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ON THE VOLUME AND PROJECTION OF CONVEX SETS CONTAINING NO LATTICE POINTS

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A convex body K in E^n containing no lattice points, has one-dimensional projections of lengths $X_1, X_2, \ldots, X_n$ on the coordinate axes. We show that for $n \geq 2$ and $1 \leq X_i \leq n/(n-1)$ $(1 \leq i \leq n)$,

$$V(K) \leq X_1X_2 \ldots X_n - (n^n/n!)(X_1-1) \ldots (X_n-1)$$

and this bound can not be improved.

1. Introducing the problem.

Let Λ be a lattice in n-dimensional Euclidean space, E^n, having determinant $d(\Lambda)$, and being generated by the n linearly independent vectors $\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n$. Let $\|\mathrm{x}_i\| = \xi_i (1 \leq i \leq n)$. Let K be a closed, bounded, convex body of volume $V(K) = V$, which contains no point of Λ in its interior. Let supporting hyperplanes to K parallel to the hyperplane spanned by x_j $(1 \leq j \leq n, j \neq i)$ cut the x_i-axis in a segment of length $X_i(K) = X_i$. We call X_i the <u>projection</u> of K on the x_i-axis.

In general terms, our problem is this:

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Given Λ and the projections $X_1, X_2, \ldots, X_n$ of K, find a function F such that

$$(1) \qquad V(K)/d(\Lambda) \le F(X_1/\xi_1, \ldots, X_n/\xi_n).$$

Since the ratios $V(K)/d(\Lambda)$, X_i/ξ_i $(1 \le i \le n)$ are invariant under affine transformation, we may assume that Λ is the integral lattice, generated by the unit vectors $e_1, e_2, \ldots, e_n$. In this case, $d(\Lambda) = 1$, $\xi_i = 1$ $(1 \le i \le n)$, and (1) simplifies to

$$V(K) \le F(X_1, \ldots, X_n).$$

To find a 'best' bound $F(X_1, X_2, \ldots, X_n)$ in n-space appears to be a complex task; even for $n = 3$ there are many cases to consider, depending on the relative sizes of $X_1, X_2, \ldots, X_n$.

THEOREM 1.

$$V(K) \le X_1 X_2 \ldots X_n .$$

Proof. This inequality is trivially true for any convex body K; equality is required when K is an $X_1 \times X_2 \times \ldots \times X_n$ rectangular parallelotope. If for some i, $1 \le i \le n$, $X_i \le 1$, then $\underline{K}$ can be placed to lie within the infinite slab $0 \le x_i \le 1$, thus containing no points of the integral lattice in its interior. It follows that in this case the bound cannot be improved.

We shall establish the following partial solution to our problem.

THEOREM 2. *If* $n \ge 2$, *and* $1 \le X_i \le n/(n-1)$ $(1 \le i \le n)$, *then*

$$V(K) \le X_1 X_2 \ldots X_n - (n^n/n!)(X_1 - 1) \ldots (X_n - 1),$$

and this bound cannot be improved.

We note that since K is a bounded convex body, for each choice of $X_1, X_2, \ldots, X_n$ the existence of a set of maximal volume is guaranteed by the Blaschke selection theorem.

2. Simplifying the problem.

We first show that for $n \geq 2$,

$$F(X_1, \ldots, X_n) = X_1X_2 \ldots X_n - (n^n/n!)(X_1-1) \ldots (X_n-1)$$

is an increasing function of each X_i in the given range. Working for example with X_1,

$$\frac{\partial F}{\partial X_1} = X_2 \ldots X_n - (n^n/n!)(X_2-1) \ldots (X_n-1).$$

For $n = 2,\ \frac{\partial F}{\partial X_1} = X_2 - 2(X_2-1) = 2 - X_2 \geq 0.$

For
$$\begin{aligned} n = 3,\ \frac{\partial F}{\partial X_1} &= X_2X_3 - (9/2)(X_2-1)(X_3-1) \\ &= (-7/2)(X_2-1)(X_3-1) + (X_2-1)+(X_3-1)+ 1 \\ &\geq 1 - (7/2)(X_2-1)(X_3-1) \\ &\geq 1 - \frac{7.1.1}{2.2.2} \\ &> 0 \end{aligned}$$

For
$$\begin{aligned} n \geq 4,\ \frac{\partial F}{\partial X_1} &\geq 1 - (n^n/n!)(n-1)^{-(n-1)} \\ &= 1 - (1 + \frac{1}{n-1})^{n-1}/(n-1)! \\ &\geq 1 - e/(n-1)! \\ &> 0 . \end{aligned}$$

Hence for each $n \geq 2$, F is an increasing function of X_1, and similarly of each X_i $(1 \leq i \leq n)$.

We shall use Steiner symmetrization to transform K into a set K' which contains no points of the integral lattice, and such that $X_i(K') \leq X_i(K)$ $(1 \leq i \leq n)$.

LEMMA 1. *If* K' *is the convex body obtained by symmetrizing* K *about the hyperplane* $H_j : x_j = \frac{1}{2}$, *then* $V(K') = V(K)$, K' *contains no lattice points,* $X_i(K') = X_i(K)$ *for* $i \neq j$, *and* $X_j(K') \leq X_j(K)$ $(1 \leq i,j \leq n)$.

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Proof. Symmetrization preserves volume. Since K contains no points of the integral lattice, the lattice lines perpendicular to H_j intercept K in segments of length not exceeding 1. Such lines intercept K' in segments of the same length; since the translated segments are centred on H_j, K' contains no lattice points.

By the definition of Steiner symmetrization, K and K' have a common pair of supporting hyperplanes normal to the x_i-axis for each $i \neq j$.

Hence $X_i(K') = X_i(K)$ for all $i \neq j$. Also, since K' is symmetric about the hyperplane $H_j : x_j = \frac{1}{2}$, $X_j(K')$ is given by the length s of the longest segment normal to H_j. Since under symmetrization this segment is translated along the line it determines, the projection $X_j(K)$ cannot be less than s. Hence $X_j(K') \le X_j(K)$.

From the lemma, we may assume that K is symmetric about each of the hyperplanes $H_i : x_i = \frac{1}{2}$ $(1 \le i \le n)$. Changing our coordinate system by translating $(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ to the origin, we henceforth assume that K is symmetric about the coordinate planes, and contains none of the vertices $(\pm\frac{1}{2}, \pm\frac{1}{2}, \dots, \pm\frac{1}{2})$ of the unit hypercube.

3. Proof of Theorem 2.

We shall need two preliminary lemmas.

LEMMA 2. *Let S be a simplex in E^n which is bounded by the n coordinate hyperplanes and a hyperplane through the point $P(p_1, p_2, \dots, p_n)$ $(p_i > 0, 1 \le i \le n)$.*
Then

$$V(S) \ge (n^n/n!)p_1p_2 \dots p_n .$$

Proof. Let the hyperplane through P have equation

$$\sum_{i=1}^{n} a_i(x_i - p_i) = 0 .$$

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This cuts the axes in (positive) lengths $(\sum_{i=1}^{n} a_i p_i)/a_j \; (1 \le j \le n)$.

Hence

$$n!\,V(S) = (\sum_{i=1}^{n} a_i p_i)^n / a_1 a_2 \ldots a_n .$$

By the inequality of the means,

$$(\sum_{i=1}^{n} a_i p_i)/n \ge (\prod_{i=1}^{n} a_i p_i)^{1/n}$$

Hence

$$n!\,V(S) = (\sum_{i=1}^{n} a_i p_i)^n / a_1 a_2 \ldots a_n$$

$$\ge n^n p_1 p_2 \ldots p_n$$

that is $$V(S) \ge (n^n/n!)\, p_1 p_2 \ldots p_n .$$

LEMMA 3. *Let T be a bounded $(n-1)$-dimensional set in E^n having centre of gravity* $\mathbf{t}$. *Let F be an $(n-2)$-dimensional flat intersecting T, and separating it into subsets U, W. Let T be rotated about F. Then (1) U, W sweep out equal volumes if and only if* $\mathbf{t}$ *lies in F*

(2) U sweeps out a larger volume than W if and only if F strictly separates $\mathbf{t}$ *from W (in the hyperplane of T).*

This lemma is proved in [2] when T is a simplex. However, the proof carries through without change for more general T.

Proof of Theorem 2. We suppose that K is a body of maximal volume for the given projections $X_1, X_2, \ldots, X_n$. The theorem is obviously true if $X_1 = X_2 = \ldots = X_n = 1$. We may therefore suppose that $X_i > 1$ for some i, $1 \le i \le n$. We consider the portion of K in the positive orthant. Since $X_i > 1$ for some i, and K is convex, K will be bounded by a hyperplane H through $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$.

Suppose first that H strictly separates at least one of the points $\frac{1}{2}(X_1, X_2, \ldots, X_{n-1}, 0), \ldots, \frac{1}{2}(0, X_2, \ldots, X_n)$ from the origin; let us take $\frac{1}{2}(0, X_2, \ldots, X_n)$ for example. Let R denote the rectangular

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parallelotope $\{(x_1, x_2, \ldots, x_n) \mid 0 \le x_i \le \frac{1}{2} X_i \ (1 \le i \le n)\}$, and let $T = R \cap H$. If F_0, F_1 denote the $(n-1)$-dimensional faces of R lying in the hyperplanes $x_1 = 0$, $x_1 = \frac{1}{2} X_1$ respectively, then $H \cap F_0$, $H \cap F_1$ are parallel, non-empty $(n-2)$-dimensional sets which form part of the boundary of T. Let F denote the $(n-2)$-dimensional flat which is parallel to $H \cap F_0$ $(H \cap F_1)$, and which passes through the point $(\frac{1}{2}, \frac{1}{2}, \ldots \frac{1}{2})$; let F separate T into parts U and W containing $H \cap F_0$, $H \cap F_1$ respectively.

Now it is well-known that the centre of gravity $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ of T can be no closer to F_1 than $(1/n)$th of the distance between F_0 and F_1, and equality can occur here only when $H \cap F_0$ is a single point [1]. Since by assumption $H \cap F_0$ is not a single point,

$$\frac{1}{2} X_1 - t_1 > \frac{1}{n} (\frac{1}{2} X_1)$$

or

$$t_1 < \frac{n-1}{n} (\frac{1}{2} X_1)$$
$$\le \frac{1}{2}$$

Thus $t_1 < \frac{1}{2}$, and F strictly separates $\mathbf{t}$ from W. Hence by Lemma 3, a small rotation of H about F which brings $H \cap F_0$ closer to the point $\frac{1}{2}(0, X_2, \ldots, X_{n-1}, X_n)$ actually increases the volume of K. Thus K does not have maximal volume.

We therefore assume that the bounding hyperplane H of K does not separate any of the points $\frac{1}{2}(X_1, \ldots, X_{n-1}, 0) \ldots, \frac{1}{2}(0, X_2, \ldots, X_n)$ from the origin. Notice that the existence of such a hyperplane is not in doubt: for example

$$\sum_{i=1}^{n} \frac{x_i}{n} = \frac{1}{2}$$

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is a hyperplane which passes through $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, and

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{X_i}{2n} \leq \frac{1}{2n} \cdot \frac{n}{n-1} \cdot \sum_{\substack{i=1 \\ i \neq j}}^{n} 1 = \frac{1}{2} \ (1 \leq j \leq n).$$

Consider now the simplex S' which is bounded by hyperplane H through $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ and hyperplanes $x_1 = \frac{1}{2} X_1, \ldots, x_n = \frac{1}{2} X_n$. From Lemma 2,

$$V(S') \geq (n^n/n!)(\tfrac{1}{2} X_1 - \tfrac{1}{2}) \ldots (\tfrac{1}{2} X_n - \tfrac{1}{2})$$

and

$$\begin{aligned} V(K) &\leq X_1 X_2 \ldots X_n - 2^n V(S') \\ &= X_1 X_2 \ldots X_n - (n^n/n!)(X_1 - 1) \ldots (X_n - 1) \end{aligned}$$

as required.

4. Further comment. The case where $X_i \geq n (1 \leq i \leq n-1)$, $X_n > 1$ is of some interest, for here the technique involving the position of the centre of gravity of T can again be applied to the symmetrized set. We can thus determine that the maximal volume of the symmetrized set occurs when the bounding hyperplane H through $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ passes through $(0, 0, \ldots, 0, X_n/2)$. Unfortunately however, the formula for this maximal volume appears to be excessively complicated, even for $n = 3$, making it unclear whether even the symmetrization is an allowable step in the proof.

For $n = 2$ there is no such problem, and we easily obtain:

THEOREM 3. *If* $n = 2$ *and* $X_1 \geq 2$, $X_2 > 1$, *the area* $A(K)$ *of* K *satisfies*

$$A(K) \leq X_1 X_2 - \tfrac{1}{2} X_1^2 (X_2 - 1)$$

and this bound cannot be improved.

We observe that this bound can also be written as

$$X_1 X_2 - 2(X_1 - 1)(X_2 - 1) - \tfrac{1}{2} (X_1 - 2)^2 (X_2 - 1) ,$$

thus incorporating the bounds of Theorems 1 and 2.

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