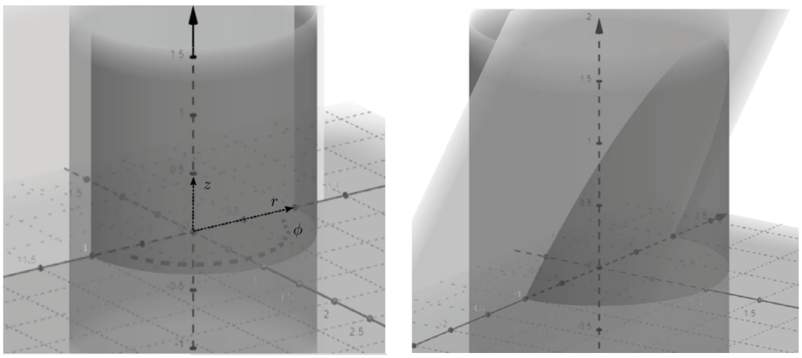
107.03 Archimedes and the ungula



(a) (b)

FIGURE 1: (a) the half-cylinder with Cartesian and cylindrical coordinate axes shown and (b) the ungula

The ungula is a geometric solid which has been studied since antiquity. In particular, Archimedes demonstrated using geometric means that the volume of an ungula is one-sixth of the volume of an enclosing cube [1]. This computation can be substantially simplified using modern calculus, and there are several examples in the literature. Lynch has shown that the calculation can be done using single integrals in the x-, y-, and z-directions separately (these directions are shown by the three mutually perpendicular dotted lines in Figure 1) [2]. In fact, Lynch does not perform these integrals explicitly, but they are written down in [3]. Another computation is found in [3] using a coordinate system designed specifically for the problem. We will find the volume by performing a triple integral in cylindrical coordinates: this computation is somewhat simpler and more transparent than others in the literature.

For the formal construction of the ungula, the reader should visualise a cylinder of radius a and height $2a$ which is cut in half through the plane $x = 0$ (Figure 1(a)). After that, cut the half-cylinder through the plane $z = 2y$ (Figure 1(b)). We then write down the standard integral for a volume in cylindrical coordinates (cylindrical coordinates are represented with dashed lines in Figure 1). The limits of the integrals should be clear, since the angle ϕ which a point makes with the y-axis goes from 0 to π for a half-cylinder (see the faint black dashed line in Figure 1). For the upper limit in the z-integral, recall from the conversion formulae to go from cartesian to cylindrical coordinates that $y = r \sin\phi$. Plugging this back in, the volume then follows from the triple integral. To recover the result of Archimedes, note that the cylinder which we started with can be embedded in a cube of sidelength $2a$, hence volume of $8a^3$.

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$$\int_{r=0}^{r=a}\int_{\phi=0}^{\phi=\pi}\int_{z=0}^{z=2r\sin\phi} r\,dz\,d\phi\,dr = \int_{r=0}^{r=a}\int_{\phi=0}^{\phi=\pi}\left[rz\right]_0^{2r\sin\phi} d\phi\,dr$$
$$= \int_{r=0}^{r=a}\int_{\phi=0}^{\phi=\pi} 2r^2\sin\phi\,d\phi\,dr$$
$$= \int_{r=0}^{r=a}\left[-2r^2\cos\phi\right]_0^{\pi} dr$$
$$= \int_{r=0}^{r=a} 4r^2\,dr$$
$$= \left[\frac{4}{3}r^3\right]_0^{\pi}$$
$$= \frac{4}{3}a^3.$$

References

1. S. H. Gould, The Method of Archimedes, *American Mathematical Monthly* 62, (1955) pp. 473-476.
2. P. Lynch, “Sharing a Pint” ThatsMaths thatsmaths.com/2012/12/13/sharing-a-pint/
3. S. B. Gray, D. Y. Ding, G. Gordillo, S. Landsberger and C. Waldman (2015) The Method of Archimedes: Propositions 13 and 14. Notices of the AMS **62** (9), pp. 1036-1040.

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HOLLIS WILLIAMS
School of Engineering,
University of Warwick,
Coventry CV4 7AL
e-mail: *Hollis.Williams@warwick.ac.uk*

107.04 A geometric telescope revisited

In 2002, Thomas Walker [1] introduced his nice and interesting result with following words,

> “The two most basic series whose sums can be computed explicitly (geometric series, telescoping series) combine forces to demonstrate the assuming fact that
>
> $$\sum_{m=2}^{\infty}\left(\zeta(m) - 1\right) = 1 \qquad (1)$$
>
> where $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$ is the Riemann zeta function.”