Dirichlet character χ, then $L(\frac{1}{2},\chi)\ll q^{1/4}$; improving this bound by replacing $1/4$ with a smaller exponent is known as a *subconvex* bound. The multiplicative version of the classical large sieve inequality combined with an approximate functional equation shows that for almost all χ, we have the bound $L(\frac{1}{2},\chi)\ll_\varepsilon q_\chi^\varepsilon$ for all $\varepsilon>0$, consistent with the generalized Lindelöf hypothesis (GLH).

For $\pi\in\mathfrak{F}_2$, GLH predicts that $L(\frac{1}{2},\pi)\ll_\varepsilon C(\pi)^\varepsilon$. Michel and Venkatesh [MV10] proved that there exists a fixed positive $\delta>0$ such that $L(\frac{1}{2},\pi)\ll_F C(\pi)^{1/4-\delta}$, the culmination of several decades of research. When $F=\mathbb{Q}$, a sharp mean value estimate for Hecke eigenvalues proved by Deshouillers and Iwaniec [DI82], in conjunction with the approximate functional equation, implies the bound $L(\frac{1}{2},\pi)\ll_\varepsilon (qT)^\varepsilon$ for almost all $\pi\in\mathfrak{F}_2$ of arithmetic conductor q, trivial central character, and archimedean complexity (Laplace eigenvalue or weight squared) lying in the dyadic interval $[T,2T]$. Note that in this case, $C(\pi)\asymp qT$.

For $\pi\in\mathfrak{F}_n$ with $n\geq 3$, the best uniform result towards the bound $L(\frac{1}{2},\pi)\ll_\varepsilon C(\pi)^\varepsilon$ predicted by GLH is that of Soundararajan and Thorner [ST19, Corollary 2.7], namely

$$L(\tfrac{1}{2},\pi)\ll C(\pi)^{1/4}(\log C(\pi))^{-1/(10^{17}n^3)}. \tag{2.1}$$

We mention three results that improve upon (2.1) in an average sense, each having complementary strengths. Jana [Jan21, Theorem 6] extended the GLH-on-average bound of Deshouillers and Iwaniec to the family of cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ of arithmetic conductor 1 and growing analytic conductor. Blomer [Blo23, Corollary 5] proved the corresponding result for the family of cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ of a large given prime arithmetic conductor q, trivial central character, and whose archimedean components are principal series representations confined to a compact subset of the unitary dual. Thorner and Zaman [TZ21, Theorem 1.3] proved that there exists a constant $c_2=c_2(n,[F:\mathbb{Q}])>0$ such that if $\varepsilon>0$, then

$$|\{\pi\in\mathfrak{F}_n(Q)\colon |L(\tfrac{1}{2},\pi)|\geq c_2 C(\pi)^{1/4-\varepsilon/10^{16}n^3}\}|\ll_F |\mathfrak{F}_n(Q)|^\varepsilon. \tag{2.2}$$

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Unlike the preceding results, (2.2) is uniform in both the arithmetic conductor and spectral aspects and holds over number fields other than $\mathbb{Q}$, but the savings over (2.1) is not comparable to GLH on average.

Given $\pi \in \mathfrak{F}_n$ and $\pi' \in \mathfrak{F}_{n'}$, Soundararajan and Thorner [ST19, Corollary 2.7] proved when $F = \mathbb{Q}$ that if $C(\pi \times \pi')$ is the analytic conductor of $L(s, \pi \times \pi')$, then

$$L(\tfrac{1}{2}, \pi \times \pi') \ll |L(\tfrac{3}{2}, \pi \times \pi')|^2 C(\pi \times \pi')^{1/4} (\log C(\pi \times \pi'))^{-1/(10^{17}(n'n)^3)}. \tag{2.3}$$

As of now, the best general upper bound for $|L(\frac{3}{2}, \pi \times \pi')|^2$ is larger than any fixed power of $\log C(\pi \times \pi')$ (see [Li10, Theorem 2]). The factor of $|L(\frac{3}{2}, \pi \times \pi')|^2$ can be removed under certain partial progress toward GRC. The bound $L(\frac{1}{2}, \pi \times \pi') \ll_\varepsilon C(\pi \times \pi')^\varepsilon$ is predicted by GLH.

In order to improve (2.3) on average with uniformity in π and π', one might first try to mimic the approach that worked well for Dirichlet L-functions and GL_2 L-functions using trace formulae, approximate functional equations, the spectral large sieve, Voronoĭ summation, etc. While such methods have seen great success for $\mathrm{GL}_n \times \mathrm{GL}_{n'}$ with $n, n' \in \{1, 2\}$, suitably uniform and flexible versions of these tools do not appear to be available yet in the general setting. The special case where $|n - n'| \leq 1$ exhibits some nice structural properties, lending itself to approaches via period integrals that completely avoids the aforementioned tools. To describe work in this direction, let $F = \mathbb{Q}$, $\mathcal{F}_n \subseteq \mathfrak{F}_n$ be the subset of cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ of arithmetic conductor 1, and $\mathcal{F}_n(Q) = \{\pi \in \mathcal{F}_n \colon C(\pi) \leq Q\}$. It follows from work of Jana [Jan22, Corollary 2.2] that if $\varepsilon > 0$, $\pi' \in \mathcal{F}_n$, and the spectral parameters of π' have real part at least $-1/(n^2+1)$ (which is far stronger than the best known unconditional lower bound $-\frac{1}{2} + 1/(n^2+1)$ due to Luo, Rudnick, and Sarnak [LRS99]), then

$$\sum_{\pi \in \mathcal{F}_n(Q)} \left| L\left(\frac{1}{2}, \pi \times \pi'\right) \right|^2 \ll_{\pi', \varepsilon} |\mathcal{F}_n(Q)|^{1+\varepsilon}. \tag{2.4}$$

Therefore, for fixed $\pi' \in \mathcal{F}_n$, GLH for $L(\frac{1}{2}, \pi \times \pi')$ holds on average over the $\pi \in \mathcal{F}_n$. Using Chebyshev's inequality, we conclude that for all $\delta > 0$, there exists a constant $c_{\pi',\delta} > 0$ such that

$$|\{\pi \in \mathcal{F}_n(Q) \colon |L(\tfrac{1}{2}, \pi \times \pi')| \geq c_{\pi',\delta} C(\pi \times \pi')^{2\delta}\}| \ll_{\pi',\delta} |\mathcal{F}_n(Q)|^{1-\delta}. \tag{2.5}$$

See also the work of Blomer [Blo12, Theorem 2], which proves a variant of (2.4) for families of 'spectrally close' Hecke–Maaß newforms on $\mathrm{SL}_n(\mathbb{Z})$.

Along the same lines as (2.2), we use (1.6) to prove the following result.

Theorem 2.1. *Let $n, n' \geq 1$ and $Q \geq 1$. If $\varepsilon > 0$ and $\pi' \in \mathfrak{F}_{n'}$, then*

$$|\{\pi \in \mathfrak{F}_n(Q) \colon |L(\tfrac{1}{2}, \pi \times \pi')| \geq C(\pi \times \pi')^{1/4 - \varepsilon/(10^{10} \max\{n, n'\})}\}| \ll_\varepsilon (C(\pi')^n |\mathfrak{F}_n(Q)|)^\varepsilon.$$

Remark 2.2. Given a subset $\mathcal{S} \subseteq \mathfrak{F}_n$, a similar result can be proved for $\pi \in \mathcal{S}(Q)$ using Theorem 1.1. Such a result would depend effectively on $\delta_\mathcal{S}$ in (1.4).

In contrast with Jana's work in (2.5), Theorem 2.1 provides a smaller power-saving improvement over (2.3), but the improvement is uniform in the arithmetic conductor and spectral aspects as well as in π'. The exceptional set in Theorem 2.1 is a much smaller than in (2.5). Theorem 2.1 removes the requirements that $n = n'$, that $q_\pi = q_{\pi'} = 1$, and that the spectral parameters of π' have real part at least $-1/(n^2+1)$. Finally, Theorem 2.1 is proved over any number field, while [Jan22] is only proved over $\mathbb{Q}$.

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2.2 Effective multiplicity one

Let $\pi = \bigotimes_v \pi_v$ and $\pi' = \bigotimes_v \pi'_v$ be cuspidal automorphic representations in $\mathfrak{F}_n(Q)$. Under the assumption of GRH for $L(s, \pi \times \widetilde{\pi})$ and $L(s, \pi \times \pi')$ and that $\pi_\mathfrak{p}$ and $\pi'_\mathfrak{p}$ are tempered for all prime ideals $\mathfrak{p} \mid \mathfrak{q}_\pi \mathfrak{q}_{\pi'}$, it is known that if $Y = (\log Q)^2$ and $\pi_\mathfrak{p} \cong \pi'_\mathfrak{p}$ for all $\mathfrak{p} \nmid \mathfrak{q}_\pi \mathfrak{q}_{\pi'}$ with $\mathrm{N}\mathfrak{p} \ll Y$, then $\pi = \pi'$ (see [IK04, Proposition 5.22]). Brumley [Bru06a], improving on work of Moreno [Mor85], proved that there exists a constant $B_n > 0$ such that this result holds unconditionally with $Y = Q^{B_n}$. This result makes effective the multiplicity one theorems of Jacquet and Shalika [JS81, Theorem 4.8] and Piatetski-Shapiro [Pia79]. Any fixed $B_n > 2n$ suffices [LW09].

When $n = 2$, we have an average result that nearly achieves what GRH predicts. Specifically, let $\mathfrak{F}_2^\flat$ be the subset of $\pi \in \mathfrak{F}_2$ with squarefree conductor and trivial central character, and let $\pi' \in \mathfrak{F}_2^\flat$. For all $\varepsilon > 0$, there exists an effectively computable constant $N_\varepsilon > 0$, depending at most on ε and $[F : \mathbb{Q}]$, such that

$$|\{\pi \in \mathfrak{F}_2^\flat(Q) \colon \pi_\mathfrak{p} \cong \pi'_\mathfrak{p} \text{ for all } \mathfrak{p} \nmid \mathfrak{q}_\pi \text{ with } \mathrm{N}\mathfrak{p} \leq (\log Q)^{N_\varepsilon}\}| \ll_\varepsilon Q^\varepsilon. \tag{2.6}$$

In particular, the implied constant does not depend on π'. This was proved by Duke and Kowalski [DK00, Theorem 3] when $F = \mathbb{Q}$ in a stronger form under the assumption of GRC. See Brumley's Ph.D. thesis [Bru04, Corollary 5.2.2] for a proof that does not use GRC.

If $\pi \in \mathfrak{F}_2$ and $\widetilde{\pi} \in \mathfrak{F}_2$ is the contragredient, then $L(s, \pi \times \widetilde{\pi}) = \zeta_F(s) L(s, \pi, \mathrm{Ad})$, where $\zeta_F(s)$ is the Dedekind zeta function of F and Ad is the adjoint square lift from a representation of $\mathrm{GL}_2(\mathbb{A}_F)$ to a representation of $\mathrm{GL}_3(\mathbb{A}_F)$. The arguments in [Bru04, DK00] rely on two key results as follows.

(1) Gelbart and Jacquet [GJ78] proved that if $\pi \in \mathfrak{F}_2$, then $L(s, \pi, \mathrm{Ad})$ is the L-function of an automorphic representation of $\mathrm{GL}_3(\mathbb{A}_F)$, denoted $\mathrm{Ad}\,\pi$, complete with a criterion by which one can determine whether $\mathrm{Ad}\,\pi$ is cuspidal (and lies in $\mathfrak{F}_3$).
(2) At most $O_\varepsilon(Q^{1/2+\varepsilon})$ representations $\pi \in \mathfrak{F}_2(Q)$ have the same adjoint lift. In addition, Ramakrishnan (see [DK00, Appendix] and [Ram00]) proved that $\mathrm{Ad} \colon \mathfrak{F}_2^\flat \to \mathfrak{F}_3$ is injective.

In attempting to generalize the strategy of Duke and Kowalski to GL_n for $n \geq 3$, one encounters some deep open problems. If $\pi \in \mathfrak{F}_n$, then for $\mathrm{Re}(s)$ sufficiently large, $L(s, \pi \times \widetilde{\pi})$ factors as $\zeta_F(s) L(s, \pi, \mathrm{Ad})$, where Ad is the adjoint square lift from GL_n to GL_{n^2-1}. Apart from some special cases, the following obstacles arise.

(1) The adjoint lift is not yet known to be automorphic for $n \geq 3$, and if it were, there is no known criterion for cuspidality.
(2) Let $\mathcal{H}_n \subseteq \mathfrak{F}_n(Q)$ be the subset of $\pi \in \mathfrak{F}_n(Q)$ such that $\mathrm{Ad}\,\pi \in \mathfrak{F}_{n^2-1}$. It is not known how many $\pi \in \mathcal{H}_n$ have the same adjoint lift.

Despite these setbacks, we can use (1.7) (more specifically, Corollary 7.1) to prove a GL_n analogue of (2.6) when π' depends mildly on Q.

THEOREM 2.3. *Let $n \geq 3$ and $Q \geq 1$. There exists an absolute, effectively computable constant $c_3 > 0$ such that if $0 < \varepsilon < 1$ and $\pi' \in \mathfrak{F}_n((\log Q)^{c_3/(n^2[F:\mathbb{Q}]^2)})$, then*

$$|\{\pi \in \mathfrak{F}_n(Q) \colon \pi_\mathfrak{p} \cong \pi'_\mathfrak{p} \text{ for all } \mathfrak{p} \nmid \mathfrak{q}_\pi \mathfrak{q}_{\pi'} \text{ with } \mathrm{N}\mathfrak{p} \leq (\log Q)^{41n^2/\varepsilon}\}| \ll_\varepsilon Q^\varepsilon.$$

Remark 2.4. Given a subset $\mathcal{S} \subseteq \mathfrak{F}_n$, a similar result can be proved for $\pi \in \mathcal{S}(Q)$ using Theorem 1.1. Such a result would depend effectively on $\delta_\mathcal{S}$ in (1.4).

The proof is very flexible. For example, if $\varepsilon > 0$ and $C(\pi') \ll_\varepsilon Q^{c_3\varepsilon^2/(n^2[F:\mathbb{Q}]^2)}$, then the same proof with minor changes in choices of parameters produces the bound

$$|\{\pi \in \mathfrak{F}_n(Q) \colon \pi_\mathfrak{p} \cong \pi'_\mathfrak{p} \text{ for all } \mathfrak{p} \nmid \mathfrak{q}_\pi \mathfrak{q}_{\pi'} \text{ with } \mathrm{N}\mathfrak{p} \leq Q^\varepsilon\}| \ll_\varepsilon Q^\varepsilon.$$

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While the number of $\mathfrak{p}$ for which one needs to check that $\pi_{\mathfrak{p}} \cong \pi'_{\mathfrak{p}}$ is larger, the range of $C(\pi')$ is greatly extended, and the threshold $\mathrm{N}\mathfrak{p} \leq Q^{\varepsilon}$ (reminiscent of Vinogradov's conjecture on the size of the least quadratic non-residue) still greatly improves on the unconditional range $\mathrm{N}\mathfrak{p} \ll_{\varepsilon} Q^{2n+\varepsilon}$ from [LW09].

2.3 Automorphic level of distribution

Let $\Lambda(m)$ be the von Mangoldt function, equal to $\log p$ if m is a power of a prime p and zero otherwise. The celebrated Bombieri–Vinogradov theorem states that if $\theta < \frac{1}{2}$ is fixed, then for all $A > 0$, we have

$$\sum_{q \leq x^{\theta}} \max_{\gcd(a,q)=1} \max_{y \leq x} \Bigg| \sum_{\substack{m \leq y \\ m \equiv a \,(\mathrm{mod}\, q)}} \Lambda(m) - \frac{y}{\varphi(q)} \Bigg| \ll_A \frac{x}{(\log x)^A}. \tag{2.7}$$

This may be viewed as an average form of GRH for Dirichlet L-functions. As part of his proof [Bom65], Bombieri proved a strong form of the zero density estimate in Theorem 1.1 for Dirichlet L-functions. We call any θ for which (2.7) holds a *level of distribution* for the primes. Elliott and Halberstam conjectured that any fixed $\theta < 1$ is a level of distribution for the primes.

Number theorists have proved several interesting extensions and variations of (2.7). For example, Murty and Murty [MM87] proved that primes in the Chebotarev density theorem have a positive level of distribution. To describe a different direction for automorphic representations over $\mathbb{Q}$, we let $n \geq 2$ and consider $\pi \in \mathfrak{F}_n$ with conductor q_{π}. Let $\Lambda(m)$ be the von Mangoldt function, and define the numbers $a_{\pi}(m)$ by

$$-\frac{L'}{L}(s,\pi) = \sum_p \sum_{k=1}^{\infty} \frac{\sum_{j=1}^{n} \alpha_{j,\pi}(p)^k \log p}{p^{ks}} = \sum_{n=1}^{\infty} \frac{a_{\pi}(m)\Lambda(m)}{m^s}, \quad \mathrm{Re}(s) > 1.$$

Note that $a_{\pi}(p) = \lambda_{\pi}(p)$. For fixed $\theta < 1/(n^2-2)$, Wong [Won20, Theorem 9] proved that if π satisfies GRC and $L(s, \pi \times (\widetilde{\pi} \otimes \chi))$ has no Landau–Siegel zero for all Dirichlet characters χ, then for any $A > 0$,

$$\sum_{q \leq x^{\theta}} \max_{\gcd(a,q)=1} \max_{y \leq x} \Bigg| \sum_{\substack{m \leq y \\ \gcd(m,q_{\pi})=1 \\ m \equiv a \,(\mathrm{mod}\, q)}} |a_{\pi}(m)|^2 \Lambda(m) - \frac{y}{\varphi(q)} \Bigg| \ll_{A,\pi} \frac{x}{(\log x)^A}. \tag{2.8}$$

This conditionally endows $L(s, \pi \times \widetilde{\pi})$ with a positive level of distribution θ. The hypotheses for (2.8) hold for π attached to non-CM holomorphic cuspidal newforms on congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

Let $\pi \in \mathfrak{F}_n$. Using (1.7), we unconditionally endow $L(s, \pi \times \widetilde{\pi})$ with a notion of positive level of distribution. In particular, we avoid recourse to unproven progress toward GRC or the absence of Landau–Siegel zeros.

Theorem 2.5. *Let $F = \mathbb{Q}$ and $\pi \in \mathfrak{F}_n$. Fix $\theta < 1/(9n^3)$. If $A > 0$, then*

$$\sum_{\substack{q \leq x^{\theta} \\ \gcd(q,q_{\pi})=1}} \max_{\gcd(a,q)=1} \max_{y \leq x} \Bigg| \sum_{\substack{m \leq y \\ m \equiv a \,(\mathrm{mod}\, q)}} |a_{\pi}(m)|^2 \Lambda(m) - \frac{y}{\varphi(q)} \Bigg| \ll_{A,\pi} \frac{x}{(\log x)^A}.$$

The implied constants are ineffective.

Remark 2.6. One can prove an analogue of Theorem 2.5 with $F \neq \mathbb{Q}$, replacing residue classes modulo q with ray classes modulo $\mathfrak{q}$. We restrict to $F = \mathbb{Q}$ for notational simplicity.

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Overview of the paper

In § 3, we recall basic properties of standard L-functions and Rankin–Selberg L-functions that we use in our proofs. In § 4, we prove a large sieve inequality for the Dirichlet coefficients of $L(s,\pi\times\pi')^{-1}$ and a corollary on mean values of Dirichlet polynomials, which we use in our proof of Theorem 1.1 in § 5. We then prove Theorem 2.1 in § 6, Theorem 2.3 in § 7, and Theorem 2.5 in § 8.

3. Properties of L-functions

We recall some standard facts about L-functions arising from automorphic representations and their Rankin–Selberg convolutions. See [Bru06a, GJ72, JPS83, MW89, ST19].

3.1 Standard L-functions

Given $\pi\in\mathfrak{F}_n$, let $\widetilde{\pi}\in\mathfrak{F}_n$ be the contragredient representation and $\mathfrak{q}_\pi$ be the conductor of π. We express π as a restricted tensor product $\bigotimes_v \pi_v$ of smooth admissible representations of $\mathrm{GL}_n(F_v)$, where v varies over places of F. When v is a non-archimedean place corresponding with a prime ideal $\mathfrak{p}$, then the local L-function $L(s,\pi_{\mathfrak{p}})$ is defined in terms of the Satake parameters $A_\pi(\mathfrak{p})=\{\alpha_{1,\pi}(\mathfrak{p}),\ldots,\alpha_{n,\pi}(\mathfrak{p})\}$ by

$$L(s,\pi_{\mathfrak{p}})=\prod_{j=1}^{n}(1-\alpha_{j,\pi}(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-s})^{-1}=\sum_{k=0}^{\infty}\frac{\lambda_\pi(\mathfrak{p}^k)}{\mathrm{N}\mathfrak{p}^{ks}}. \tag{3.1}$$

We have $\alpha_{j,\pi}(\mathfrak{p})\neq 0$ for all j whenever $\mathfrak{p}\nmid\mathfrak{q}_\pi$, and when $\mathfrak{p}\mid\mathfrak{q}_\pi$, it might be the case that there exist j such that $\alpha_{j,\pi}(\mathfrak{p})=0$. The standard L-function $L(s,\pi)$ associated to π is of the form

$$L(s,\pi)=\prod_{\mathfrak{p}}L(s,\pi_{\mathfrak{p}})=\sum_{\mathfrak{n}}\frac{\lambda_\pi(\mathfrak{n})}{\mathrm{N}\mathfrak{n}^s}.$$

The Euler product and Dirichlet series converge absolutely when $\mathrm{Re}(s)>1$.

At each archimedean place v of F, there are n Langlands parameters $\mu_{j,\pi}(v)\in\mathbb{C}$ such that

$$L(s,\pi_\infty)=\prod_{v\mid\infty}\prod_{j=1}^{n}\Gamma_v(s+\mu_{j,\pi}(v)),\quad \Gamma_v(s):=\begin{cases}\pi^{-s/2}\Gamma(s/2) & \text{if } F_v=\mathbb{R},\\ 2(2\pi)^{-s}\Gamma(s) & \text{if } F_v=\mathbb{C}.\end{cases}$$

By combining the work in [BB11, BB13, LRS99, MS04], we know that there exists

$$0\le\theta_n\le\begin{cases}0 & \text{if } n=1,\\ 7/64 & \text{if } n=2,\\ 5/14 & \text{if } n=3,\\ 9/22 & \text{if } n=4,\\ 1/2-1/(n^2+1) & \text{if } n\ge 5,\end{cases} \tag{3.2}$$

such that

$$|\alpha_{j,\pi}(\mathfrak{p})|\le\mathrm{N}\mathfrak{p}^{\theta_n}\quad\text{and}\quad\mathrm{Re}(\mu_{j,\pi}(v))\ge-\theta_n. \tag{3.3}$$

GRC asserts that in (3.2), one may take $\theta_n=0$. We have $\mathfrak{q}_\pi=\mathfrak{q}_{\widetilde{\pi}}$, and for each $\mathfrak{p}$ and each v, we have the equalities of sets $\{\alpha_{j,\widetilde{\pi}}(\mathfrak{p})\}=\{\overline{\alpha_{j,\pi}(\mathfrak{p})}\}$ and $\{\mu_{j,\widetilde{\pi}}(v)\}=\{\overline{\mu_{j,\pi}(v)}\}$.

Let r_π be the order of the pole of $L(s,\pi)$ at $s=1$. The completed L-function

$$\Lambda(s,\pi)=(s(s-1))^{r_\pi}(D_F^n\mathrm{N}\mathfrak{q}_\pi)^{s/2}L(s,\pi)L(s,\pi_\infty)$$

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is entire of order 1, and there exists a complex number $W(\pi)$ of modulus 1 such that for all $s\in\mathbb{C}$, we have the functional equation $\Lambda(s,\pi)=W(\pi)\Lambda(1-s,\widetilde{\pi})$. Let $d(v)=1$ if $F_v=\mathbb{R}$ and $d(v)=2$ if $F_v=\mathbb{C}$. The analytic conductor of π (see [IS00]) is given by

$$C(\pi,t) := D_F^n \mathrm{N}\mathfrak{q}_\pi \prod_{v|\infty}\prod_{j=1}^{n}(3+|it+\mu_{j,\pi}(v)|^{d(v)}),\quad C(\pi):=C(\pi,0). \tag{3.4}$$

Since $\Lambda(s,\pi)$ is entire of order 1, there exist complex numbers a_π and b_π such that

$$\Lambda(s,\pi)=e^{a_\pi+b_\pi s}\prod_{\Lambda(\rho,\pi)=0}\Big(1-\frac{s}{\rho}\Big)e^{s/\rho}.$$

The zeros ρ in the above Hadamard product are the non-trivial zeros of $L(s,\pi)$, and the zeros of $L(s,\pi)$ that arise as poles of $s^{r_\pi}L(s,\pi_\infty)$ are the trivial zeros.

3.2 Rankin–Selberg *L*-functions

Let $\pi\in\mathfrak{F}_n$ and $\pi'\in\mathfrak{F}_{n'}$. At each prime ideal $\mathfrak{p}$, Jacquet *et al.* [JPS83] associate to $\pi_\mathfrak{p}$ and $\pi'_\mathfrak{p}$ a local Rankin–Selberg L-function

$$L(s,\pi_\mathfrak{p}\times\pi'_\mathfrak{p})=\prod_{j=1}^{n}\prod_{j'=1}^{n'}(1-\alpha_{j,j',\pi\times\pi'}(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-s})^{-1}=\sum_{k=0}^{\infty}\frac{\lambda_{\pi\times\pi'}(\mathfrak{p}^k)}{\mathrm{N}\mathfrak{p}^{ks}} \tag{3.5}$$

and a local conductor $\mathfrak{q}_{\pi_\mathfrak{p}\times\pi'_\mathfrak{p}}$. If $\mathfrak{p}\nmid\mathfrak{q}_\pi\mathfrak{q}_{\pi'}$, then we have the equality of sets

$$\{\alpha_{j,j',\pi\times\pi'}(\mathfrak{p})\}=\{\alpha_{j,\pi}(\mathfrak{p})\alpha_{j',\pi'}(\mathfrak{p})\}. \tag{3.6}$$

The Rankin–Selberg L-function $L(s,\pi\times\pi')$ associated to π and π' and its arithmetic conductor are

$$L(s,\pi\times\pi')=\prod_{\mathfrak{p}}L(s,\pi_\mathfrak{p}\times\pi'_\mathfrak{p})=\sum_{\mathfrak{n}}\frac{\lambda_{\pi\times\pi'}(\mathfrak{n})}{\mathrm{N}\mathfrak{n}^s},\quad \mathfrak{q}_{\pi\times\pi'}=\prod_{\mathfrak{p}}\mathfrak{q}_{\pi_\mathfrak{p}\times\pi'_\mathfrak{p}}.$$

At an archimedean place v of F, Jacquet, Piatetski-Shapiro, and Shalika associate $n'n$ complex Langlands parameters $\mu_{j,j',\pi\times\pi'}(v)$ to π_v and π'_v, from which one defines

$$L(s,\pi_\infty\times\pi'_\infty)=\prod_{v|\infty}\prod_{j=1}^{n}\prod_{j'=1}^{n'}\Gamma_v(s+\mu_{j,j',\pi\times\pi'}(v)).$$

Using the explicit descriptions of $\alpha_{j,j',\pi\times\pi'}(\mathfrak{p})$ and $\mu_{j,j',\pi\times\pi'}(v)$ in [Hum19, ST19], one sees that

$$|\alpha_{j,j',\pi\times\pi'}(\mathfrak{p})|\le \mathrm{N}\mathfrak{p}^{\theta_n+\theta_{n'}},\quad \mathrm{Re}(\mu_{j,j',\pi\times\pi'}(v))\ge-\theta_n-\theta_{n'}. \tag{3.7}$$

Let $r_{\pi\times\pi'}=-\mathrm{ord}_{s=1}L(s,\pi\times\pi')$. By our normalization for the central characters of π and π', we have that $r_{\pi\times\pi'}=0$ if and only if $\pi\neq\widetilde{\pi}'$, and $r_{\pi\times\widetilde{\pi}}=1$ otherwise. The function

$$\Lambda(s,\pi\times\pi')=(s(s-1))^{r_{\pi\times\pi'}}(D_F^{n'n}\mathrm{N}\mathfrak{q}_{\pi\times\pi'})^{s/2}L(s,\pi\times\pi')L(s,\pi_\infty\times\pi'_\infty) \tag{3.8}$$

is entire of order 1, and there exists a complex number $W(\pi\times\pi')$ of modulus 1 such that $\Lambda(s,\pi\times\pi')$ satisfies the functional equation $\Lambda(s,\pi\times\pi')=W(\pi\times\pi')\Lambda(1-s,\widetilde{\pi}\times\widetilde{\pi}')$. As with $L(s,\pi)$, the analytic conductor of $L(s,\pi\times\pi')$ is given by

$$C(\pi\times\pi',t):=D_F^{n'n}\mathrm{N}\mathfrak{q}_{\pi\times\pi'}\prod_{v|\infty}\prod_{j=1}^{n}\prod_{j'=1}^{n'}(3+|it+\mu_{j,j',\pi\times\pi'}(v)|^{d(v)}),\quad C(\pi\times\pi'):=C(0,\pi\times\pi'). \tag{3.9}$$

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The combined work of Bushnell and Henniart [BH97] and Brumley [Hum19, Appendix] yields

$$C(\pi\times\pi',t)\ll C(\pi\times\pi')(3+|t|)^{[F:\mathbb{Q}]n'n},\quad C(\pi\times\pi')\ll C(\pi)^{n'}C(\pi')^{n}. \tag{3.10}$$

Since $\Lambda(s,\pi\times\pi')$ is entire of order 1, there exist complex numbers $a_{\pi\times\pi'}$ and $b_{\pi\times\pi'}$ such that the Hadamard factorization

$$\Lambda(s,\pi\times\pi')=e^{a_{\pi\times\pi'}+b_{\pi\times\pi'}s}\prod_{\Lambda(\rho,\pi\times\pi')=0}\Big(1-\frac{s}{\rho}\Big)e^{s/\rho} \tag{3.11}$$

holds. The zeros ρ in (3.11) are the non-trivial zeros of $L(s,\pi\times\pi')$, and the zeros of $L(s,\pi\times\pi')$ that arise as poles of $s^{r_{\pi\times\pi'}}L(s,\pi_\infty\times\pi'_\infty)$ are the trivial zeros.

It follows from work of Li [Li10, Theorem 2] (with minor adjustments when $F\neq\mathbb{Q}$) that there exists an absolute and effectively computable constant $c_4>0$, which we assume to be sufficiently large for future convenience, such that

$$\lim_{\sigma_0\to\sigma}(\sigma_0-1)^{r_{\pi\times\pi'}}L(\sigma_0,\pi\times\pi')\ll\exp\Big(c_4n'n[F:\mathbb{Q}]\frac{\log C(\pi\times\pi')}{\log\log C(\pi\times\pi')}\Big),\quad \sigma\in[1,3]. \tag{3.12}$$

We can change π' to $\pi'\otimes|\det|^{it}$; at the archimedean places, this has the effect of adding it to each $\mu_{j,j',\pi\times\pi'}(v)$. We then apply functional equation, the Phragmén–Lindelöf convexity principle, and (3.10) to obtain for all $\sigma\geq 0$

$$\begin{aligned}&\lim_{\sigma_0\to\sigma}\Big(\frac{\sigma_0+it-1}{\sigma_0+it+1}\Big)^{r_{\pi\times\pi'}}L(\sigma_0+it,\pi\times\pi')\\ &\quad\ll_\varepsilon C(\pi\times\pi',t)^{\max\{1-\sigma,0\}/2+\varepsilon/n'n[F:\mathbb{Q}]}\\ &\quad\ll_\varepsilon (C(\pi)^{n'}C(\pi')^{n}(3+|t|)^{n'n[F:\mathbb{Q}]})^{\max\{1-\sigma,0\}/2+\varepsilon/n'n[F:\mathbb{Q}]}.\end{aligned} \tag{3.13}$$

LEMMA 3.1. *If $\pi\in\mathfrak{F}_n$, $X\geq 3$, and $\varepsilon>0$, then $\sum_{\mathrm{N}\mathfrak{n}\leq X}\lambda_{\pi\times\widetilde{\pi}}(\mathfrak{n})/\mathrm{N}\mathfrak{n}\ll_\varepsilon C(\pi)^\varepsilon\log X$.*

Proof. Since $\lambda_{\pi\times\widetilde{\pi}}(\mathfrak{n})\geq 0$ for all $\mathfrak{n}$ by [HR95, Lemma a], we observe by (3.12) that

$$\sum_{\mathrm{N}\mathfrak{n}\leq X}\frac{\lambda_{\pi\times\widetilde{\pi}}(\mathfrak{n})}{\mathrm{N}\mathfrak{n}}\leq e\sum_{\mathfrak{n}}\frac{\lambda_{\pi\times\widetilde{\pi}}(\mathfrak{n})}{\mathrm{N}\mathfrak{n}^{1+1/\log X}}\ll(\log X)\operatorname*{Res}_{s=1}L(s,\pi\times\widetilde{\pi}).$$

The desired bounded now follows from (3.13) with $\sigma=1$, $t=0$, and $\pi'=\widetilde{\pi}$. □

LEMMA 3.2. *Let $J\geq 1$ be an integer. For all $j\in\{1,\dots,J\}$, let $t_j\in\mathbb{R}$; n_j,n_j' be positive integers; and $\pi_j\in\mathfrak{F}_{n_j}$ and $\pi_j'\in\mathfrak{F}_{n_j'}$. Let*

$$D(s)=\prod_{j=1}^{J}L(s+it_j,\pi_j\times\widetilde{\pi}_j'),\quad \Delta(s)=\prod_{j=1}^{J}\Lambda(s+it_j,\pi_j\times\widetilde{\pi}_j'),\quad Q=\prod_{j=1}^{J}C(\pi_j)^{n_j'}C(\pi_j')^{n_j}.$$

Let $R=-\mathrm{ord}_{s=1}D(s)$. If $1<\sigma<2$ and the $\mathfrak{n}$th Dirichlet coefficient of $-(D'/D)(s)$ is non-negative when $\gcd(\mathfrak{n},\prod_{j=1}^{J}\mathfrak{q}_{\pi_j}\mathfrak{q}_{\pi_j'})=\mathcal{O}_F$, then

$$\sum_{\Delta(\rho)=0}\mathrm{Re}\Big(\frac{1}{\sigma-\rho}\Big)<\frac{R}{\sigma-1}+\sum_{\substack{1\leq j\leq J\\ \pi_j=\pi_j',\ t_j\neq 0}}\frac{\sigma-1}{(\sigma-1)^2+t_j^2}+O(\log Q).$$

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Proof. Let $1 < \sigma < 2$ and

$$D_\infty(s) = \prod_{j=1}^{J} L(s, (\pi_j)_\infty \times (\widetilde{\pi}'_j)_\infty), \quad \mathfrak{q}_D = \prod_{j=1}^{J} \mathfrak{q}_{\pi_j \times \widetilde{\pi}'_j}.$$

By comparing the logarithmic derivative of $\Delta(s)$ with the logarithmic derivative of its Hadamard factorization

$$\Delta(s) = e^{a_D + b_D s} \prod_{\Delta(\rho)=0} \Big(1 - \frac{s}{\rho}\Big) e^{s/\rho},$$

we find that

$$\begin{aligned} &\sum_{\Delta(\rho)=0} \Big(\frac{1}{\sigma-\rho} + \frac{1}{\rho}\Big) + b_D \\ &\quad = \frac{D'}{D}(\sigma) + \frac{\log \mathrm{N}\mathfrak{q}_D}{2} + \frac{R}{\sigma-1} + \sum_{\substack{1 \le j \le J \\ \pi_j = \pi'_j}} \Big(\frac{1}{\sigma + it_j - 1} + \frac{1}{\sigma + it_j}\Big) + \frac{D'_\infty}{D_\infty}(\sigma). \end{aligned}$$

Since $\mathrm{Re}(b_D) = -\sum_{\Delta(\rho)=0} \mathrm{Re}(\rho^{-1})$ (see [IK04, Proposition 5.7(3)]), we take real parts and obtain

$$\sum_{\Delta(\rho)=0} \mathrm{Re}\Big(\frac{1}{\sigma-\rho}\Big) = \mathrm{Re}\Big(\frac{D'}{D}(s) + \frac{\log \mathrm{N}\mathfrak{q}_D}{2} + \frac{R}{\sigma-1} + \sum_{\substack{1 \le j \le J \\ \pi_j = \pi'_j \\ t_j \neq 0}} \frac{\sigma-1}{(\sigma-1)^2 + t_j^2} + \frac{D'_\infty}{D_\infty}(s)\Big) + O(1).$$

By (3.3), (3.7), and (3.10), the bound $\mathrm{Re}((D'/D)(\sigma)) \ll \log Q$ (respectively, $(D'_\infty/D_\infty)(\sigma) \ll \log Q$) follows from our hypothesis on the Dirichlet coefficients of $-(D'/D)(s)$ (respectively, Stirling's formula). ☐

3.3 Rankin–Selberg combinatorics

A partition $\mu = (\mu_i)_{i=1}^\infty$ is a sequence of non-increasing nonnegative integers $\mu_1 \ge \mu_2 \ge \cdots$ with only finitely many non-zero entries. For a partition μ, let $\ell(\mu)$ be the number of non-zero μ_i, and let $|\mu| = \sum_{i=1}^\infty \mu_i$. For a set $\{\alpha_1, \ldots, \alpha_n\}$ of real numbers and a partition μ with $\ell(\mu) \le n$, let $s_\mu(\{\alpha_1, \ldots, \alpha_n\})$ be the Schur polynomial $\det[(\alpha_i^{\lambda(j)+n-j})_{ij}]/\det[(\alpha_i^{n-j})_{ij}]$ associated to μ. If $|\mu| = 0$, then $s_\mu(\{\alpha_1, \ldots, \alpha_n\})$ is identically one. By convention, if $\ell(\mu) > n$, then $s_\mu(\{\alpha_1, \ldots, \alpha_n\})$ is identically zero.

Let $\pi \in \mathfrak{F}_n$ and $\pi' \in \mathfrak{F}_{n'}$. By (3.5), (3.6), and Cauchy's identity [Bum13, Theorem 38.1], we have

$$\sum_{k=0}^\infty \frac{\lambda_{\pi \times \pi'}(\mathfrak{p}^k)}{\mathrm{N}\mathfrak{p}^{ks}} = L(s, \pi_\mathfrak{p} \times \pi'_\mathfrak{p}) = \sum_\mu \frac{s_\mu(A_\pi(\mathfrak{p})) s_\mu(A_{\pi'}(\mathfrak{p}))}{\mathrm{N}\mathfrak{p}^{s|\mu|}}, \quad \mathfrak{p} \nmid \mathfrak{q}_\pi \mathfrak{q}_{\pi'},$$

where the sum ranges over all partitions. This yields

$$\lambda_{\pi \times \pi'}(\mathfrak{p}^k) = \sum_{|\mu|=k} s_\mu(A_\pi(\mathfrak{p})) s_\mu(A_{\pi'}(\mathfrak{p})), \quad \mathfrak{p} \nmid \mathfrak{q}_\pi \mathfrak{q}_{\pi'}.$$

For an integral ideal $\mathfrak{n}$ with factorization $\mathfrak{n} = \prod_\mathfrak{p} \mathfrak{p}^{\mathrm{ord}_\mathfrak{p}(\mathfrak{n})}$ (with $\mathrm{ord}_\mathfrak{p}(\mathfrak{n}) = 0$ for all but finitely many $\mathfrak{p}$), the multiplicativity of $\lambda_{\pi \times \pi'}(\mathfrak{n})$ tells us that if $\gcd(\mathfrak{n}, \mathfrak{q}_\pi \mathfrak{q}_{\pi'}) = \mathcal{O}_F$, then

$$\lambda_{\pi \times \pi'}(\mathfrak{n}) = \prod_\mathfrak{p} \lambda_{\pi \times \pi'}(\mathfrak{p}^{\mathrm{ord}_\mathfrak{p}(\mathfrak{n})}) = \sum_{(\mu_\mathfrak{p})_\mathfrak{p} \in \underline{\mu}[\mathfrak{n}]} \prod_\mathfrak{p} s_{\mu_\mathfrak{p}}(A_\pi(\mathfrak{p})) s_{\mu_\mathfrak{p}}(A_{\pi'}(\mathfrak{p})), \tag{3.14}$$

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where $(\mu_{\mathfrak{p}})_{\mathfrak{p}}$ denotes a sequence of partitions indexed by prime ideals and

$$\underline{\mu}[\mathfrak{n}] := \{(\mu_{\mathfrak{p}})_{\mathfrak{p}} \colon |\mu_{\mathfrak{p}}| = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{n}) \text{ for all } \mathfrak{p}\}. \tag{3.15}$$

Define the numbers $\mu_{\pi\times\pi'}(\mathfrak{n})$ on unramified prime powers by

$$\sum_{k=0}^{\infty} \frac{\mu_{\pi\times\pi'}(\mathfrak{p}^k)}{\mathrm{N}\mathfrak{p}^{ks}} = L(s, \pi_{\mathfrak{p}} \times \pi'_{\mathfrak{p}})^{-1} = \prod_{j=1}^{n} \prod_{j'=1}^{n'} (1 - \alpha_{j,\pi}(\mathfrak{p})\alpha_{j',\pi'}(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-s}), \quad \mathfrak{p} \nmid \mathfrak{q}_{\pi}\mathfrak{q}_{\pi'}.$$

By multiplicativity, this defines $\mu_{\pi\times\pi'}(\mathfrak{n})$ when $\gcd(\mathfrak{n}, \mathfrak{q}_{\pi}\mathfrak{q}_{\pi'}) = \mathcal{O}_F$. For a partition $\mu = (\mu_i)_{i=1}^{\infty}$, let $\mu^* = (\mu_i^*)_{i=1}^{\infty}$ be the dual partition defined by $\mu_i^* = |\{j \colon \mu_j \geq i\}|$. It follows from the dual Cauchy identity [Bum13, Chapter 38] and (3.6) that

$$\sum_{k=0}^{\infty} \frac{\mu_{\pi\times\pi'}(\mathfrak{p}^k)}{\mathrm{N}\mathfrak{p}^{ks}} = \sum_{\mu} \frac{s_{\mu}(A_{\pi}(\mathfrak{p}))s_{\mu^*}(-A_{\pi'}(\mathfrak{p}))}{\mathrm{N}\mathfrak{p}^{|\mu|s}}, \quad \mathfrak{p} \nmid \mathfrak{q}_{\pi}\mathfrak{q}_{\pi'},$$

where $-A_{\pi'}(\mathfrak{p}) = \{-\alpha_{1,\pi'}(\mathfrak{p}), \ldots, -\alpha_{n,\pi'}(\mathfrak{p})\}$. Hence, we have

$$\mu_{\pi\times\pi'}(\mathfrak{n}) = \sum_{(\mu_{\mathfrak{p}})_{\mathfrak{p}} \in \underline{\mu}[\mathfrak{n}]} \prod_{\mathfrak{p}} s_{\mu_{\mathfrak{p}}}(A_{\pi}(\mathfrak{p}))s_{\mu_{\mathfrak{p}}^*}(-A_{\pi}(\mathfrak{p})), \quad \gcd(\mathfrak{n}, \mathfrak{q}_{\pi}\mathfrak{q}_{\pi'}) = \mathcal{O}_F. \tag{3.16}$$

LEMMA 3.3. *If $\gcd(\mathfrak{n}, \mathfrak{q}_{\pi}\mathfrak{q}_{\pi'}) = \mathcal{O}_F$, then we have $|\mu_{\pi\times\pi'}(\mathfrak{n})| \leq \frac{1}{2}(\lambda_{\pi\times\widetilde{\pi}}(\mathfrak{n}) + \lambda_{\pi'\times\widetilde{\pi}'}(\mathfrak{n}))$.*

Proof. We apply the inequality of arithmetic and geometric means to (3.16):

$$|\mu_{\pi\times\pi'}(\mathfrak{n})| \leq \frac{1}{2}\bigg(\sum_{(\mu_{\mathfrak{p}})_{\mathfrak{p}} \in \underline{\mu}[\mathfrak{n}]} \Big| \prod_{\mathfrak{p}} s_{\mu_{\mathfrak{p}}}(A_{\pi}(\mathfrak{p})) \Big|^2 + \sum_{(\mu_{\mathfrak{p}})_{\mathfrak{p}} \in \underline{\mu}[\mathfrak{n}]} \Big| \prod_{\mathfrak{p}} s_{\mu_{\mathfrak{p}}^*}(-A_{\pi'}(\mathfrak{p})) \Big|^2 \bigg).$$

The first sum equals $\lambda_{\pi\times\widetilde{\pi}}(\mathfrak{n})$ by (3.14). For the second sum, note that since $|\mu| = |\mu^*|$, we have $(\mu_{\mathfrak{p}})_{\mathfrak{p}} \in \underline{\mu}[\mathfrak{n}]$ if and only if $(\mu_{\mathfrak{p}}^*)_{\mathfrak{p}} \in \underline{\mu}[\mathfrak{n}]$. Hence, by rearranging, we see that

$$\sum_{(\mu_{\mathfrak{p}})_{\mathfrak{p}} \in \underline{\mu}[\mathfrak{n}]} \Big| \prod_{\mathfrak{p}} s_{\mu_{\mathfrak{p}}^*}(-A_{\pi'}(\mathfrak{p})) \Big|^2 = \sum_{(\mu_{\mathfrak{p}})_{\mathfrak{p}} \in \underline{\mu}[\mathfrak{n}]} \Big| \prod_{\mathfrak{p}} s_{\mu_{\mathfrak{p}}}(-A_{\pi'}(\mathfrak{p})) \Big|^2 = \lambda_{\pi\times\widetilde{\pi}'}(\mathfrak{n}), \tag{3.17}$$

where the last equality holds because $\alpha_{j,\pi'}(\mathfrak{p})\overline{\alpha_{j',\pi'}(\mathfrak{p})} = (-\alpha_{j,\pi'}(\mathfrak{p}))\overline{(-\alpha_{j',\pi'}(\mathfrak{p}))}$. □

4. A new mean value estimate

Our proof of Theorem 1.1 uses the following new mean value estimate for the Dirichlet coefficients of $L(s, \pi \times \pi')$ and $L(s, \pi \times \pi')^{-1}$. Let $\mathcal{S} \subseteq \mathfrak{F}_n$, and let $\mathcal{S}(Q) = \{\pi \in \mathcal{S} \colon C(\pi) \leq Q\}$.

THEOREM 4.1. *Let b be a complex-valued function supported on the integral ideals of $\mathcal{O}_F$. Let $n, n' \geq 1$, and let $\pi' \in \mathfrak{F}_{n'}$. Let $Q, T \geq 1$, $\varepsilon > 0$, and $x \geq 1$. Both*

$$\sum_{\pi \in \mathcal{S}(Q)} \Big| \sum_{\substack{\mathrm{N}\mathfrak{n} \in (x, xe^{1/T}] \\ \gcd(\mathfrak{n}, \mathfrak{q}_{\pi}\mathfrak{q}_{\pi'}) = \mathcal{O}_F}} \mu_{\pi\times\pi'}(\mathfrak{n})b(\mathfrak{n}) \Big|^2 \quad \text{and} \quad \sum_{\pi \in \mathcal{S}(Q)} \Big| \sum_{\substack{\mathrm{N}\mathfrak{n} \in (x, xe^{1/T}] \\ \gcd(\mathfrak{n}, \mathfrak{q}_{\pi}\mathfrak{q}_{\pi'}) = \mathcal{O}_F}} \lambda_{\pi\times\pi'}(\mathfrak{n})b(\mathfrak{n}) \Big|^2$$

are

$$\ll_{\varepsilon} Q^{\varepsilon}\Big(\frac{x}{T} + Q^{4n^2\theta_n + n}T^{(1/2)n^2[F:\mathbb{Q}]+\varepsilon}|\mathcal{S}(Q)|\Big) \sum_{\substack{\mathrm{N}\mathfrak{n} \in (x, xe^{1/T}] \\ \gcd(\mathfrak{n}, \mathfrak{q}_{\pi'}) = \mathcal{O}_F}} \lambda_{\pi'\times\widetilde{\pi}'}(\mathfrak{n})|b(\mathfrak{n})|^2,$$

where $\theta_n \in [0, \frac{1}{2} - 1/(n^2+1)]$ is the best known exponent towards GRC for $\pi \in \mathcal{S}$.

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Remark 4.2. If $\pi' = \mathbb{1}$ and $\mathcal{S} = \mathfrak{F}_n$, then Theorem 4.1 recovers [TZ21, Theorem 1.1]:[3]

$$\sum_{\pi \in \mathfrak{F}_n(Q)} \Big| \sum_{\substack{\mathrm{N}\mathfrak{n} \in (x, xe^{1/T}] \\ \gcd(\mathfrak{n}, \mathfrak{q}_\pi) = \mathcal{O}_F}} \lambda_\pi(\mathfrak{n}) b(\mathfrak{n}) \Big|^2$$
$$\ll_\varepsilon Q^\varepsilon \left(\frac{x}{T} + Q^{4n^2\theta_n + n} T^{(1/2)n^2[F:\mathbb{Q}] + \varepsilon} |\mathfrak{F}_n(Q)| \right) \sum_{\substack{\mathrm{N}\mathfrak{n} \in (x, xe^{1/T}] \\ \gcd(\mathfrak{n}, \mathfrak{q}_{\pi'}) = \mathcal{O}_F}} |b(\mathfrak{n})|^2.$$

One could try to prove Theorem 4.1 starting with this, replacing $b(\mathfrak{n})$ with $\lambda_{\pi'}(\mathfrak{n})\mu_F(\mathfrak{n})b(\mathfrak{n})$ (where $\mu_F(\mathfrak{n})$ is the $\mathfrak{n}$th Dirichlet coefficient of $\zeta_F(s)^{-1}$), and try to recover a version of Theorem 4.1 with $\mu_{\pi\times\pi'}(\mathfrak{n})$ replaced by $\lambda_\pi(\mathfrak{n})\lambda_{\pi'}(\mathfrak{n})\mu_F(\mathfrak{n})$. Note that $\mu_{\pi\times\pi'}(\mathfrak{n}) = \lambda_\pi(\mathfrak{n})\lambda_{\pi'}(\mathfrak{n})\mu_F(\mathfrak{n})$ when $\mathfrak{n}$ is squarefree and coprime to $\mathfrak{q}_\pi\mathfrak{q}_{\pi'}$. Otherwise, equality is not guaranteed. If one wants to approximate $L(s, \pi \times \pi')$ with $\sum_{\mathfrak{n} \text{ squarefree}} \lambda_\pi(\mathfrak{n})\lambda_{\pi'}(\mathfrak{n})\mathrm{N}\mathfrak{n}^{-s}$ and extend into the critical strip when $\pi, \pi' \in \mathfrak{F}_n$ and $n \geq 5$, then one must have progress towards GRC well beyond what is known unconditionally [Bru06b, DK00]. Such progress would then be a hypothesis for Theorem 1.1.

Theorem 4.1 provides the first non-trivial unconditional mean value estimates of large sieve type for the Dirichlet coefficients $\lambda_{\pi\times\pi'}(\mathfrak{n})$ or $\mu_{\pi\times\pi'}(\mathfrak{n})$ for arbitrary n and n'. Theorem 4.1 follows from a more general result, Proposition 4.3, for sequences of products of Schur polynomials evaluated on the set $A_\pi(\mathfrak{p})$ of Satake parameters of π at $\mathfrak{p}$. We begin with a mean value estimate for the Satake parameters of π as $\pi \in \mathcal{S}$ varies. Let

$$a : \bigcup_{x < \mathrm{N}\mathfrak{n} \leq xe^{1/T}} \underline{\mu}[\mathfrak{n}] \to \mathbb{C}, \quad \alpha : \mathcal{S}(Q) \to \mathbb{C} \tag{4.1}$$

be functions that are not identically zero. Their ℓ^2 norms $\|a\|_2$ and $\|\alpha\|_2$ are defined by

$$\|a\|_2 = \Bigg(\sum_{x < \mathrm{N}\mathfrak{n} \leq xe^{1/T}} \sum_{(\mu_\mathfrak{p})_\mathfrak{p} \in \underline{\mu}[\mathfrak{n}]} |a((\mu_\mathfrak{p})_\mathfrak{p})|^2 \Bigg)^{1/2}, \quad \|\alpha\|_2 = \Bigg(\sum_{\pi \in \mathcal{S}(Q)} |\alpha(\pi)|^2 \Bigg)^{1/2}.$$

For convenience, we define $\mathbf{1}_{(\mathfrak{n},\mathfrak{q})}$ to equal one when $\gcd(\mathfrak{n}, \mathfrak{q}) = \mathcal{O}_F$ and zero otherwise.

Proposition 4.3. *Let $x \geq 1$ and $Q, T \geq 1$. Define*

$$C(Q,T,x) := \sup_{\|a\|_2 \neq 0} \frac{1}{\|a\|_2^2} \sum_{\pi \in \mathcal{S}(Q)} \Bigg| \sum_{\substack{x < \mathrm{N}\mathfrak{n} \leq xe^{1/T} \\ \gcd(\mathfrak{n}, \mathfrak{q}_\pi) = \mathcal{O}_F}} \sum_{(\mu_\mathfrak{p})_\mathfrak{p} \in \underline{\mu}[\mathfrak{n}]} \Bigg[\prod_\mathfrak{p} s_{\mu_\mathfrak{p}}(A_\pi(\mathfrak{p})) \Bigg] a((\mu_\mathfrak{p})_\mathfrak{p}) \Bigg|^2.$$

We have the bound $C(Q,T,x) \ll_\varepsilon Q^\varepsilon(T^{-1}x + Q^{4n^2\theta_n + n} T^{n^2[F:\mathbb{Q}]/2+\varepsilon} |\mathcal{S}(Q)|)$.

Proof. We observe that

$$C(Q,T,x) = \sup_{\|a\|_2 = 1} \sum_{\pi \in \mathcal{S}(Q)} \Bigg| \sum_{x < \mathrm{N}\mathfrak{n} \leq xe^{1/T}} \sum_{(\mu_\mathfrak{p})_\mathfrak{p} \in \underline{\mu}[\mathfrak{n}]} \Bigg[\prod_\mathfrak{p} s_{\mu_\mathfrak{p}}(A_\pi(\mathfrak{p})) \Bigg] \mathbf{1}_{(\mathfrak{n},\mathfrak{q}_\pi)} a((\mu_\mathfrak{p})_\mathfrak{p}) \Bigg|^2. \tag{4.2}$$

By the duality principle for bilinear forms, (4.2) is bounded by the supremum over the functions $\alpha \colon \mathcal{S}(Q) \to \mathbb{C}$ such that $\|\alpha\|_2 = 1$ of

$$\sum_{x < \mathrm{N}\mathfrak{n} \leq xe^{1/T}} \sum_{(\mu_\mathfrak{p})_\mathfrak{p} \in \underline{\mu}[\mathfrak{n}]} \Bigg| \sum_{\pi \in \mathcal{S}(Q)} \Bigg[\prod_\mathfrak{p} s_{\mu_\mathfrak{p}}(A_\pi(\mathfrak{p})) \Bigg] \mathbf{1}_{(\mathfrak{n},\mathfrak{q}_\pi)} \alpha(\pi) \Bigg|^2. \tag{4.3}$$

[3] The factor $Q^{4n^2\theta_n+n}$ fixes a minor error in the proof of [TZ21, Theorem 1.1], which led to a factor of Q^{n^2+n} instead.

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Let ϕ be a fixed smooth test function, supported in a compact subset of $[-2,2]$, such that $\phi(t)=1$ for $t\in[0,1]$ and $\phi(t)\in[0,1)$ otherwise. Then (4.3) is at most

$$\sum_{\mathfrak{n}}\sum_{(\mu_{\mathfrak{p}})_{\mathfrak{p}}\in\underline{\mu}[\mathfrak{n}]}\Big|\sum_{\pi\in\mathcal{S}(Q)}\Big[\prod_{\mathfrak{p}}s_{\mu_{\mathfrak{p}}}(A_{\pi}(\mathfrak{p}))\Big]\mathbf{1}_{(\mathfrak{n},\mathfrak{q}_{\pi})}\alpha(\pi)\Big|^{2}\phi\Big(T\log\frac{\mathrm{N}\mathfrak{n}}{x}\Big). \tag{4.4}$$

We expand the square, interchange the order of summation, and apply (3.14) to find that (4.4) equals

$$\begin{aligned}&\sum_{\pi,\pi'\in\mathcal{S}(Q)}\alpha(\pi)\overline{\alpha(\pi')}\sum_{\mathfrak{n}}\sum_{(\mu_{\mathfrak{p}})_{\mathfrak{p}}\in\underline{\mu}[\mathfrak{n}]}\Big[\prod_{\mathfrak{p}}s_{\mu_{\mathfrak{p}}}(A_{\pi}(\mathfrak{p}))\Big]\overline{\Big[\prod_{\mathfrak{p}}s_{\mu_{\mathfrak{p}}}(A_{\pi'}(\mathfrak{p}))\Big]}\mathbf{1}_{(\mathfrak{n},\mathfrak{q}_{\pi})}\mathbf{1}_{(\mathfrak{n},\mathfrak{q}_{\pi'})}\phi\Big(T\log\frac{\mathrm{N}\mathfrak{n}}{x}\Big)\\&=\sum_{\pi,\pi'\in\mathcal{S}(Q)}\alpha(\pi)\overline{\alpha(\pi')}\sum_{\gcd(\mathfrak{n},\mathfrak{q}_{\pi}\mathfrak{q}_{\pi'})=\mathcal{O}_F}\lambda_{\pi\times\widetilde{\pi}'}(\mathfrak{n})\phi\Big(T\log\frac{\mathrm{N}\mathfrak{n}}{x}\Big).\end{aligned} \tag{4.5}$$

Let $\kappa_{\pi\times\pi'}=\mathrm{Res}_{s=1}L(s,\pi\times\pi')\prod_{\mathfrak{p}|\mathfrak{q}_{\pi}\mathfrak{q}_{\pi'}}L(s,\pi_{\mathfrak{p}}\times\pi'_{\mathfrak{p}})^{-1}$. Note that $\kappa_{\pi\times\pi'}\geq 0$, with equality if and only if $\pi'\neq\widetilde{\pi}$. Since $|\alpha_{j,j',\pi\times\pi'}(\mathfrak{p})|\leq\mathrm{N}\mathfrak{p}$, we have the bound

$$\prod_{\mathfrak{p}|\mathfrak{q}_{\pi}\mathfrak{q}_{\pi'}}|L(1,\pi_{\mathfrak{p}}\times\pi'_{\mathfrak{p}})|^{-1}=\prod_{\mathfrak{p}|\mathfrak{q}_{\pi}\mathfrak{q}_{\pi'}}\prod_{j=1}^{n}\prod_{j'=1}^{n'}\Big|1-\frac{\alpha_{j,j',\pi\times\pi'}(\mathfrak{p})}{\mathrm{N}\mathfrak{p}}\Big|\leq\prod_{\mathfrak{p}|\mathfrak{q}_{\pi}\mathfrak{q}_{\pi'}}2^{n'n}.$$

Since $|\{\mathfrak{p}\colon\mathfrak{p}\mid\mathfrak{n}\}|\ll(\log\mathrm{N}\mathfrak{n})/\log\log\mathrm{N}\mathfrak{n}$ (see [Wei83, Lemma 1.13b]), it follows from (3.12) that

$$\kappa_{\pi\times\widetilde{\pi}}\ll_{\varepsilon}C(\pi)^{\varepsilon}. \tag{4.6}$$

Let $\widehat{\phi}(s)=\int_{\mathbb{R}}\phi(y)e^{sy}\,dy$. It follows from a standard contour integral calculation using (3.7) and (3.13) that (4.5) equals

$$\begin{aligned}&\sum_{\pi,\pi'\in\mathcal{S}(Q)}\alpha(\pi)\overline{\alpha(\pi')}\Big(\frac{1}{2\pi iT}\int_{3-i\infty}^{3+i\infty}\frac{L(s,\pi\times\widetilde{\pi}')}{\prod_{\mathfrak{p}|\mathfrak{q}_{\pi}\mathfrak{q}_{\pi'}}L(s,\pi_{\mathfrak{p}}\times\widetilde{\pi}'_{\mathfrak{p}})}x^{s}\widehat{\phi}(s/T)\,ds\Big)\\&=\sum_{\pi,\pi'\in\mathcal{S}(Q)}\alpha(\pi)\overline{\alpha(\pi')}\Big(\kappa_{\pi\times\widetilde{\pi}'}x\frac{\widehat{\phi}(1/T)}{T}+\frac{1}{2\pi iT}\int_{1/4\log(ex)-i\infty}^{1/4\log(ex)+i\infty}\frac{L(s,\pi\times\widetilde{\pi}')x^{s}\widehat{\phi}(s/T)}{\prod_{\mathfrak{p}|\mathfrak{q}_{\pi}\mathfrak{q}_{\pi'}}L(s,\pi_{\mathfrak{p}}\times\widetilde{\pi}'_{\mathfrak{p}})}\,ds\Big)\\&=\sum_{\pi,\pi'\in\mathcal{S}(Q)}\alpha(\pi)\overline{\alpha(\pi')}\Big(\kappa_{\pi\times\widetilde{\pi}'}x\frac{\widehat{\phi}(1/T)}{T}+O_{\phi,\varepsilon}(Q^{4n^2\theta_n+n+\varepsilon}T^{(1/2)n^2[F:\mathbb{Q}]+\varepsilon})\Big).\end{aligned} \tag{4.7}$$

Recall that $\kappa_{\pi\times\widetilde{\pi}'}>0$ when $\pi=\pi'$, and $\kappa_{\pi\times\widetilde{\pi}'}=0$ otherwise. Since $\|\alpha\|_2=1$ and ϕ is fixed, it follows from the inequality of arithmetic and geometric means that (4.7) equals

$$\begin{aligned}&\frac{\widehat{\phi}(1/T)}{T}x\sum_{\pi\in\mathcal{S}(Q)}|\alpha(\pi)|^{2}\kappa_{\pi\times\widetilde{\pi}'}+O_{\phi}(Q^{4n^2\theta_n+n+\varepsilon}T^{(1/2)n^2[F:\mathbb{Q}]+\varepsilon}|\mathcal{S}(Q)|)\\&\ll_{\phi,\varepsilon}\frac{x}{T}\max_{\pi\in\mathcal{S}(Q)}\kappa_{\pi\times\widetilde{\pi}'}+Q^{4n^2\theta_n+n+\varepsilon}T^{(1/2)n^2[F:\mathbb{Q}]+\varepsilon}|\mathcal{S}(Q)|.\end{aligned}$$

We estimate the maximum using (4.6), and the desired result follows. □

We use Proposition 4.3 to prove Theorem 4.1.

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Proof of Theorem 4.1. For the sum involving $\mu_{\pi\times\pi'}(\mathfrak{n})$, we apply Proposition 4.3 with

$$a((\mu_{\mathfrak{p}})_{\mathfrak{p}}) = b\Big(\prod_{\mathfrak{p}} \mathrm{N}\mathfrak{p}^{|\mu_{\mathfrak{p}}|}\Big)\mathbf{1}_{(\prod_{\mathfrak{p}} \mathrm{N}\mathfrak{p}^{|\mu_{\mathfrak{p}}|},\mathfrak{q}_{\pi'})}\prod_{\mathfrak{p}} s_{\mu_{\mathfrak{p}}^*}(-A_{\pi'}(\mathfrak{p})).$$

If $(\mu_{\mathfrak{p}})_{\mathfrak{p}} \in \underline{\mu}[\mathfrak{n}]$, then by (3.15), we have that $|\mu_{\mathfrak{p}}| = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})$ and

$$a((\mu_{\mathfrak{p}})_{\mathfrak{p}}) = b(\mathfrak{n})\mathbf{1}_{(\mathfrak{n},\mathfrak{q}_{\pi'})}\prod_{\mathfrak{p}} s_{\mu_{\mathfrak{p}}^*}(-A_{\pi'}(\mathfrak{p})).$$

By (3.16), the left-hand side of Proposition 4.3 becomes

$$\begin{aligned}
&\sum_{\pi\in\mathcal{S}(Q)}\Bigg|\sum_{\substack{x<\mathrm{N}\mathfrak{n}\le xe^{1/T}\\ \gcd(\mathfrak{n},\mathfrak{q}_\pi)=\mathcal{O}_F}}\ \sum_{(\mu_{\mathfrak{p}})_{\mathfrak{p}}\in\underline{\mu}[\mathfrak{n}]}\Big[\prod_{\mathfrak{p}} s_{\mu_{\mathfrak{p}}}(A_\pi(\mathfrak{p}))\Big]a((\mu_{\mathfrak{p}})_{\mathfrak{p}})\Bigg|^2\\
&=\sum_{\pi\in\mathcal{S}(Q)}\Bigg|\sum_{\substack{x<\mathrm{N}\mathfrak{n}\le xe^{1/T}\\ \gcd(\mathfrak{n},\mathfrak{q}_\pi)=\mathcal{O}_F}} b(\mathfrak{n})\mathbf{1}_{(\mathfrak{n},\mathfrak{q}_{\pi'})}\sum_{(\mu_{\mathfrak{p}})_{\mathfrak{p}}\in\underline{\mu}[\mathfrak{n}]}\prod_{\mathfrak{p}} s_{\mu_{\mathfrak{p}}}(A_\pi(\mathfrak{p}))s_{\mu_{\mathfrak{p}}^*}(-A_{\pi'}(\mathfrak{p})\Bigg|^2\\
&=\sum_{\pi\in\mathcal{S}(Q)}\Bigg|\sum_{\substack{x<\mathrm{N}\mathfrak{n}\le xe^{1/T}\\ \gcd(\mathfrak{n},\mathfrak{q}_\pi\mathfrak{q}_{\pi'})=\mathcal{O}_F}} \mu_{\pi\times\pi'}(\mathfrak{n})b(\mathfrak{n})\Bigg|^2.
\end{aligned}$$

Similarly, the right-hand side of Proposition 4.3 becomes

$$Q^{\varepsilon}\Big(\frac{x}{T} + Q^{4n^2\theta_n+n}T^{(1/2)n^2[F:\mathbb{Q}]+\varepsilon}|\mathcal{S}(Q)|\Big)\sum_{\substack{\mathrm{N}\mathfrak{n}\in(x,xe^{1/T}]\\ \gcd(\mathfrak{n},\mathfrak{q}_{\pi'})=\mathcal{O}_F}}|b(\mathfrak{n})|^2\sum_{(\mu_{\mathfrak{p}})_{\mathfrak{p}}\in\underline{\mu}[\mathfrak{n}]}\Big|\prod_{\mathfrak{p}} s_{\mu_{\mathfrak{p}}^*}(-A_{\pi'}(\mathfrak{p}))\Big|^2.$$

The desired result now follows from (3.17). Apart from the new choice of

$$a((\mu_{\mathfrak{p}})_{\mathfrak{p}}) = b\Big(\prod_{\mathfrak{p}} \mathrm{N}\mathfrak{p}^{|\mu_{\mathfrak{p}}|}\Big)\mathbf{1}_{(\prod_{\mathfrak{p}} \mathrm{N}\mathfrak{p}^{|\mu_{\mathfrak{p}}|},\mathfrak{q}_{\pi'})}\prod_{\mathfrak{p}} s_{\mu_{\mathfrak{p}}}(A_{\pi'}(\mathfrak{p})),$$

the result for the sum involving $\lambda_{\pi\times\pi'}(\mathfrak{n})$ is handled in the same manner. □

COROLLARY 4.4. *Let $\pi' \in \mathfrak{F}_{n'}$, $Q, T \ge 1$, and $\varepsilon > 0$. If $Y \ge e$ and*

$$X \ge Q^{4n^2\theta_n+n+\varepsilon}T^{(1/2)n^2[F:\mathbb{Q}]+1+\varepsilon}|\mathcal{S}(Q)|, \quad \log Y \asymp_{\varepsilon} \log X,$$

then

$$\sum_{\pi\in\mathcal{S}(Q)}\int_{-T}^{T}\Bigg|\sum_{\substack{\mathrm{N}\mathfrak{n}>X\\ \gcd(\mathfrak{n},\mathfrak{q}_\pi\mathfrak{q}_{\pi'})=\mathcal{O}_F}}\frac{\mu_{\pi\times\pi'}(\mathfrak{n})}{\mathrm{N}\mathfrak{n}^{1+1/\log Y+iv}}\Bigg|^2 dv \ll_{\varepsilon} C(\pi')^{\varepsilon}Q^{\varepsilon}\log X,$$

$$\sum_{\pi\in\mathcal{S}(Q)}\int_{-T}^{T}\Bigg|\sum_{\substack{\mathrm{N}\mathfrak{n}\le X\\ \gcd(\mathfrak{n},\mathfrak{q}_\pi\mathfrak{q}_{\pi'})=\mathcal{O}_F}}\frac{\mu_{\pi\times\pi'}(\mathfrak{n})}{\mathrm{N}\mathfrak{n}^{1/2+iv}}\Bigg|^2 dv \ll_{\varepsilon} C(\pi')^{\varepsilon}Q^{\varepsilon}X\log X.$$

Proof. We prove the first result; the second result is proved completely analogously. A formal generalization of [Gal70, Theorem 1] to number fields tells us that if $c(\mathfrak{n})$ is a complex-valued

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function supported on the integral ideals of $\mathcal{O}_F$ with $\sum_{\mathfrak{n}}|c(\mathfrak{n})|<\infty$, then

$$\int_{-T}^{T}\Big|\sum_{\mathfrak{n}}\frac{c(\mathfrak{n})}{\mathrm{N}\mathfrak{n}^{it}}\Big|^2 dt \ll T^2\int_0^\infty\Big|\sum_{\mathrm{N}\mathfrak{n}\in(x,xe^{1/T}]}c(\mathfrak{n})\Big|^2\frac{dx}{x}.$$

We choose $b(\mathfrak{n})=\mathrm{N}\mathfrak{n}^{-1-1/\log Y}$ if $\mathrm{N}\mathfrak{n}>X$ and $b(\mathfrak{n})=0$ otherwise, which leads to

$$\begin{aligned}\sum_{\pi\in\mathcal{S}(Q)}\int_{-T}^{T}\Big|&\sum_{\substack{\mathrm{N}\mathfrak{n}>X\\ \gcd(\mathfrak{n},\mathfrak{q}_\pi\mathfrak{q}_{\pi'})=\mathcal{O}_F}}\frac{\mu_{\pi\times\pi'}(\mathfrak{n})}{\mathrm{N}\mathfrak{n}^{1+1/\log Y+it}}\Big|^2 dt\\ &\ll T^2\int_0^\infty\sum_{\pi\in\mathcal{S}(Q)}\Big|\sum_{\mathrm{N}\mathfrak{n}\in(x,xe^{1/T}]}\mu_{\pi\times\pi'}(\mathfrak{n})b(\mathfrak{n})\Big|^2\frac{dx}{x}.\end{aligned}$$

Theorem 4.1 and the fact that $\lambda_{\pi'\times\widetilde{\pi}'}(\mathfrak{n})\geq 0$ for all $\mathfrak{n}$ imply that the above display is

$$\begin{aligned}&\ll_\varepsilon Q^\varepsilon\sum_{\substack{\mathrm{N}\mathfrak{n}>X\\ \gcd(\mathfrak{n},\mathfrak{q}_{\pi'})=\mathcal{O}_F}}\frac{\lambda_{\pi'\times\widetilde{\pi}'}(\mathfrak{n})}{\mathrm{N}\mathfrak{n}^{1+2/\log Y}}\Big(1+\frac{Q^{4n^2\theta_n+n}T^{(1/2)n^2[F:\mathbb{Q}]+1+\varepsilon}|\mathcal{S}(Q)|}{\mathrm{N}\mathfrak{n}}\Big)\\ &\ll_\varepsilon Q^\varepsilon\sum_{\mathrm{N}\mathfrak{n}>X}\frac{\lambda_{\pi'\times\widetilde{\pi}'}(\mathfrak{n})}{\mathrm{N}\mathfrak{n}^{1+2/\log Y}}.\end{aligned}$$

The desired result now follows from Lemma 3.1 and our choices of X and Y. □

5. Proof of Theorem 1.1

Let $n,n'\geq 1$, $\varepsilon>0$, $T\geq 3$, $Q\geq 3$, $\pi\in\mathcal{S}(Q)$, and $\pi'\in\mathfrak{F}_{n'}$. Define

$$\begin{aligned}X &\coloneqq C(\pi')^\varepsilon Q^{4n^2\theta_n+n+\varepsilon}T^{(1/2)n^2[F:\mathbb{Q}]+1+\varepsilon}|\mathcal{S}(Q)|,\\ Y &\coloneqq \big((C(\pi')^{n/2}Q^{n'/2}(C(\pi')Q)^{2n'n\max\{0,\theta_n+\theta_{n'}-1/2\}}|\mathcal{S}(Q)|T^{n'n[F:\mathbb{Q}]/2+1})^{1+\varepsilon/4n'n[F:\mathbb{Q}]}X\big)^{1/(3-2\sigma)},\\ L^{\mathrm{ur}}(s,\pi\times\pi') &\coloneqq L(s,\pi\times\pi')\prod_{\mathfrak{p}|\mathfrak{q}_\pi\mathfrak{q}_{\pi'}}L(s,\pi_{\mathfrak{p}}\times\pi'_{\mathfrak{p}})^{-1},\\ M_X(s,\pi\times\pi') &\coloneqq \sum_{\substack{\mathrm{N}\mathfrak{n}\leq X\\ \gcd(\mathfrak{n},\mathfrak{q}_\pi\mathfrak{q}_{\pi'})=\mathcal{O}_F}}\frac{\mu_{\pi\times\pi'}(\mathfrak{n})}{\mathrm{N}\mathfrak{n}^s},\\ LM_X(s,\pi\times\pi') &\coloneqq L^{\mathrm{ur}}(s,\pi\times\pi')M_X(s,\pi\times\pi').\end{aligned}\tag{5.1}$$

Note that $\log(C(\pi)C(\pi')T)\ll\log X\asymp_\varepsilon\log Y$. We assume that $T\geq 2(\log Y)^2$ since the proof does not change appreciably otherwise. In this section, ε might vary from line to line, and terms of size $(C(\pi')QT^{[F:\mathbb{Q}]}|\mathcal{S}(Q)|)^\varepsilon$ and polynomials in $\log(C(\pi')QT^{[F:\mathbb{Q}]}|\mathcal{S}(Q)|)$ might be bounded by X^ε without mention.

If $t\in[-T,T]$, then

$$|\{\rho=\beta+i\gamma\colon \beta\geq 0,\ |\gamma-t|\leq 1\}|\ll\log C(\pi\times\pi',t)\ll\log(C(\pi)C(\pi')T)$$

by [IK04, Proposition 5.7]. We observe that the rectangle $[\sigma,1]\times[-T,T]$ is covered by $O(T)$ boxes of the form $[\sigma,1]\times[y,y+2(\log Y)^2]$, each containing $O((\log(C(\pi)C(\pi')T))^3)$ zeros. If we

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write $n_{\pi\times\pi'}$ for the number of such boxes containing at least one zero ρ of $L(s,\pi\times\pi')$, then

$$N_{\pi\times\pi'}(\sigma,T)\ll(\log(C(\pi')QT))^3 n_{\pi\times\pi'}.$$

If $\pi'=\widetilde{\pi}$, then it suffices to assume that $|\rho-1|>1$ since there are $O(\log(C(\pi')QT))$ zeros ρ such that $|\rho-1|\le 1$.

Let $\rho=\beta+i\gamma$ be a zero of $L(s,\pi\times\pi')$, in which case $LM_X(\rho,\pi\times\pi')=0$. We compute

$$\begin{aligned}e^{-1/Y}&=\frac{1}{2\pi i}\int_{1-\beta+1/\log Y-i\infty}^{1-\beta+1/\log Y+i\infty}(1-LM_X(\rho+w,\pi\times\pi')\Gamma(w)Y^w\,dw\\&\quad+\frac{1}{2\pi i}\int_{1/2-\beta-i\infty}^{1/2-\beta+i\infty}LM_X(\rho+w,\pi\times\pi')\Gamma(w)Y^w\,dw\\&\quad+\kappa_{\pi\times\pi'}\Gamma(1-\rho)Y^{1-\rho}\sum_{\substack{\mathrm{N}\mathfrak{n}\le X\\ \gcd(\mathfrak{n},\mathfrak{q}_\pi\mathfrak{q}_{\pi'})=\mathcal{O}_F}}\frac{\mu_{\pi\times\pi'}(\mathfrak{n})}{\mathrm{N}\mathfrak{n}}.\end{aligned}\tag{5.2}$$

It follows from (3.13) and Lemmas 3.1 and 3.3 that if $\mathrm{Re}(w)=1-\beta+1/\log Y$, then

$$\begin{aligned}&|1-LM_X(\rho+w,\pi\times\pi')|\\&\quad=|L^{\mathrm{ur}}(\rho+w,\pi\times\pi')|\cdot|L^{\mathrm{ur}}(\rho+w,\pi\times\pi')^{-1}-M_X(\rho+w,\pi\times\pi')|\\&\quad\ll_\varepsilon C(\pi\times\pi',|\gamma+\mathrm{Im}(w)|+1)^\varepsilon\sum_{\substack{\mathrm{N}\mathfrak{n}>X\\ \gcd(\mathfrak{n},\mathfrak{q}_\pi\mathfrak{q}_{\pi'})=\mathcal{O}_F}}\frac{|\mu_{\pi\times\pi'}(\mathfrak{n})|}{\mathrm{N}\mathfrak{n}^{1+1/\log Y}}\\&\quad\ll_\varepsilon C(\pi\times\pi',|\gamma+\mathrm{Im}(w)|+1)^\varepsilon\sum_{\substack{\mathrm{N}\mathfrak{n}>X\\ \gcd(\mathfrak{n},\mathfrak{q}_\pi\mathfrak{q}_{\pi'})=\mathcal{O}_F}}\frac{\lambda_{\pi\times\widetilde{\pi}}(\mathfrak{n})+\lambda_{\pi'\times\widetilde{\pi}'}(\mathfrak{n})}{\mathrm{N}\mathfrak{n}^{1+1/\log Y}}\\&\quad\ll_\varepsilon C(\pi)^\varepsilon C(\pi')^\varepsilon(|\gamma+\mathrm{Im}(w)|+1)^\varepsilon\log X.\end{aligned}\tag{5.3}$$

Therefore, we deduce from Stirling's formula that

$$\begin{aligned}&\bigg|\frac{1}{2\pi i}\int_{1-\beta+1/\log Y-i\infty}^{1-\beta+1/\log Y+i\infty}(1-LM_X(\rho+w,\pi\times\pi'))\Gamma(w)Y^w\,dw\\&\quad-\frac{1}{2\pi i}\int_{1-\beta+1/\log Y-i(\log Y)^2}^{1-\beta+1/\log Y+i(\log Y)^2}(1-LM_X(\rho+w,\pi\times\pi'))\Gamma(w)Y^w\,dw\bigg|\ll\frac{1}{Y}.\end{aligned}$$

The other terms in (5.2) are handled similarly. Since $e^{-1/Y}=1+O(1/Y)$, it follows that

$$\begin{aligned}1&\ll\frac{1}{2\pi i}\int_{1-\beta+1/\log Y-i(\log Y)^2}^{1-\beta+1/\log Y+i(\log Y)^2}(1-LM_X(\rho+w,\pi\times\pi'))\Gamma(w)Y^w\,dw\\&\quad+\frac{1}{2\pi i}\int_{1/2-\beta-i(\log Y)^2}^{1/2-\beta+i(\log Y)^2}LM_X(\rho+w,\pi\times\pi')\Gamma(w)Y^w\,dw+\kappa_{\pi\times\pi'}Y^{1-\sigma}\frac{\log X}{\max\{1,|\gamma|^3\}}.\end{aligned}\tag{5.4}$$

Write the first integral in (5.4) as I_1 and the second as I_2. A simple optimization calculation shows that if $c\ge 1$ and $c^{-1}\le|I_1|+|I_2|$, then $c^{-1}\le 2c(|I_1|^2+|I_2|)$. We estimate $|I_1|^2$ using the

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Cauchy–Schwarz inequality, thus obtaining the bound

$$1 \ll (\log Y)^2 Y^{2(1-\sigma)} \int_{\gamma-(\log Y)^2}^{\gamma+(\log Y)^2} \Big|1 - LM_X\Big(1+\frac{1}{\log Y}+iv, \pi\times\pi'\Big)\Big|^2 dv$$
$$+ Y^{1/2-\sigma} \int_{\gamma-(\log Y)^2}^{\gamma+(\log Y)^2} \Big|LM_X\Big(\frac{1}{2}+iv, \pi\times\pi'\Big)\Big| dv + \kappa_{\pi\times\pi'} Y^{1-\sigma}\frac{\log X}{\max\{1,|\gamma|^3\}}.$$

Since $T \geq 2(\log Y)^2$ by hypothesis, we conclude that

$$n_{\pi\times\pi'} \ll (\log Y)^2 Y^{2(1-\sigma)} \int_{-T}^{T} \Big|1 - L\Big(1+\frac{1}{\log Y}+iv\Big)M_X\Big(1+\frac{1}{\log Y}+iv\Big)\Big|^2 dv$$
$$+ Y^{1/2-\sigma} \int_{-T}^{T} \Big|L\Big(\frac{1}{2}+iv\Big)M_X\Big(\frac{1}{2}+iv\Big)\Big| dv + \kappa_{\pi\times\pi'} Y^{1-\sigma}\log X + 1. \tag{5.5}$$

Note that $\kappa_{\pi\times\pi'} = 0$ unless $\pi' = \widetilde{\pi}$, which occurs for at most one $\pi \in \mathcal{S}(Q)$. When $\kappa_{\pi\times\pi'} \neq 0$, it satisfies $\kappa_{\pi\times\pi'} = \kappa_{\pi\times\widetilde{\pi}} \ll_\varepsilon C(\pi)^\varepsilon$ per (4.6). Therefore, summing (5.5) over $\pi \in \mathcal{S}(Q)$, we find that

$$\sum_{\pi\in\mathcal{S}(Q)} N_{\pi\times\pi'}(\sigma,T) \ll (\log C(\pi)C(\pi')T)^3 \sum_{\pi\in\mathcal{S}(Q)} n_{\pi\times\pi'}$$
$$\ll_\varepsilon X^\varepsilon\Big(Y^{2(1-\sigma)} \int_{-T}^{T} \sum_{\pi\in\mathcal{S}(Q)} \Big|1 - LM_X\Big(1+\frac{1}{\log Y}+iv, \pi\times\pi'\Big)\Big|^2 dv$$
$$+ Y^{1/2-\sigma} \int_{-T}^{T} \sum_{\pi\in\mathcal{S}(Q)} \Big|LM_X\Big(\frac{1}{2}+iv, \pi\times\pi'\Big)\Big| dv + Y^{1-\sigma}\Big). \tag{5.6}$$

For the first integral in (5.6), we deduce from (3.13) that

$$\Big|1 - LM_X\Big(1+\frac{1}{\log Y}+it, \pi\times\pi'\Big)\Big|^2$$
$$= \Big|L^{\mathrm{ur}}\Big(1+\frac{1}{\log Y}+it, \pi\times\pi'\Big)\Big|^2 \Big|\frac{1}{L^{\mathrm{ur}}(1+\frac{1}{\log Y}+it, \pi\times\pi')} - M_X\Big(1+\frac{1}{\log Y}+it, \pi\times\pi'\Big)\Big|^2$$
$$\ll_\varepsilon X^\varepsilon \Big| \sum_{\substack{\mathrm{N}\mathfrak{n}>X \\ \gcd(\mathfrak{n},\mathfrak{q}_\pi\mathfrak{q}_{\pi'})=\mathcal{O}_F}} \frac{\mu_{\pi\times\pi'}(\mathfrak{n})}{\mathrm{N}\mathfrak{n}^{1+1/\log Y}}\Big|^2.$$

For the second integral in (5.6), it follows from the Cauchy–Schwarz inequality that

$$\int_{-T}^{T} \sum_{\pi\in\mathcal{S}(Q)} \Big|LM_X\Big(\frac{1}{2}+iv, \pi\times\pi'\Big)\Big| dv \leq \Big(\int_{-T}^{T} \sum_{\pi\in\mathcal{S}(Q)} \Big|L^{\mathrm{ur}}\Big(\frac{1}{2}+iv, \pi\times\pi'\Big)\Big|^2 dv\Big)^{1/2}$$
$$\times \Big(\int_{-T}^{T} \sum_{\pi\in\mathcal{S}(Q)} \Big| \sum_{\substack{\mathrm{N}\mathfrak{n}\leq X \\ \gcd(\mathfrak{n},\mathfrak{q}_\pi\mathfrak{q}_{\pi'})=\mathcal{O}_F}} \frac{\mu_{\pi\times\pi'}(\mathfrak{n})}{\mathrm{N}\mathfrak{n}^{1/2+iv}}\Big|^2 dv\Big)^{1/2}.$$

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We bound the second moment of $L^{\mathrm{ur}}(\frac{1}{2}+iv,\pi\times\pi')$ trivially as

$$\int_{-T}^{T}\sum_{\pi\in\mathcal{S}(Q)}\left|L^{\mathrm{ur}}\left(\frac{1}{2}+iv,\pi\times\pi'\right)\right|^2 dv$$
$$\ll_\varepsilon (QC(\pi'))^{2n'n\max\{0,\theta_n+\theta_{n'}-1/2\}+\varepsilon}Q^{n'/2}C(\pi')^{n/2}T^{n'n[F:\mathbb{Q}]/2+1+\varepsilon}|\mathcal{S}(Q)| \ll_\varepsilon \frac{Y^{3-2\sigma}}{X}$$

using (3.7) (to bound the ramified Euler factors) and (3.13). In summary, we find that

$$\sum_{\pi\in\mathcal{S}(Q)}N_{\pi\times\pi'}(\sigma,T)\ll_\varepsilon X^\varepsilon\Bigg(Y^{2(1-\sigma)}\int_{-T}^{T}\sum_{\pi\in\mathcal{S}(Q)}\Bigg|\sum_{\substack{\mathrm{N}\mathfrak{n}>X\\ \gcd(\mathfrak{n},\mathfrak{q}_\pi\mathfrak{q}_{\pi'})=\mathcal{O}_F}}\frac{\mu_{\pi\times\pi'}(\mathfrak{n})}{\mathrm{N}\mathfrak{n}^{1+1/\log Y+iv}}\Bigg|^2 dv$$
$$+Y^{1/2-\sigma}\Bigg(\frac{Y^{3-2\sigma}}{X}\int_{-T}^{T}\sum_{\pi\in\mathcal{S}(Q)}\Bigg|\sum_{\substack{\mathrm{N}\mathfrak{n}\le X\\ \gcd(\mathfrak{n},\mathfrak{q}_\pi\mathfrak{q}_{\pi'})=\mathcal{O}_F}}\frac{\mu_{\pi\times\pi'}(\mathfrak{n})}{\mathrm{N}\mathfrak{n}^{1/2+iv}}\Bigg|^2 dv\Bigg)^{1/2}+Y^{1-\sigma}\Bigg).$$

We bound the two v-integrals using Corollary 4.4 and (5.1), thus obtaining

$$\sum_{\pi\in\mathcal{S}(Q)}N_{\pi\times\pi'}(\sigma,T)\ll_\varepsilon Y^{2(1-\sigma)}X^\varepsilon.$$

Theorem 1.1 now follows from (5.1), the bounds for θ_n and $\theta_{n'}$ in (3.2), and (1.5).

6. Bounds for Rankin–Selberg L-functions

In this section, we prove Theorem 2.1. Let $\pi'\in\mathfrak{F}_{n'}$ and $Q\ge 1$. In (1.6), we rescale ε to $\varepsilon/72$ and define $\alpha=\varepsilon/(7.2\max\{n,n'\})$ so that

$$|\{\pi\in\mathfrak{F}_n(Q)\colon N_{\pi\times\pi'}(1-\alpha,6)=0\}|\le\sum_{\pi\in\mathfrak{F}_n(Q)}N_{\pi\times\pi'}(1-\alpha,6)\ll_{F,\varepsilon}(C(\pi')^n|\mathfrak{F}_n(Q)|)^\varepsilon. \tag{6.1}$$

Let $\pi\in\mathfrak{F}_n(Q)$. Proceeding as in the proof of [ST19, Theorem 1.1], we obtain[4]

$$\log\left|L\left(\frac{1}{2},\pi\times\pi'\right)\right|\le\left(\frac{1}{4}-\frac{\alpha}{10^9}\right)\log C(\pi\times\pi')+2\log\left|L\left(\frac{3}{2},\pi\times\pi'\right)\right|$$
$$+\frac{\alpha}{10^7}N_{\pi\times\pi'}(1-\alpha,6)+O(1).$$

By (3.12) and (3.13), we find that there exists an effectively computable constant $c_5=c_5(n,n',[F:\mathbb{Q}],\varepsilon)>0$ such that if $C(\pi\times\pi')\ge c_5$, then

$$\log\left|L\left(\frac{1}{2},\pi\times\pi'\right)\right|\le\left(\frac{1}{4}-\frac{9\alpha}{10^{10}}\right)\log C(\pi\times\pi')+\frac{\alpha}{10^7}N_{\pi\times\pi'}(1-\alpha,6). \tag{6.2}$$

Theorem 2.1 now follows from (6.1) and (6.2).

7. Effective multiplicity one on average

In this section, we prove Theorem 2.3. We consider the family of cuspidal automorphic representations $\mathfrak{F}_n(Q)$ over a number field F. Recall our convention that implied constants are allowed to depend on $n\ge 3$, $[F:\mathbb{Q}]$, and ε unless specifically mentioned otherwise.

[4] Small modifications are required to work over $F\neq\mathbb{Q}$, none of which substantially changes the proof.

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To prove Theorem 2.3, we use Theorem 1.1 to build large zero-free regions for almost all L-functions $L(s,\pi\times\pi')$ with $\pi\in\mathfrak{F}_n$ varying. If $\pi'\in\{\widetilde{\pi},\widetilde{\pi}'\}$, then $L(s,\pi\times\pi')$ has the standard zero-free region (1.1) apart from a possible Landau–Siegel zero, which must be both real and simple. In all other cases, only Brumley's much narrower zero-free region is known (see [Bru06a] and [Lap13, Appendix]). We can use Theorem 1.1 to establish a much stronger zero-free region for $L(s,\pi\times\pi')$ for all $\pi\in\mathfrak{F}_n(Q)$ with very few exceptions. Here is a simple example.

COROLLARY 7.1. *Let $\varepsilon>0$, $n\geq 3$, and $\pi'\in\mathfrak{F}_n(Q)$. For all $\pi\in\mathfrak{F}_n(Q)$ with $O_\varepsilon(Q^\varepsilon)$ exceptions, the L-function $L(s,\pi\times\pi')$ is non-zero in the region*

$$\mathrm{Re}(s)\geq 1-\frac{\varepsilon}{20n^2},\quad |\mathrm{Im}(s)|\leq\log Q.$$

Proof. This follows from (1.7) (once we rescale ε to $\varepsilon/21$) with $C(\pi')\leq Q$, $T=\log Q$, and $\sigma=1-\varepsilon/(20n^2)$. $\square$

For $\pi_1,\pi_2\in\mathfrak{F}_n$ define the numbers $\Lambda_{\pi_1\times\pi_2}(\mathfrak{n})$ by the Dirichlet series identity (for $\mathrm{Re}(s)>1$)

$$\sum_{\mathfrak{n}}\frac{\Lambda_{\pi_1\times\pi_2}(\mathfrak{n})}{\mathrm{N}\mathfrak{n}^s}=-\frac{L'}{L}(s,\pi_1\times\pi_2)=\sum_{\mathfrak{p}}\sum_{k=1}^{\infty}\frac{\sum_{j=1}^{n}\sum_{j'=1}^{n}\alpha_{j,j',\pi_1\times\pi_2}(\mathfrak{p})^k\log\mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{ks}}. \tag{7.1}$$

LEMMA 7.2. *Let $\pi'\in\mathfrak{F}_n$. There exist absolute and effectively computable constants $c_6\in(0,1)$, $c_7,c_8,c_9,c_{10}\geq 1$, and $c_{11}\in(0,1)$ such that the following are true.*

(1) *The L-function $L(s,\pi'\times\widetilde{\pi}')$ has at most one zero, say β_1, in the region*

$$\mathrm{Re}(s)\geq 1-\frac{c_6}{\log(C(\pi')^n(|\mathrm{Im}(s)|+e)^{n^2[F:\mathbb{Q}]})}.$$

If β_1 exists, then it must be real and simple and satisfy $\beta_1\leq 1-C(\pi')^{-c_7 n}$.

(2) *If $A\geq c_8$, $\log\log C(\pi')\geq c_9 n^4[F:\mathbb{Q}]^2$, and $x\geq C(\pi')^{c_{10}A^2n^3[F:\mathbb{Q}]\log(en[F:\mathbb{Q}])}$, then*

$$\sum_{x/2<\mathrm{N}\mathfrak{n}\leq x}\Lambda_{\pi'\times\widetilde{\pi}'}(\mathfrak{n})=\begin{cases}\dfrac{x}{2}(1-\xi^{\beta_1-1})(1+O(e^{-c_{11}A})) & \text{if } \beta_1 \text{ exists},\\ \dfrac{x}{2}(1+O(e^{-c_{11}A})) & \text{otherwise},\end{cases}$$

where $\xi\in[x/2,x]$ satisfies $x^{\beta_1}-(x/2)^{\beta_1}=\beta_1(x/2)\xi^{\beta_1-1}$ and the implied constants are absolute and effectively computable.

Proof. This is [HT22, Theorem 2.1] with $\delta=0$ and x replaced with $x/2$. $\square$

Lemma 7.2 informs the following choices of parameters that we will use throughout the rest of this section. Let $0<\varepsilon<1$, let Q be sufficiently large with respect to ε, and let

$$\begin{gathered}c_7\geq\frac{c_8^2c_{10}}{3240},\quad A=\sqrt{\frac{3240c_7}{c_{10}\varepsilon}},\quad B=\frac{41n^2}{\varepsilon},\quad x=\frac{1}{2}(\log Q)^B,\\ \pi\in\mathfrak{F}_n(Q),\quad \pi'\in\mathfrak{F}_n\big(x^{1/A^2c_{10}n^4[F:\mathbb{Q}]^2}\big)=\mathfrak{F}_n\big((2^{-\varepsilon/41n^2}\log Q)^{41/3240c_7n^2[F:\mathbb{Q}]^2}\big).\end{gathered} \tag{7.2}$$

COROLLARY 7.3. *Under the notation and hypotheses of (7.2), there holds*

$$\sum_{\substack{x/2<\mathrm{N}\mathfrak{n}\leq x\\ \gcd(\mathfrak{n},\mathfrak{q}_\pi\mathfrak{q}_{\pi'})=\mathcal{O}_F}}\Lambda_{\pi'\times\widetilde{\pi}'}(\mathfrak{n})\gg_\varepsilon x^{1-2c_7/A^2c_{10}n^3[F:\mathbb{Q}]^2}.$$

Proof. Let $\pi,\pi'\in\mathfrak{F}_n(Q)$. In Lemma 7.2, the lower bound on $C(\pi')$ only serves to ensure that the implied constants are absolute. This was pertinent in [HT22], but it is not pertinent here. Thus, we may replace the two conditions $\log\log C(\pi')\geq c_9n^4[F:\mathbb{Q}]^2$ and

$x \geq C(\pi')^{A^2 c_{10} n^3 [F:\mathbb{Q}] \log(en[F:\mathbb{Q}])}$ with the single condition $C(\pi') \ll x^{1/(A^2 c_{10} n^4 [F:\mathbb{Q}]^2)}$. We want to refine Lemma 7.2 so that one only sums over $\mathfrak{n}$ such that $\gcd(\mathfrak{n}, \mathfrak{q}_\pi \mathfrak{q}_{\pi'}) = \mathcal{O}_F$. Using (3.3) and (3.7), we find that the contribution to Lemma 7.2(2) arising from $\mathfrak{n}$ such that $\gcd(\mathfrak{n}, \mathfrak{q}_\pi \mathfrak{q}_{\pi'}) \neq \mathcal{O}_F$ is

$$\sum_{\substack{x/2 < \mathrm{N}\mathfrak{n} \leq x \\ \gcd(\mathfrak{n}, \mathfrak{q}_\pi \mathfrak{q}_{\pi'}) \neq \mathcal{O}_F}} \Lambda_{\pi' \times \widetilde{\pi}'}(\mathfrak{n}) \ll \sum_{\mathfrak{p} | \mathfrak{q}_\pi \mathfrak{q}_{\pi'}} \sum_{\log(x/2)/\log \mathrm{N}\mathfrak{p} < j \leq \log x/\log \mathrm{N}\mathfrak{p}} \mathrm{N}\mathfrak{p}^{j(1-2/(n^2+1))}$$

$$\ll \frac{\log Q}{\log\log Q} x^{1-2/(n^2+1)},$$

which is $\ll x^{1-1/(n^2+1)}$. In the worst case, where β_1 in Lemma 7.2(1) exists, it follows from Lemma 7.2(2) and the above discussion that if $C(\pi') \ll_{n,[F:\mathbb{Q}]} x^{1/(A^2 c_{10} n^3 [F:\mathbb{Q}] \log(en[F:\mathbb{Q}]))}$, then

$$\sum_{\substack{x < \mathrm{N}\mathfrak{n} \leq 2x \\ \gcd(\mathfrak{n}, \mathfrak{q}_\pi \mathfrak{q}_{\pi'}) = \mathcal{O}_F}} \Lambda_{\pi' \times \widetilde{\pi}'}(\mathfrak{n}) \gg x(1 - \xi^{\beta_1 - 1}) \gg_{n,[F:\mathbb{Q}]} x \min\{1, (1-\beta_1) \log C(\pi')\} \gg \frac{x \log C(\pi')}{C(\pi')^{c_7 n}}.$$

The desired result now follows. □

Lemma 7.4. *Let $n \geq 3$. Assume the notation and hypotheses in (7.2). Let Φ be a fixed smooth function supported on a compact subset of $[\frac{1}{4}, 2]$ such that $0 \leq \Phi(t) \leq 1$ for all $t \in [\frac{1}{4}, 2]$ and $\Phi(t) = 1$ for $t \in (\frac{1}{2}, 1]$. If $L(s, \pi \times \pi')$ is entire and does not vanish in the region $\mathrm{Re}(s) \geq 1 - \varepsilon/(20n^2)$ and $|\mathrm{Im}(s)| \leq \log Q$, then*

$$\sum_{\gcd(\mathfrak{n}, \mathfrak{q}_\pi \mathfrak{q}_{\pi'}) = \mathcal{O}_F} \Lambda_{\pi \times \widetilde{\pi}'}(\mathfrak{n}) \Phi(\mathrm{N}\mathfrak{n}/x) \ll_B x^{1+2/B-\varepsilon/20n^2}.$$

Proof. Writing $\widehat{\Phi}(s) = \int_0^\infty \Phi(t) t^{s-1}\, dt$ for the Mellin transform of Φ (which is an entire function of s), we observe via Mellin inversion that

$$\sum_{\gcd(\mathfrak{n}, \mathfrak{q}_\pi \mathfrak{q}_{\pi'}) = \mathcal{O}_F} \Lambda_{\pi \times \widetilde{\pi}'}(\mathfrak{n}) \Phi(\mathrm{N}\mathfrak{n}/x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} -\frac{(L^{\mathrm{ur}})'}{(L^{\mathrm{ur}})}(s, \pi \times \widetilde{\pi}') \widehat{\Phi}(s) x^s\, ds,$$

where $L^{\mathrm{ur}}(s, \pi \times \widetilde{\pi}') = L(s, \pi \times \widetilde{\pi}') \prod_{\mathfrak{p} | \mathfrak{q}_\pi \mathfrak{q}_{\pi'}} L(s, \pi_\mathfrak{p} \times \widetilde{\pi}'_\mathfrak{p})^{-1}$. A standard contour integral calculation using the argument principle shows that the above display equals

$$-\sum_{L^{\mathrm{ur}}(\rho, \pi \times \widetilde{\pi}') = 0} \widehat{\Phi}(\rho) x^\rho,$$

where ρ ranges over all zeros of $L^{\mathrm{ur}}(s, \pi \times \widetilde{\pi}')$.

Since Φ is compactly supported and $\widehat{\Phi}$ is entire, it follows that for any $R \geq 2$, we have $|\widehat{\Phi}(s)| \ll_R \min\{1, |s|^{-R} e^{\mathrm{Re}(s)}\}$. Note that the reciprocals of the Euler factors of $L(s, \pi \times \widetilde{\pi}')$ at prime ideals $\mathfrak{p} \mid \mathfrak{q}_\pi \mathfrak{q}_{\pi'}$ and all of the trivial zeros of $L(s, \pi \times \widetilde{\pi}')$ have real part no larger than $1 - 2/(n^2+1)$ per (3.3) and (3.7). Since for any $t \in \mathbb{R}$ there are $\ll \log Q + \log(|t|+2)$ zeros $\rho = \beta + i\gamma$ of $L^{\mathrm{ur}}(s, \pi \times \widetilde{\pi}')$ that satisfy $0 < \beta < 1$ and $|\gamma - t| \leq 1$, we find for any $T \geq 1$, $R \geq 2$, and $\sigma_0 \geq 1 - 2/(n^2+1)$ that

$$\sum_{L^{\mathrm{ur}}(\rho, \pi \times \widetilde{\pi}') = 0} \widehat{\Phi}(\rho) x^\rho = \sum_{\substack{L^{\mathrm{ur}}(\rho, \pi \times \widetilde{\pi}') = 0 \\ \beta \geq \sigma_0 \\ |\gamma| \leq T}} \widehat{\Phi}(\rho) x^\rho + O(T^{1-R} \log(QT) x + T \log(QT) x^{\sigma_0}). \tag{7.3}$$

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We choose $T = \log Q = (2x)^{1/B}$ and $\sigma_0 = 1 - \varepsilon/(20n^2)$, in which case our hypotheses imply that the sum over zeros on the right-hand side of (7.3) is empty and

$$\Bigg|\sum_{L^{\mathrm{ur}}(\rho,\pi\times\widetilde{\pi}')=0} \widehat{\Phi}(\rho)x^{\rho}\Bigg| \ll_B x^{1+(2-R)/B} + x^{2/B+\sigma_0}.$$

The desired result follows from choosing $R = \max\{B(1-\sigma_0), 3\}$. □

Proof of Theorem 2.3. Suppose to the contrary that $\pi \neq \pi'$ and $\pi_{\mathfrak{p}} \cong \pi'_{\mathfrak{p}}$ for all $\mathfrak{p} \nmid \mathfrak{q}_\pi\mathfrak{q}_{\pi'}$ with $\mathrm{N}\mathfrak{p} \le 2x$. Then $A_\pi(\mathfrak{p}) = A_{\pi'}(\mathfrak{p})$ for all $\mathfrak{p} \nmid \mathfrak{q}_\pi\mathfrak{q}_{\pi'}$ with $\mathrm{N}\mathfrak{p} \le 2x$. By (3.6) and (7.1), it follows that $\Lambda_{\pi\times\widetilde{\pi}'}(\mathfrak{n}) = \Lambda_{\pi'\times\widetilde{\pi}'}(\mathfrak{n})$ for all $\mathfrak{n}$ such that $\gcd(\mathfrak{n}, \mathfrak{q}_\pi\mathfrak{q}_{\pi'}) = \mathcal{O}_F$ and $\mathrm{N}\mathfrak{n} \le x$. Since $\Lambda_{\pi'\times\widetilde{\pi}'}(\mathfrak{n}) \ge 0$ for all such $\mathfrak{n}$, the same holds for $\Lambda_{\pi\times\widetilde{\pi}'}(\mathfrak{n})$. It follows that

$$\sum_{\substack{x/2<\mathrm{N}\mathfrak{n}\le x\\ \gcd(\mathfrak{n},\mathfrak{q}_\pi\mathfrak{q}_{\pi'})=\mathcal{O}_F}} \Lambda_{\pi'\times\widetilde{\pi}'}(\mathfrak{n}) = \sum_{\substack{x/2<\mathrm{N}\mathfrak{n}\le x\\ \gcd(\mathfrak{n},\mathfrak{q}_\pi\mathfrak{q}_{\pi'})=\mathcal{O}_F}} \Lambda_{\pi\times\widetilde{\pi}'}(\mathfrak{n}).$$

By (7.2), we have that

$$C(\pi') \ll x^{1/(A^2c_{10}n^4[F:\mathbb{Q}]^2)}. \tag{7.4}$$

Therefore, if $L(s, \pi\times\widetilde{\pi}') \neq 0$ in the region $\{s \in \mathbb{C}\colon \mathrm{Re}(s) \ge 1 - \varepsilon/(20n^2),\ |\mathrm{Im}(s)| \le \log Q\}$, then by Corollary 7.3 and Lemma 7.4, we have that

$$x^{1-2c_7/A^2c_{10}n^3[F:\mathbb{Q}]^2} \ll \sum_{\substack{x/2<\mathrm{N}\mathfrak{n}\le x\\ \gcd(\mathfrak{n},\mathfrak{q}_\pi\mathfrak{q}_{\pi'})=\mathcal{O}_F}} \Lambda_{\pi'\times\widetilde{\pi}'}(\mathfrak{n}) = \sum_{\substack{x/2<\mathrm{N}\mathfrak{n}\le x\\ \gcd(\mathfrak{n},\mathfrak{q}_\pi\mathfrak{q}_{\pi'})=\mathcal{O}_F}} \Lambda_{\pi\times\widetilde{\pi}'}(\mathfrak{n}) \ll_B x^{1+2/B-\varepsilon/20n^2}.$$

This implies that

$$1 - \frac{2c_7}{A^2c_{10}n^3[F:\mathbb{Q}]^2} \le 1 + \frac{2}{B} - \frac{\varepsilon}{20n^2},$$

which contradicts our choices of A and B in (7.2). The desired result now follows from Corollary 7.1. □

8. Automorphic level of distribution

In this section, we prove Theorem 2.5. In what follows, let $F = \mathbb{Q}$. If $n = 2$, then $L(s, \pi\times\widetilde{\pi})$ is the L-function of an isobaric automorphic representation of $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$ whose cuspidal constituents have rank at most 3. Since the L-function any Dirichlet character twist of any cuspidal constituents of rank 2 or 3 have no Landau–Siegel zero by [Ban97, HR95], a stronger result than Theorem 2.5 follows from a minor variation of the proof of [JLTW23, Theorem 1.1]. Therefore, we may restrict our consideration to $n \ge 3$.

Let $\pi \in \mathfrak{F}_n$ have arithmetic conductor q_π, and let χ be a primitive Dirichlet character modulo q. We allow π to be fixed, so for notational compactness, we introduce $L_\chi(s) := L(s, \pi\times(\widetilde{\pi}\otimes\chi))$ and $L_1(s) := L(s, \pi\times\widetilde{\pi})$. If q and q_π are coprime, then $L_\chi(s)$ is entire if and only if χ is non-trivial, since $\widetilde{\pi}\otimes\chi \neq \widetilde{\pi}$, which is clear by comparing the arithmetic conductors of $q_{\widetilde{\pi}\otimes\chi}$ and $q_{\widetilde{\pi}}$. If χ is trivial, then $L_\chi(s) = L_1(s)$. We also define $\Lambda_\chi(s) := \Lambda(s, \pi\times(\widetilde{\pi}\otimes\chi))$ and $\Lambda_1(s) := \Lambda(s, \pi\times\widetilde{\pi})$.

8.1 Preliminaries

We define $a_\pi(m)$ and $a_{\pi\times\widetilde{\pi}}(m)$ by the Dirichlet series identities

$$\sum_{n=1}^{\infty} \frac{a_\pi(m)\Lambda(m)}{m^s} = -\frac{L'}{L}(s,\pi), \quad \sum_{n=1}^{\infty} \frac{a_{\pi\times\widetilde{\pi}}(m)\Lambda(m)}{m^s} = -\frac{L'}{L}(s,\pi\times\widetilde{\pi}), \quad \mathrm{Re}(s) > 1.$$

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The local calculations in [LRS95, Lemma 2.1] show that if $\chi \pmod q$ is a primitive Dirichlet character and $\gcd(q, q_\pi) = 1$, then for all primes p, we have that

$$L(s, \pi_p \times (\widetilde{\pi} \otimes \chi)_p) = \prod_{j=1}^{n} \prod_{j'=1}^{n} (1 - \alpha_{j,j',\pi\times\widetilde{\pi}}(p)\chi(p)p^{-s})^{-1}. \tag{8.1}$$

It follows from (3.6) and (8.1) that if $\gcd(m, q_\pi) = 1$, then $a_{\pi\times(\widetilde{\pi}\otimes\chi)}(m) = |a_\pi(m)|^2\chi(m)$. We now provide a convenient expression for $-L_\chi'(s)/L_\chi(s)$.

LEMMA 8.1. *Let $\pi \in \mathfrak{F}_n$. Let $\psi \pmod q$ be a Dirichlet character with q such that $\gcd(q, q_\pi) = 1$ and χ be the primitive Dirichlet character that induces ψ. Let $\delta(\chi) = 1$ if χ is trivial and $\delta(\chi) = 0$ otherwise. There exists a function $H_\pi(s; \chi, \psi)$ such that in the region $\mathrm{Re}(s) \geq 1 - 1/n^2$:*

(1) *$H_\pi(s; \chi, \psi)$ is holomorphic;*
(2) *$|H_\pi(s; \chi, \psi)| \ll_\pi \log(q(3 + |\mathrm{Im}(s)|))$; and*
(3) *we have the identity*

$$\sum_{m=1}^{\infty} \frac{|a_\pi(m)|^2\psi(m)\Lambda(m)}{m^s} = \frac{\delta(\chi)}{s-1} - \frac{\Lambda_\chi'}{\Lambda_\chi}(s) + H_\pi(s; \chi, \psi).$$

Proof. Suppose first that ψ is primitive, in which case $\psi = \chi$ and the function $H_\pi(s; \chi, \psi)$ is

$$\frac{\log q_{\pi\times(\widetilde{\pi}\otimes\chi)}}{2} + \frac{L'}{L}(s, \pi_\infty \times (\widetilde{\pi} \otimes \chi)_\infty) + \frac{\delta(\chi)}{s} + \sum_{p|q_\pi q} \frac{L'}{L}(s, \pi_p \times (\widetilde{\pi}' \otimes \chi)_p)$$
$$- \sum_{p|q_\pi q} \sum_{\substack{1\leq j\leq n \\ 1\leq j'\leq n}} \frac{\alpha_{j,\pi}(p)\overline{\alpha_{j',\pi'}(p)}\chi(p)\log p}{p^s - \alpha_{j,\pi}(p)\overline{\alpha_{j',\pi}(p)}\chi(p)}.$$

This is holomorphic and bounded as claimed for $\mathrm{Re}(s) \geq 1 - 1/n^2$ by (3.7), (3.10), and Stirling's formula. If ψ is not primitive and χ is the primitive character that induces ψ, then in the same region, we have

$$\left| \sum_{m=1}^{\infty} \frac{|a_\pi(m)|^2\psi(m)\Lambda(m)}{m^s} - \sum_{m=1}^{\infty} \frac{|a_\pi(m)|^2\chi(m)\Lambda(m)}{m^s} \right| \leq \sum_{p|q} \sum_{k\geq 1} \frac{|a_\pi(p^k)|^2\log p}{p^{k\,\mathrm{Re}(s)}},$$

which is bounded as desired using (3.7) again. □

We use the following zero-free region for $L_\chi(s)$ and Siegel-type bound on any real exceptional zeros. We prove this theorem later.

THEOREM 8.2. *Let $Q \geq 3$ and $\pi \in \mathfrak{F}_n$. There exists an effectively computable constant $c_{12} = c_{12}(\pi) > 0$ such that for all primitive Dirichlet characters $\chi \pmod q$ with $q \leq Q$ and $\gcd(q, q_\pi) = 1$ with at most one exception, the L-function $L(s, \pi \times (\widetilde{\pi} \otimes \chi))$ does not vanish in the region*

$$\mathrm{Re}(s) \geq 1 - \frac{c_{12}}{\log(Q(|\mathrm{Im}(s)| + 3))}.$$

If the exceptional character $\chi_1 \pmod{q_1}$ exists, then $L(s, \pi \times (\widetilde{\pi} \otimes \chi_1))$ has at most one zero, say β_1, in this region; β_1 is both real and simple; and χ_1 must be quadratic. Moreover, for all $\varepsilon > 0$, there exists an ineffective constant $c_\pi(\varepsilon) > 0$ such that $\beta_1 \leq 1 - c_\pi(\varepsilon)q_1^{-\varepsilon}$.

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8.2 Proof of Theorem 2.5

We follow Gallagher's proof of the Bombieri–Vinogradov theorem in [Gal68], with $n \geq 2$ and $\pi \in \mathfrak{F}_n$ fixed at the onset. Note that the function

$$\psi_k(y;q,a) \coloneqq \frac{1}{k!} \sum_{\substack{m \leq y \\ m \equiv a \pmod{q}}} |a_\pi(m)|^2 \Lambda(m) \Big(\log \frac{y}{m} \Big)^k$$

is monotonically increasing as a function of y for each $k \geq 0$. Thus, if $0 < \lambda \leq 1$, then

$$\frac{1}{\lambda} \int_{e^{-\lambda} y}^{y} \psi_{k-1}(t;q,a) \frac{dt}{t} \leq \psi_{k-1}(y;q,a) \leq \frac{1}{\lambda} \int_{y}^{e^{\lambda} y} \psi_{k-1}(t;q,a) \frac{dt}{t}.$$

The integrals, which equal $\psi_k(y;q,a) - \psi_k(e^{-\lambda} y;q,a)$ and $\psi_k(e^{\lambda} y;q,a) - \psi_k(y;q,a)$, respectively, both have the same asymptotic expansion, namely

$$(\lambda + O(\lambda^2)) \frac{y}{\varphi(q)} + O\Big(\max_{y \leq ex} |r_k(y;q,a)| \Big), \quad r_k(y;q,a) \coloneqq \psi_k(y;q,a) - \frac{y}{\varphi(q)},$$

where φ is Euler's totient function. Thus, we have the bounds

$$\max_{y \leq x} |r_{k-1}(y;q,a)| \ll \frac{\lambda x}{\varphi(q)} + \frac{1}{\lambda} \max_{y \leq ex} |r_k(y;q,a)|$$

and

$$\sum_{\substack{q \leq x^\theta \\ \gcd(q,q_\pi)=1}} \max_{\gcd(a,q)=1} \max_{y \leq x} |r_{k-1}(y;q,a)| \ll \lambda x \log x + \frac{1}{\lambda} \sum_{\substack{q \leq x^\theta \\ \gcd(q,q_\pi)=1}} \max_{\gcd(a,q)=1} \max_{y \leq ex} |r_k(y;q,a)|.$$

It follows by induction on k that

$$\sum_{\substack{q \leq x^\theta \\ \gcd(q,q_\pi)=1}} \max_{\gcd(a,q)=1} \max_{y \leq x} |r_0(y;q,a)| \ll_k \lambda x \log x + \frac{1}{\lambda^{2^k-1}} \sum_{\substack{q \leq x^\theta \\ \gcd(q,q_\pi)=1}} \max_{\gcd(a,q)=1} \max_{y \leq ex} |r_k(y;q,a)|.$$

Proposition 8.3. *If $\theta < 1/(9n^3)$ is fixed and $k = 9n^2+1$, then for any $B > 0$, we have*

$$\sum_{\substack{q \leq x^\theta \\ \gcd(q,q_\pi)=1}} \max_{\gcd(a,q)=1} \max_{y \leq ex} |r_k(y;q,a)| \ll_{\pi,B} \frac{x}{(\log x)^B}.$$

Proposition 8.3 implies Theorem 2.5. It follows from Proposition 8.3 that

$$\sum_{\substack{q \leq x^\theta \\ \gcd(q,q_\pi)=1}} \max_{\gcd(a,q)=1} \max_{y \leq x} |r_0(y;q,a)| \ll_{\pi,B} \lambda x \log x + \frac{1}{\lambda^{2^k-1}} \frac{x}{(\log x)^B}.$$

To finish, we choose $B = 2^k(A+1)-1$ and $\lambda = (\log x)^{-(B+1)/2^k}$. ☐

8.3 Proof of Proposition 8.3

Let $\pi \in \mathfrak{F}_n$, $k = 9n^2+1$, $Q = x^\theta$ for some fixed $0 < \theta < 1/(9n^3)$, and $q \leq Q$. We have the decomposition

$$\frac{1}{k!} \sum_{\substack{m \leq y \\ m \equiv a \pmod{q}}} |a_\pi(m)|^2 \Lambda(m) \Big(\log \frac{y}{m} \Big)^k = \frac{1}{k!} \sum_{\psi \pmod{q}} \frac{\overline{\psi}(a)}{\varphi(q)} \sum_{y \leq m} |a_\pi(m)|^2 \Lambda(m) \psi(n) \Big(\log \frac{y}{m} \Big)^k.$$

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By Lemma 8.1 and Mellin inversion, this equals

$$\frac{1}{\varphi(q)} \sum_{\psi \,(\mathrm{mod}\ q)} \sum_{\substack{\chi \text{ primitive} \\ \chi \text{ induces } \psi}} \frac{\overline{\chi}(a)}{2\pi i} \int_{3-i\infty}^{3+i\infty} \left(-\frac{\Lambda_\chi'}{\Lambda_\chi}(s) + H_\pi(s;\chi,\psi) \right) \frac{y^s}{s^{k+1}}\, ds.$$

Since a primitive character $\chi \,(\mathrm{mod}\ q)$ induces characters to moduli that are a multiple of q, it follows from the bound $\sum_{f \le Q,\ q|f} \varphi(f)^{-1} \ll (\log Q)/\varphi(q)$ that

$$\begin{aligned} &\sum_{\substack{q \le Q \\ \gcd(q,q_\pi)=1}} \max_{\gcd(a,q)=1} \max_{y \le x} \left| \frac{1}{k!} \sum_{\substack{m \le y \\ m \equiv a \,(\mathrm{mod}\ q)}} |a_\pi(m)|^2 \Lambda(m) \left(\log \frac{y}{m} \right)^k - \frac{y}{\varphi(q)} \right| \\ &\ll \sum_{\substack{q \le Q \\ \gcd(q,q_\pi)=1}} \frac{\log Q}{\varphi(q)} \sum_{\substack{\chi \,(\mathrm{mod}\ q) \\ \chi \text{ primitive}}} \max_{y \le x} \left| \frac{1}{2\pi i} \int_{3-i\infty}^{3+i\infty} \left(-\frac{\Lambda_\chi'}{\Lambda_\chi}(s) + H_\pi(s;\chi,\chi) \right) \frac{y^s}{s^{k+1}}\, ds \right|. \end{aligned} \tag{8.2}$$

Observe that by Lemma 8.1 and our range of θ, (8.2) equals

$$\begin{aligned} &\sum_{\substack{q \le Q \\ \gcd(q,q_\pi)=1}} \frac{\log Q}{\varphi(q)} \sum_{\substack{\chi \,(\mathrm{mod}\ q) \\ \chi \text{ primitive}}} \max_{y \le x} \left| -\sum_{L_\chi(\rho)=0} \frac{y^\rho}{\rho^{k+1}} + \frac{1}{2\pi i} \int_{1-1/n^2-i\infty}^{1-1/n^2+i\infty} H_\pi(s;\chi,\chi) \frac{y^s}{s^{k+1}}\, ds \right| \\ &\ll_{\pi,B} \log Q \sum_{\substack{q \le Q \\ \gcd(q,q_\pi)=1}} \frac{1}{\varphi(q)} \sum_{\substack{\chi \,(\mathrm{mod}\ q) \\ \chi \text{ primitive}}} \sum_{L_\chi(\rho)=0} \frac{y^\beta}{|\rho|^{k+1}} + \frac{x}{(\log x)^B}, \end{aligned} \tag{8.3}$$

where $\rho = \beta + i\gamma$ ranges over the non-trivial zeros of $L_\chi(s)$. In light of the bounds $1/q \le 1/\varphi(q) \ll (\log(eq))/q$, we dyadically decompose $[1, Q]$ into $O(\log Q)$ subintervals and find that (8.3) is

$$\begin{aligned} &\ll_{\pi,B} x(\log Q)^2 \sup_{3 \le R \le Q} \sum_{\substack{q \le R \\ \gcd(q,q_\pi)=1}} \frac{1}{\varphi(q)} \sum_{\substack{\chi \,(\mathrm{mod}\ q) \\ \chi \text{ primitive}}} \sum_{\rho=\beta+i\gamma} \frac{x^{\beta-1}}{|\rho|^k} + \frac{x}{(\log x)^B} \\ &\ll_{\pi,B} x(\log Q)^3 \sup_{3 \le R \le x^\theta} \frac{1}{R} \sum_{\substack{q \le R \\ \gcd(q,q_\pi)=1}} \sum_{\substack{\chi \,(\mathrm{mod}\ q) \\ \chi \text{ primitive}}} \sum_{\rho=\beta+i\gamma} \frac{x^{\beta-1}}{|\rho|^k} + \frac{x}{(\log x)^B} \\ &\ll_{\pi,B} x(\log x)^3 \sup_{3 \le R \le x^\theta} \frac{1}{R} \left(\frac{x^{\beta_1-1}}{\beta_1^k} + \sum_{\substack{q \le R \\ \gcd(q,q_\pi)=1}} \sum_{\substack{\chi \,(\mathrm{mod}\ q) \\ \chi \text{ primitive}}} \sum_{\rho=\beta+i\gamma \ne \beta_1} \frac{x^{\beta-1}}{|\rho|^k} \right) + \frac{x}{(\log x)^B}. \end{aligned} \tag{8.4}$$

where β_1 is the exceptional zero in Theorem 8.2. The term x^{β_1-1}/β_1^k is omitted if β_1 does not exist.

If β_1 exists as in Theorem 8.2 and the supremum is achieved when $R \le (\log x)^{4B}$, then we apply Theorem 8.2 with $\varepsilon = 1/8B$ and conclude that the contribution from such a zero is absorbed in our error term. If the supremum is achieved when $R > (\log x)^{4B}$, then contribution from β_1 is trivially absorbed in our error term. Hence, (8.4) is

$$\ll_{\pi,B} x(\log x)^3 \sup_{3 \le R \le x^\theta} \frac{1}{R} \sum_{\substack{q \le R \\ \gcd(q,q_\pi)=1}} \sum_{\substack{\chi \,(\mathrm{mod}\ q) \\ \chi \text{ primitive}}} \sum_{\rho=\beta+i\gamma \ne \beta_1} \frac{x^{\beta-1}}{|\rho|^k} + \frac{x}{(\log x)^B}. \tag{8.5}$$

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Now let us consider the zeros ρ with $|\rho| < \frac{1}{4}$. The number of such zeros is $\ll R^2 \log R$. From the consideration of the corresponding zeros $1-\rho$ of $L_{\overline{\chi}}(s)$, we deduce that $|\rho| \gg x^{-1/(4k)}$. Thus, the contribution from these zeros is $\ll Rx^{1/4+k/4k} \log R \ll Qx^{1/2} \log Q \ll_B x(\log x)^{-B}$. Define $T_0 = 0$ and $T_j = 2^{j-1}$ for $j \geq 1$. The above discussion shows that (8.5) is

$$\ll_{\pi,B} x(\log x)^3 \sup_{3\leq R\leq x^\theta} \frac{1}{R} \sum_{\substack{q\leq R \\ \gcd(q,q_\pi)=1}} \sum_{\substack{\chi \,(\mathrm{mod}\ q) \\ \chi \text{ primitive}}} \sum_{\substack{\rho=\beta+i\gamma\neq\beta_1 \\ |\rho|>\frac{1}{4}}} \frac{x^{\beta-1}}{|\rho|^k} + \frac{x}{(\log x)^B}$$

$$\ll_{\pi,B} x(\log x)^3 \sup_{3\leq R\leq x^\theta} \frac{1}{R} \sum_{j=1}^{\infty} \sum_{\substack{q\leq R \\ \gcd(q,q_\pi)=1}} \sum_{\substack{\chi \,(\mathrm{mod}\ q) \\ \chi \text{ primitive}}} {\sum_{\substack{\rho=\beta+i\gamma\neq\beta_1 \\ |\rho|\geq\frac{1}{4} \\ T_{j-1}\leq|\gamma|\leq T_j}}}' \frac{x^{\beta-1}}{|\rho|^k} + \frac{x}{(\log x)^B}. \tag{8.6}$$

If $|\rho| \geq \frac{1}{4}$ and $T_{j-1} \leq |\gamma| \leq T_j$, then $|\rho| \geq \max\{T_{j-1}, \frac{1}{4}\} \geq T_j/4$ and $|\rho| \gg |\gamma| + 3$. Therefore, if $\delta = \min\{1 - 9n^3\theta, \frac{1}{2}\}$, then

$$\begin{aligned} x^{\beta-1}|\rho|^{-k} &\ll T_j^{-1/2}(|\gamma|+1)^{-1/2} x^{-(1-\beta)\delta}(x^{1-\delta}T_j^{k-1})^{-(1-\beta)} \\ &\ll T_j^{-1/2}(|\gamma|+1)^{-1/2} x^{-(1-\beta)\delta}(R^{(1-\delta)/\theta}T_j^{k-1})^{\beta-1}. \end{aligned}$$

Since $\rho = \beta + i\gamma \neq \beta_1$, it follows from Theorem 8.2 that

$$(|\gamma|+1)^{-1/2} x^{-\delta(1-\beta)} \leq e^{-\delta\eta_\pi(x,R)}, \quad \eta_\pi(x,R) \coloneqq \inf_{t\geq 3}\Big[c_\pi \frac{\log x}{\log(Rt)} + \log t\Big],$$

so (8.6) is

$$\begin{aligned} &\ll_{\pi,B} x(\log x)^3 \sup_{3\leq R\leq x^\theta} \frac{e^{-\delta\eta(x,R)}}{R} \sum_{j=1}^{\infty} T_j^{-1/2} \\ &\quad\times \sum_{\substack{q\leq R \\ \gcd(q,q_\pi)=1}} \sum_{\substack{\chi \,(\mathrm{mod}\ q) \\ \chi \text{ primitive}}} \sum_{\substack{\rho=\beta+i\gamma\neq\beta_1 \\ |\gamma|\leq T_j}} (R^{(1-\delta)/\theta}T_j^{k-1})^{\beta-1} + \frac{x}{(\log x)^B}. \end{aligned}$$

Define

$$N_\pi^*(\sigma,T,R) = \sum_{\substack{q\leq R \\ \gcd(q,q_\pi)=1}} \sum_{\substack{\chi \bmod q \\ \chi \text{ primitive}}} |\{\rho = \beta+i\gamma\colon \beta\geq\sigma,\ |\gamma|\leq T,\ L_\chi(\rho)=0\}|.$$

By (3.10), there exists an effectively computable constant $c_{13} = c_{13}(n) > 0$ such that

$$\{\widetilde{\pi}\otimes\chi\colon \chi \,(\mathrm{mod}\ q_\chi) \text{ primitive}, \gcd(q_\chi, q_\pi) = 1,\ q_\chi \leq Q\} \subseteq \mathfrak{F}_n(c_{13}C(\pi)Q^n).$$

Thus, by (1.7), we have $N_\pi^*(\sigma,T,R) \ll_{\pi,\varepsilon} (R^nT)^{9n^2(1-\sigma)+\varepsilon}$. Partial summation yields

$$\begin{aligned} &\sum_{\substack{q\leq R \\ \gcd(q,q_\pi)=1}} \sum_{\substack{\chi \,(\mathrm{mod}\ q) \\ \chi \text{ primitive}}} \sum_{\substack{\rho=\beta+i\gamma\neq\beta_1 \\ |\gamma|\leq T_j}} (R^{(1-\delta)/\theta}T_j^{k-1})^{-(1-\beta)} \\ &\quad\ll (R^{(1-\delta)/\theta}T_j^{k-1})^{-1}N_\pi^*(0,T_j,R) + \log(RT_j)\int_0^1 (R^{(1-\delta)/\theta}T_j^{k-1})^{-\sigma}N_\pi^*(1-\sigma,T_j,R)\,d\sigma \\ &\quad\ll_{\pi,\varepsilon} (RT_j)^\varepsilon\Big((R^{(1-\delta)/\theta}T_j^{k-1})^{-1}RT_j + \int_0^1 (R^{(1-\delta)/\theta}T_j^{k-1})^{-\sigma}(R^nT_j)^{9n^2\sigma}d\sigma\Big). \end{aligned} \tag{8.7}$$

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Our choices for θ, k, and δ ensure that (8.7) is $\ll_{\pi,\varepsilon} (RT_j)^{\varepsilon}$. Thus, (8.6) is

$$\begin{aligned}
&\ll_{\pi,B,\varepsilon} x(\log x)^3 \sup_{3\le R\le x^{\theta}} e^{-\delta\eta_\pi(x,R)}\frac{R^{\varepsilon}}{R}\sum_{j=1}^{\infty} T_j^{-1/2+\varepsilon} + \frac{x}{(\log x)^B}\\
&\ll_{\pi,B,\varepsilon} x(\log x)^3 \sup_{3\le R\le x^{\theta}} e^{-\delta\eta_\pi(x,R)}\frac{R^{\varepsilon}}{R} + \frac{x}{(\log x)^B}. \qquad (8.8)
\end{aligned}$$

A small calculation (cf. [TZ19, § 4]) shows that there is a constant $c_{14} = c_{14}(\pi) > 0$ such that $e^{-\delta\eta_\pi(x,R)} \ll_{\pi,B,\varepsilon} \exp(-c_{14}\log x/\log R) + \exp(-c_{14}\sqrt{\log x})$, and Theorem 2.5 follows.

8.4 Proof of Theorem 8.2

Although the proof of Theorem 8.2 contains only standard techniques, such zero-free regions for $L_\chi(s)$ are new even in the case when χ is trivial. The case when χ is trivial was only recently handled unconditionally in [HT22].

Lemma 8.4 [HT22, Theorem 2.1(1)]. *Let $\pi \in \mathfrak{F}_n$. There exists an absolute and effectively computable constant $c_{15} > 0$ such that $L_1(s)$ has at most one zero, say β_1, in the region $\mathrm{Re}(s) \ge 1 - c_{15}/\log(C(\pi)^n(|\mathrm{Im}(s)|+e)^{n^2})$. If β_1 exists, then it must be real and simple, and there exists an absolute and effectively computable constant c_{16} such that $\beta_1 \le 1 - C(\pi)^{-c_{16}n}$.*

We prove the corresponding result for $L_\chi(s)$. The ideas in [HT22] inform our approach here.

Lemma 8.5. *Let $\pi \in \mathfrak{F}_n$. Let $\chi \pmod q$ be a non-trivial primitive Dirichlet character such that $\gcd(q, q_\pi) = 1$. There exists an effectively computable constant $c_{17} = c_{17}(n) > 0$ such that $L_\chi(s)$ has at most one zero, say β_1, in the region*

$$\mathrm{Re}(s) \ge 1 - \frac{c_{17}}{\log(qC(\pi)(|\mathrm{Im}(s)|+3))}. \qquad (8.9)$$

If the exceptional zero β_1 exists, then it is real and simple, and χ is quadratic.

Proof. Let $\chi \pmod q$ be a primitive non-trivial Dirichlet character, let ψ be the primitive character that induces χ^2, and let $\beta + i\gamma$ be a non-trivial zero of $L_\chi(s)$. Define $\Pi_\chi = \pi \boxplus \pi \otimes \chi|\cdot|^{it} \boxplus \pi \otimes \overline{\chi}|\cdot|^{-it}$, and define

$$D(s) = L(s, \Pi_\chi \times \widetilde{\Pi}_\chi) = L_1(s)^3 L_\chi(s+i\gamma)^2 L_{\overline{\chi}}(s-i\gamma)^2 L_\psi(s+2i\gamma) L_{\overline{\psi}}(s-2i\gamma). \qquad (8.10)$$

The factor $L_1(s)^3$ has a pole of order 3 at $s = 1$, and the hypothesis that $\gcd(q, q_\pi) = 1$ ensures that $L_\chi(s+i\gamma)^2 L_{\overline{\chi}}(s-i\gamma)^2$ is entire. If ψ is complex, then $L_\psi(s+2i\gamma)L_{\overline{\psi}}(s-2i\gamma)$ is entire; otherwise, it has poles of order 1 at $s = 1 \pm 2i\gamma$. The additional poles when ψ is real require some additional casework when γ is close to zero. For notational compactness, let $\mathcal{Q}_\gamma = qC(\pi)(|\gamma|+3)$. Note that $-(D'/D)(s)$ has non-negative Dirichlet coefficients per [HR95, Lemma a].

The functional equation for $L_\chi(s)$ together with the fact that $L(s, \pi \times \widetilde{\pi})$ is a self-dual L-function (even if $L(s,\pi)$ itself is not self-dual) implies that if ρ is a zero of $L_\chi(s)$, then $\overline{\rho}$ is a zero of $L_{\overline{\chi}}(s)$. Thus, we have that

$$\operatorname*{ord}_{s=\beta} D(s) \ge 4. \qquad (8.11)$$

Let ω denote a non-trivial zero of $D(s)$, $\delta(\psi) = 1$ if ψ is trivial, and $\delta(\psi) = 0$ otherwise. We apply Lemma 3.2 and (3.10) to (8.10), concluding that if $1 < \sigma < 2$, then

$$\sum_{\omega} \mathrm{Re}\Big(\frac{1}{\sigma-\omega}\Big) < \frac{3}{\sigma-1} + \delta(\psi)\frac{2(\sigma-1)}{(\sigma-1)^2+4\gamma^2} + c_{18}\log\mathcal{Q}_\gamma, \qquad (8.12)$$

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where $c_{18} = c_{18}(n) > 1$ is a suitable implied constant. Since $\beta < 1$ is, by hypothesis, one of the zeros in the sum in (8.12), we have by (8.11) and non-negativity that

$$\frac{4}{\sigma-\beta} < \frac{3}{\sigma-1} + \delta(\psi)\frac{2(\sigma-1)}{4\gamma^2+(\sigma-1)^2} + c_{18}\log \mathcal{Q}_\gamma. \tag{8.13}$$

Case 1: Either χ is real and $|\gamma| \geq 1/7c_{22}\log \mathcal{Q}_0$ or χ is complex. If $\sigma = 1 + 1/5c_{18}\log\mathcal{Q}_\gamma$, then

$$\delta(\psi)\frac{2(\sigma-1)}{4\gamma^2+(\sigma-1)^2} \leq \frac{490}{149}c_{18}\log\mathcal{Q}_\gamma.$$

Thus, (8.13) becomes

$$\frac{4}{1+1/5c_{18}\log\mathcal{Q}_\gamma - \beta} \leq \frac{2874}{149}c_{18}\log\mathcal{Q}_\gamma.$$

Upon solving for β, we conclude that $\beta \leq 1 - 1/136c_{18}\mathcal{Q}_\gamma$.

Case 2: χ is real and $\gamma = 0$. We start at (8.12) with $\delta(\psi) = 1$, $\gamma = 0$, and $\sigma = 1 + 1/3c_{18}\log\mathcal{Q}_0$. If there are N zeros β (with multiplicity) of $D(s)$ such that $\beta \geq 1 - 1/96c_{18}\log\mathcal{Q}_0$, then by (8.12),

$$\frac{32}{11}c_{18}N\log\mathcal{Q}_0 = \frac{N}{\sigma-(1-1/96c_{18}\log\mathcal{Q}_0)} \leq \frac{5}{\sigma-1} + c_{18}\log\mathcal{Q}_0 = 16c_{18}\log\mathcal{Q}_0.$$

It follows that (since N is an integer) $N \leq \lfloor\frac{11}{2}\rfloor = 5$. By the bound (8.11), $L_\chi(s)$ has at most one real zero β, necessarily simple, satisfying $\beta \geq 1 - 1/96c_{18}\log\mathcal{Q}_0$.

Case 3: χ is quadratic and $0 < |\gamma| < 1/7c_{22}\log\mathcal{Q}_0$. We apply Lemma 3.2 to $L_1(s)L_\chi(s)$. The only singularity is a simple pole at $s = 1$. Both $L_1(s)$ and $L_\chi(s)$ are self-dual, so if $\beta + i\gamma$ is a non-trivial zero of $L_1(s)L_\chi(s) = 0$, then so is $\beta - i\gamma$. By (8.1), the mth Dirichlet coefficient of $-L_1'/L_1(s) - L_\chi'/L_\chi(s)$ is $(1+\chi(m))a_{\pi\times\widetilde{\pi}}(m)\Lambda(m) \geq 0$, so Lemma 3.2 yields

$$2\frac{\sigma-\beta}{(\sigma-\beta)^2+\gamma^2} \leq \frac{1}{\sigma-1} + c_{18}\log\mathcal{Q}_\gamma. \tag{8.14}$$

If $\sigma = 1 + 1/2c_{18}\log\mathcal{Q}_\gamma$ and $0 < |\gamma| < 1/7c_{18}\log\mathcal{Q}_0$ in (8.14), then $\beta \leq 1 - 1/8c_{18}\log\mathcal{Q}_\gamma$. □

We now prove a Siegel-type bound for β_1 in Lemma 8.5 (if it exists) using the ideas of Hoffstein and Lockhart [HL94]. This is new for all $\pi \in \mathfrak{F}_n$ with $n \geq 3$. We begin with an auxiliary calculation. Let $\chi \pmod q$ and $\chi' \pmod{q'}$ be distinct non-trivial quadratic Dirichlet characters such that $\gcd(q'q, q_\pi) = 1$, and let ψ be the primitive character that induces $\chi'\chi$ (whose conductor is necessarily coprime to q_π). These coprimality restrictions ensure that $\pi \neq \pi\otimes\chi$, $\pi \neq \pi\otimes\chi'$, $\pi \neq \pi\otimes\psi$, $q_{\pi\otimes\chi} = q_\pi q^n$, $q_{\pi\otimes\chi'} = q_\pi(q')^n$, and $q_{\pi\otimes\psi} = q_\psi q^n$. We also have that $q_\psi | q_\chi q_{\chi'}$. Let

$$L(s,\Pi^\star) = L_1(s)L_\chi(s)L_{\chi'}(s)L_\psi(s). \tag{8.15}$$

By the above discussion, $L(s,\Pi^\star)$ is holomorphic apart from a simple pole at $s = 1$. It follows from (8.1) that if $k \geq 1$, then the p^kth Dirichlet coefficient of $\log L(s,\Pi^\star)$ equals

$$k^{-1}a_{\pi\times\widetilde{\pi}}(p^k)(1+\chi(p^k)+\chi'(p^k)+\psi(p^k)) \geq 0.$$

The non-negativity of $1+\chi(p^k)+\chi'(p^k)+\psi(p^k)$ follows from the fact that this sum is a Dirichlet coefficient of the Dedekind zeta function of a biquadratic field. Upon exponentiating, we find that the mth Dirichlet coefficient $\lambda_{\Pi^\star}(m)$ of $L(s,\Pi^\star)$ is non-negative.

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Lemma 8.6. *If $\pi \in \mathfrak{F}_n$ and χ is a primitive non-trivial real Dirichlet character, then $L_\chi(1) > 0$ and the Dirichlet coefficients of $L_1(s)L_\chi(s)$ are non-negative.*

Proof. Let K be the quadratic field associated to χ. If π_{BC} is the base change of π to an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_K)$, then $L(s, \pi_{\mathrm{BC}} \times \widetilde{\pi}_{\mathrm{BC}}) = L_1(s)L_\chi(s)$. Since $L(s, \pi_{\mathrm{BC}} \times \widetilde{\pi}_{\mathrm{BC}})$ is holomorphic on $\mathbb{C} - \{1\}$ apart from a simple pole at $s = 1$, the same holds for $L_1(s)L_\chi(s)$. Since $\gcd(q, q_\pi) = 1$ (hence, $L_\chi(s)$ is entire), the residue of $L(s, \pi_{\mathrm{BC}} \times \widetilde{\pi}_{\mathrm{BC}})$ at $s = 1$, which is positive, equals $L_\chi(1)\mathrm{Res}_{s=1}L_1(s)$. Since $\mathrm{Res}_{s=1}L_1(s) > 0$, it follows that $L_\chi(1) > 0$. The Dirichlet coefficients of $L_1(s)L_\chi(s)$ are non-negative per case 3 in the proof of Lemma 8.5. □

Let $0 < \varepsilon < 1$, and let $\beta \in (1 - \varepsilon, 1)$. By (3.13) and the above discussion, we have

$$\begin{aligned} L(1/2 + it, \Pi^\star) &\ll_{\pi,\varepsilon} (q'q)^{n^2/2+\varepsilon}(3 + |t|)^{n^2+\varepsilon}, \\ \operatorname*{Res}_{s=1-\beta} L(s + \beta, \Pi^\star)x^s\Gamma(s) &\ll_{\pi,\varepsilon} L_\chi(1)(q'q)^\varepsilon(1 - \beta)^{-1}x^{1-\beta}. \end{aligned} \tag{8.16}$$

If $x \geq 3$, then since $\lambda_{\Pi^\star}(1) = 1$, we use (8.16) to deduce that

$$\begin{aligned} \frac{1}{e} &\leq \sum_{m=1}^{\infty} \frac{\lambda_{\Pi^\star}(m)}{m^\beta} e^{-m/x} \\ &= \frac{1}{2\pi i} \int_{3-i\infty}^{3+i\infty} L(s + \beta, \Pi^\star)x^s\Gamma(s)\, ds \\ &= \operatorname*{Res}_{s=1-\beta} L(s + \beta, \Pi^\star)x^s\Gamma(s) + L(\beta, \Pi^\star) + \frac{1}{2\pi i} \int_{1/2-\beta-i\infty}^{1/2-\beta+i\infty} L(s + \beta, \Pi^\star)x^s\Gamma(s)\, ds \\ &\ll_{\pi,\varepsilon} L_\chi(1)(q'q)^\varepsilon(1 - \beta)^{-1}x^{1-\beta} + L(\beta, \Pi^\star) + (q'q)^{n^2/2+\varepsilon}x^{1/2-\beta}. \end{aligned} \tag{8.17}$$

Proposition 8.7. *Recall the notation and hypotheses of Lemma 8.5. If β_1 exists for a primitive character $\chi \pmod q$ such that $\gcd(q, q_\pi) = 1$, then for all $\varepsilon > 0$, there exists an (ineffective) constant $c'_\pi(\varepsilon) > 0$ such that $L_\chi(1) \geq c'_\pi(\varepsilon)q^{-\varepsilon}$.*

Proof. It suffices to take $0 < \varepsilon < 1$. Let $\chi \pmod q$ and $\chi' \pmod{q'}$ be primitive quadratic Dirichlet characters with, let ψ be the primitive character that induces $\chi'\chi$, and recall the definition of $L(s, \Pi^\star)$ from (8.15). Our proof consists of two cases.

First, suppose that there exists no primitive quadratic Dirichlet character ω such that $L_\omega(s)$ does not vanish for $s \in (1 - \varepsilon/2, 1)$. It then follows that there exists a constant $c_{19} = c_{19}(\pi) > 0$ such that $L_1(s)L_\chi(s) \neq 0$ in the interval $s \in (1 - c_{19}/\log q, 1)$. Since the Dirichlet coefficients of $L_1(s)L_\chi(s)$ are non-negative (using Lemma 8.6) and the residue of $L_1(s)L_\chi(s)$ at $s = 1$ is $L_\chi(1)\mathrm{Res}_{s=1}L_1(s)$, it follows from [HL94, Proposition 1.1] and (3.13) that $L_\chi(1)\mathrm{Res}_{s=1}L_1(s) \gg_\pi (\log q)^{-1}$. Since each term on the left-hand side is positive (using Lemma 8.6), the desired result follows.

Second, suppose that there exists $\chi' \pmod{q'}$ and $\beta \in (1 - \varepsilon/2, 1)$ such that $L_{\chi'}(\beta) = 0$. We may assume that q' is minimal, subject to this condition. Now, let χ be arbitrary. If $q < q'$, then the minimality of q' ensures that $L_\chi(s) \neq 0$ for $s \in (1 - \varepsilon/2, 1)$, and the preceding case implies the desired result. Suppose now that $q \geq q'$. If $L_\chi(s)$ has no real zero within a distance of $c_{17}/\log(3q'qC(\pi))$ of $s = 1$, then since $q \geq q'$, $L_\chi(s)$ has no real zero within a distance of $\frac{1}{2}c_{17}/\log(3qC(\pi))$ of $s = 1$. Again, the desired result follows by the preceding case. Finally, suppose that $L_\chi(s)$ has a real zero within a distance of $c_{17}/\log(3q'qC(\pi))$ of $s = 1$. At this stage, we assume that $\chi \neq \chi'$.

It follows from analysis nearly identical to the second case in Lemma 8.5 (with $L(s,\Pi^{\star})$ replacing $D(s)$) that $L(s,\Pi^{\star})$ has at most one real zero within distance $c_{17}/\log(3q'qC(\pi))$ from 1. Since we have supposed that $L_{\chi}(s)$ has a real zero within distance $c_{17}/\log(3q'qC(\pi))$ of $s=1$, the above discussion indicates that this is the sole real zero for $L(s,\Pi^{\star})$ within a distance of $c_{17}/\log(3q'qC(\pi))$ of $s=1$. It follows that the zero β of $L_{\chi'}(s)$ must satisfy

$$\beta \leq 1-\frac{c_{17}}{\log(3q'qC(\pi))},\quad \beta\in\Big(1-\frac{\varepsilon}{2},1\Big).$$

Since $L_{\chi'}(\beta)=0$, it follows that $L(\beta,\Pi^{\star})=0$. Using (8.17), the above bounds on β, and the fact that $q\geq q'$, we find that

$$1\ll_{\pi,\varepsilon} L_{\chi}(1)q^{2\varepsilon}x^{1-\beta}+q^{n^2+2\varepsilon}x^{1/2-\beta}\ll_{\pi,\chi',\varepsilon} L_{\chi}(1)q^{2\varepsilon}x^{\varepsilon/2}+q^{n^2+2\varepsilon}x^{\varepsilon/2-1/2}.$$

Choosing $x=q^{2n^2}/L_{\chi}(1)^2$ (which is at least 3 by (3.10) and (3.13)) and solving for $L_{\chi}(1)$, we find that for all $\chi\neq\chi'$ and all $0<\varepsilon<1$, there exists a constant $d_{\pi}(\varepsilon)>0$ such that $L_{\chi}(1)\geq d_{\pi}(\varepsilon)q^{-\varepsilon(n^2+2)/(1-\varepsilon)}$. Upon rescaling ε to $\varepsilon/(n^2+2+\varepsilon)$, we have $L_{\chi}(1)\geq d_{\pi}(\varepsilon/(n^2+2+\varepsilon))q^{-\varepsilon}$. As long as $\chi\neq\chi'$, the constant $d_{\pi}(\varepsilon/(n^2+2+\varepsilon))$ is effective. Once we decrease $d_{\pi}(\varepsilon/(n^2+2+\varepsilon))$ suitably to account for the case where $\chi=\chi'$, the claimed result holds for arbitrary χ. $\square$

COROLLARY 8.8. *Recall the notation and hypotheses of Lemma 8.5. If β_1 exists, then for all $\varepsilon>0$, there exists an (ineffective) constant $c_{\pi}(\varepsilon)>0$ such that $\beta_1\leq 1-c_{\pi}(\varepsilon)q^{-\varepsilon}$.*

Proof. If β_1 exists, then there exists $\sigma\in[\beta_1,1]$ such that $L_{\chi}'(\sigma)(1-\beta_1)=L_{\chi}(1)\geq c_{\pi}'(\varepsilon/2)q^{-\varepsilon/2}$ by Proposition 8.7 and the mean value theorem. The result follows once we establish the bound $L_{\chi}'(\sigma)\ll_{\pi,\varepsilon} q^{\frac{\varepsilon}{2}}$ for $\sigma\in[1-b_n/\log(3qC(\pi)),1]$, where $b_n>0$ is a suitable constant depending at most on n. To prove this, we observe via Cauchy's integral formula that

$$L_{\chi}'(1)=\frac{1}{2\pi i}\int_{|z-1|=1/\log q}\frac{L_{\chi}(z)}{(z-1)^2}\,dz\ll(\log q)\max_{|\xi-1|\leq 1/\log q}|L_{\chi}(\xi)|,$$

in which case the desired bound follows from (3.13). $\square$

We show that among the primitive characters $\chi\,(\mathrm{mod}\ q)$ with $q\leq Q$, we encounter very few with the property that $L_{\chi}(s)$ has an exceptional zero.

LEMMA 8.9. *Let $Q\geq 3$. There exists an effectively computable constant $c_{20}=c_{20}(n)>0$ such that there is at most one real non-trivial primitive Dirichlet character $\chi_1\,(\mathrm{mod}\ q_1)$ with $q_1\leq Q$ such that $L_{\chi_1}(s)$ has a real zero β_1 satisfying $\beta_1>1-c_{20}/\log(C(\pi)Q)$.*

Proof. Suppose to the contrary that $\chi\,(\mathrm{mod}\ q)$ and $\chi'\,(\mathrm{mod}\ q')$ are two distinct such characters with $q,q'\leq Q$. Let $\Pi=\pi\boxplus\pi\otimes\chi\boxplus\pi\otimes\chi'$, let ψ be the primitive character that induces $\chi'\chi$, and let

$$F(s)=L(s,\Pi\times\widetilde{\Pi})=L_1(s)^3L_{\chi}(s)^2L_{\chi'}(s)^2L_{\psi}(s)^2.$$

By [HR95, Lemma a], the Dirichlet coefficients of $-(F'/F)(s)$ are non-negative. By Lemma 3.2, there exists a constant $c_{21}=c_{21}(n)\geq 1$ such that if ω runs through the nontrivial zeros of $F(s)$ and $1<\sigma<2$, then

$$\sum_{\omega}\mathrm{Re}\Big(\frac{1}{\sigma-\omega}\Big)<\frac{3}{\sigma-1}+c_{21}\log(C(\pi)Q). \tag{8.18}$$

If $\sigma = 1 + 1/2c_{21}\log(C(\pi)Q)$ and M is the number (necessarily an integer) of real zeros (counting multiplicity) of $F(s)$ that are at least $1 - 1/14c_{21}\log(C(\pi)Q)$, then it follows from (8.18) that

$$\frac{M}{1 + 1/2c_{21}\log(C(\pi)Q) - (1 - 1/14c_{21}\log(C(\pi)Q))} < \frac{3}{(1 + 1/2c_{21}\log(C(\pi)Q)) - 1} + c_{21}\log(C(\pi)Q).$$

This implies that $M \leq 3$. However, if $L_\chi(s)$ and $L_{\chi'}(s)$ both have real zeros that are larger than $1 - 1/14c_{21}\log(C(\pi)Q)$, then $F(s)$ has four such zeros, a contradiction. The lemma now follows. □

Proof of Theorem 8.2. This follows from Corollary 8.8 and Lemmas 8.4, 8.5, and 8.9. □

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CONFLICTS OF INTEREST

None.

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