FREE PRODUCTS IN MAPPING CLASS GROUPS GENERATED BY DEHN TWISTS

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1. Introduction. Let F be an orientable surface with or without boundary and let $M(F)$ be the mapping class group of F, i.e. the group of isotopy classes of orientation preserving diffeomorphisms of F. To each essential simple closed curve c on F we can associate an element C of $M(F)$ called the Dehn twist about c. We refer the reader to [**1**] for definitions. It is well known (see [**1**]) that, at least in the case where F has no more than one boundary component, $M(F)$ is generated by Dehn twists. Further, there are important subgroups of $M(F)$ which are also generated by Dehn twists or simple products of Dehn twists; for example the Torelli group, the kernel of the homology action map $M(F) \to \mathrm{Aut}(H_1(F; Z)) = \mathrm{Sp}(H_1(F; Z))$, where $\mathrm{Sp}(H_1(F; Z))$ denotes the symplectic group, is known to be generated by Dehn twists about bounding curves and by "bounding pairs". See [**8**] for proofs and definitions. Also Dehn twists crop up as geometric monodromy maps associated to Picard–Lefschetz vanishing cycles for plane curve singularities (see [**5**]).

In this paper we seek to understand the subgroup structure of groups generated by sets of Dehn twists and in particular we obtain conditions on a set of curves $\{c_1, \ldots, c_n\}$ in F which guarantee that the subgroup $\langle C_1, \ldots, C_n \rangle$ of $M(F)$ generated by the Dehn twists $C_1, \ldots, C_n$ is a free product of free abelian groups. First we remark that each non-trivial Dehn twist has infinite order and so generates a subgroup isomorphic to Z. This deals with the case $n = 1$. If $n = 2$ (with c_1 and c_2 distinct) it is well known that C_1 and C_2 generate Z^2 if $i(c_1, c_2) = 0$, B_3 or $SL_2(Z)$ if $i(c_1, c_2) = 1$ and $Z * Z$ if $i(c_1, c_2) > 1$. Here $i(c_1, c_2)$ is the geometric intersection number of c_1 and c_2 and B_3 is the braid group on three strands. This deals with the case $n = 2$; however even in this case we remark that just knowing the algebraic intersection numbers of the two curves is not sufficient to determine the subgroup structure of $\langle C_1, C_2 \rangle$. If $n > 2$, then it turns out that just knowing the geometric intersection numbers $i(c_i, c_j)$ of the curves $c_1, \ldots, c_n$ is not sufficient to completely determine the nature of $\langle C_1, \ldots, C_n \rangle$; one also needs to know how the curves are embedded in F. We will give examples of this later on. The type of thing we prove is indicated in

THEOREM 1.1. *Let $c_1, \ldots, c_n$ be distinct essential simple closed curves in F such that $i(c_i, c_j) > 1$ for $i \neq j$ and no component of $F \backslash (\bigcup c_i)$ is a disc. Then $\langle C_1, \ldots, C_n \rangle$ is a free group of rank n.*

However in some situations, when the geometric intersection numbers $i(c_i, c_j)$ or algebraic intersection numbers $c_i \cdot c_j$ of the curves in question satisfy certain strong conditions, then it is true that the $i(c_i, c_j)$ or $c_i \cdot c_j$ completely determine the nature of the group $\langle C_1, \ldots, C_n \rangle$. For example we prove, using the homology representation mentioned above where each Dehn twist is sent to a symplective transvection:

THEOREM 1.2. *If the homology classes in $H_1(F; Z)$ of the essential simple closed curves $c_1, \ldots, c_n$ are independent and the algebraic intersection numbers $c_i \cdot c_j$ satisfy either*

(a) $|c_i \cdot c_j c_k \cdot c_j| \geq 6\,|c_i \cdot c_k|$ *for all distinct i, j, k*

or

(b) $|c_i \cdot c_j c_k \cdot c_j| \geq 2\,|c_i \cdot c_k|$ *for all distinct i, j, k and* $|c_i \cdot c_j| \geq 7$ *for all distinct i, j then* $\langle C_1, \ldots, C_n \rangle$ *is isomorphic to a free group of rank n.*

We prove a similar result in the case where the $i(c_i, c_j)$ satisfy this type of condition also (see Theorem 2.8).

Recall that if H is a subgroup of a linear group $GL_n(F)$ where F is a field of characteristic zero, then a result of Tits [9] says that either H is soluble by finite or H contains a rank 2 free subgroup. As far as subgroups H of mapping class groups generated by Dehn twists are concerned it easily follows that either (i) H is abelian (and so soluble) or (ii) that $\langle C_i^\lambda, C_j^\lambda \rangle$ is a rank 2 free group for some $i \neq j$ and integer λ. The first case is where $i(c_i, c_j) = 0$ for all i, j and (ii) occurs when $i(c_i, c_j) \neq 0$ for some $i \neq j$. Thus we obtain no information on the (non)-linearity of such subgroups by these methods.

2. General results. All simple closed curves referred to will be assumed essential and non-boundary parallel. Further, since the definition of a Dehn twist does not depend on the orientation of the curve involved (but only on the orientation of the surface—see [1]) we will not assume that curves are oriented. Let $S = \{c_1, \ldots, c_n\}$ be a set of distinct simple closed curves on F and let $S_1, \ldots, S_k$ be a partition of S such that $i(c_i, c_j) = 0$ for all c_i, $c_j \in S_q$ and $i(c_i, c_j) > 1$ for all $c_i \in S_q$, $c_j \in S_p$ with $p \neq q$. We call the partition $S_1, \ldots, S_k$ a *complete* partition of S in this case. We first prove a result which will imply Theorem 1.1 as a special case.

THEOREM 2.1. *Let $S = \{c_1, \ldots, c_n\}$ be a set of distinct simple closed curves on F and $S_1, \ldots, S_k$ a complete partition of S. Let n_i be the cardinality of S_i and assume that no component of $F \backslash (\bigcup c_i)$ is a disc. Then $\langle C_1, \ldots, C_n \rangle$ is isomorphic to the free product $Z^{n_1} * Z^{n_2} * \ldots * Z^{n_k}$.*

Proof. Without loss, and for notational convenience, we assume that each component of $F \backslash (\bigcup c_i)$ is an annulus. Let Γ be the graph whose vertices correspond to the components $E_1, \ldots, E_t$ of $F \backslash (\bigcup c_i)$ and where each unoriented edge from E_i to E_j corresponds to a common edge of their closures. We realise Γ as a subset of F by identifying each vertex E_i with the boundary component ∂E_i of the annulus E_i. Any two such components (vertices) belonging to the same edge in Γ are connected in F by a simple arc running from one boundary component to another which crosses $\bigcup c_i$ once. We call this realisation Γ also. Let $e_1, \ldots, e_p$ denote the edges of Γ with ordering as indicated. We use these ordered edges to give coordinates to any curve $\alpha(c_1)$ where $\alpha \in \langle C_1, \ldots, C_n \rangle$: note that for each c_i there are edges $e_{i(1)}, \ldots, e_{i(r(i))}$ such that c_i is isotopic to $e_{i(1)}\cup\ldots\cup e_{i(r(i))}$ and so if γ is a simple closed curve, then $i(\gamma, c_i)=\sum_{j=1}^{r(i)} i(\gamma, e_{i(j)})$. Now if $c=\alpha(c_1)$ where $\alpha\in\langle C_1,\ldots,C_n\rangle$, then c is isotopic to a curve c' in Γ which never doubles back on itself. Now count the number of times c' completely traverses an edge e_i of Γ (in either direction) and call this the e_i coordinate of c. Denote it by $e_i(c)$. Note that since no component of $F\backslash\bigcup c_i$ is a disc this coordinate is unique. Let $e(c)=(e_1(c),\ldots,e_p(c))$ be the coordinates of c.

LEMMA 2.2. *For c as above we have $e_j(C_i^{\pm1}(c))=e_j(c)$ if $j\neq i(k)$ for all $k=1,\ldots,r(i)$ and otherwise $e_j(C_i^{\pm1}(c))\geq|e_j(c)-i(c_i,c)|$, where $i(c_i,c)=\sum_p e_p(c)i(c_i,c_p)$.*

Proof. Since the only non-trivial coordinates of c_i are $e_{i(1)},\ldots,e_{i(r(i))}$ we see that these are the only coordinates which are changed by C_i. Further, if c and c_i have minimal intersection (which we can assume by an isotopy), then C_i adds in a copy of c_i at each such intersection point. Thus we add in $i(c_i,c)$ copies of each of $e_{i(1)},\ldots,e_{i(r(i))}$ in this way and at most $e_j(c)$ of these can cancel with ones already there. The Lemma now follows.

Let $S_q(c)=\max(\min(e_{i(j)}, j=1,\ldots,r(i)))$, where the maximum is taken over all i such that $c_i\in S_q$. Then by the above result we have $S_j(C_i^{\pm1}(c))=S_j(c)$ if c_i does not belong to S_j and $S_j(C_i^{\pm1}(c))\geq|S_j(c)-i(c_i,c)|$ if $c_i\in S_j$, where $i(c_i,c)\geq\sum_p S_p(c)i(c_i,c_p)$. Let $S(c)=(S_1(c),\ldots,S_k(c))$. We will use the $S(c)$ as (lower bounds for) coordinates in what follows. Note that $S(c)\in(Z^+)^k$.

In order to prove that the group $\langle C_1,\ldots,C_n\rangle$ is free we will need the following freeness criterion (see [**6**, Chapter 3]):

LEMMA 2.3. *Let G be a group generated by subgroups $G_1,\ldots,G_m$ and acting on a set Ω. Let $\Omega_0=\{x_0\},\Omega_1,\ldots,\Omega_m$ be disjoint subsets of Ω such that if $\alpha\in G_i$, $\alpha\neq$ id, then $\alpha(\Omega_s)\subset\Omega_i$ for all $i\neq s$. Then G is the free product $G_1*G_2*\ldots*G_m$ of the subgroups $G_1,\ldots,G_m$.*

We apply this result to the situation $G=\langle C_1,\ldots,C_n\rangle$, $G_i=\langle C_j\mid c_j\in S_i\rangle$, $i=1,\ldots,k$. Note that by previous remarks each of the subgroups G_i is free abelian of rank card$(S_i)=n_i$. We let $\Omega_0=\{c_1\}$, and for $t>0$ we let Ω_t be the set of all curves c which satisfy the conditions (i) $S_q(c)>1$ for some $q=1,\ldots,k$ and (ii) $S_t(c)\geq S_q(c)$ for all $q=1,\ldots,k$. We now prove that the G_i and the Ω_j satisfy the conditions of Lemma 2.3. Let $\alpha=C_i^p$ where $c_i\in S_j$. Then we have to show that $S(\alpha(c))\in\Omega_j$. Here we can assume that $c=\beta(c_1)$ where β is a freely reduced product $\beta_1\beta_2$ with β_2 a freely reduced word whose last letter does not lie in G_j and $\beta_1\in G_j$ but does not contain the letters $C_i^{\pm1}$. By the above remarks we have $S_j(\alpha(c))\geq|S_j(c)-i(c_i,c)|$ if $c_i\in S_j$, where $i(c_i,c)\geq\sum_p S_p(c)i(c_i,c_p)$. Since we are assuming that $i(c_i,c_p)\geq2$ for all c_p not in S_j we have $i(c_i,c)\geq\sum_{p\neq j}2S_p(c)$ and so $S_j(\alpha(c))\geq\sum_{p\neq j}2S_p(c)-S_j(c)$. Now since $c\in\Omega_p$ where $p\neq j$, then the result follows easily from this last statement. If $c\in\Omega_j$, then since $c=\beta_1\beta_2(c_1)$

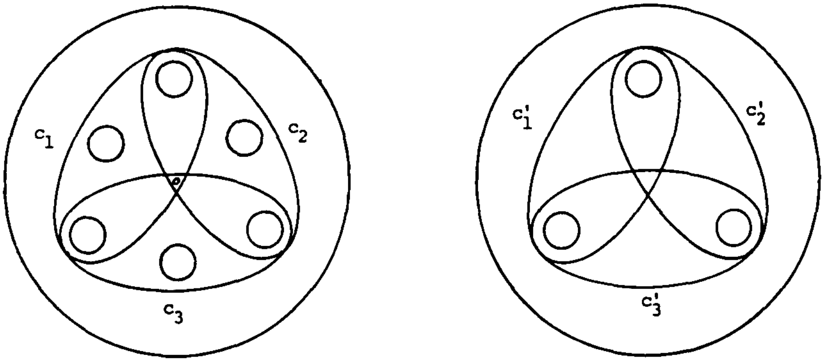

Figure 1

with β_1 and β_2 as above we see from the definition of $S_j(c)$ and the fact that the curves in S_j are disjoint that $S_j(\alpha(c)) \geq S_j(c)$. This gives the desired condition in this case and completes the proof of Theorem 2.1.

EXAMPLE 2.5. We now give two examples of three simple closed curves c_1, c_2, c_3 (respectively c_1', c_2', c_3') each intersecting the other twice geometrically but zero times algebraically whose groups are different, one being free of rank three the other free of rank two. These curves are shown in Figure 1.

Since no component of $F\backslash\{c_1, c_2, c_3\}$ is a disc it follows from Theorem 2.1 that $\langle C_1, C_2, C_3\rangle$ is a free group of rank three. As for the group $\langle C_1', C_2', C_3'\rangle$ we note that F is a three punctured disc and it follows that $M(F)$ is the quotient group of the braid group B_3 by its centre (the Dehn twist about the outside curve). It follows that $\langle C_1', C_2', C_3'\rangle$ is a subgroup of the group B_3/centre. It so happens that B_3 is generated by C_1', C_2', and C_3' (see [7, pages 173–174]) and that the centre is generated by $C_1'C_2'C_3'$. Thus $\langle C_1', C_2', C_3'\rangle = \langle C_1', C_2'\rangle$ is a free group of rank two.

Next we consider the homology representation $\varphi_F: M(F) \to \mathrm{Sp}(H_1(F;R))$. It certainly follows that if $\varphi_F(C_1), \ldots, \varphi_F(C_n)$ generate a free subgroup of $\mathrm{Sp}(H_1(F;R))$ of rank n, then $C_1, \ldots, C_n$ generate a free subgroup of $M(F)$ of rank n. We use this idea to give further conditions guaranteeing the freeness of the group $\langle C_1, \ldots, C_n\rangle$. First note that is c is a simple closed curve and C the associated Dehn twist, then the action of a $\varphi_F(C)$ on $H_1(F;R)$ is given by the following simple rule:

$$\varphi_F(C)([d]) = [C^k(d)] = [d] + k(c \cdot d)[c],$$

where $[d]$ denotes the homology class of d in $H_1(F;R)$. Thus $\varphi_F(C)$ is a symplectic transvection, i.e. an element of $\mathrm{Sp}(H_1(F;R))$ representable in matrix form as $I + A$ where I denotes the identity matrix and A has rank 1. We prove:

THEOREM 2.6. *For simple closed curves* $c_1, \ldots, c_n$ *we define* $\lambda_{ijk} = |(c_i \cdot c_j)(c_k \cdot c_j)|/|c_i \cdot c_k|$ *for all distinct* i, j, k. *Then the group* $\langle C_1, \ldots, C_n\rangle$ *is free of rank*

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n if the homology classes $[c_1]$, $[c_2], \ldots, [c_n]$ *are linearly independent in* $H_1(F; R)$ *and either*

(i) $\lambda_{ijk} \geq 6$ *for all distinct* i, j, k;

or

(ii) $n = 3$, $\lambda_{ijk} \geq 2$ *for all distinct* i, j, k *and* $|c_i \cdot c_j| \geq 7$ *for all distinct* i, j;

or

(iii) $n = 3$, $\lambda_{132} \geq 33$, $\lambda_{123} \geq 3{\cdot}6$, $\lambda_{213} \geq 4{\cdot}2$, $|c_1 \cdot c_2|^2 \geq 15$, $|c_1 \cdot c_3|^2 \geq 200$, $|c_2 \cdot c_3|^2 \geq 240$;

or

(iv) $n = 3$, $|c_1 \cdot c_2/c|$, $|c_2 \cdot c_3/c|$, $|c_1 \cdot c_3/c| \geq 200$, *where* $c = |(c_1 \cdot c_2)(c_1 \cdot c_3)|$;

or

(v) $n = 3$, $\lambda_{ijk} \geq 2$ *for all* i, j, k, $|c_i \cdot c_j| \geq 60$ *for all* $i \neq j$ *with the possible exception of* $|c_1 \cdot c_2| < 12$.

Proof. Choose a basis for $H_1(F, R)$ whose first n elements are $[c_1]$, $[c_2], \ldots, [c_n]$. Then with respect to this basis the matrix of $\varphi_F(C_i)$ differs from the identity matrix only in the i^{th} row, whose (i, j) entry is $c_i \cdot c_j$ if $i \neq j$ and 1 if $i = j$. The theorem now follows from results of [**3**, **4**], which give conditions guaranteeing that a set of transvections generates a free group. The idea of the proof of these results (in [**3**, **4**]) is to show that for any freely reduced word α in the letters $\varphi_F(C_1), \ldots, \varphi_F(C_n)$, there is a vector e such that the length of the vector $\alpha(e)$ is greater than the length of the vector $\beta(e)$ where β is any proper initial subword of α.

From [**2**, Exposé 4] we obtain

LEMMA 2.7. *If* c, d, e *are simple closed curves on the surface* F, *then*

$$i(C(d), e) \geq |i(c, d)i(c, e) - i(d, e)|.$$

We use this inequality to give conditions on the numbers $i(c_i, c_j)$ which guarantee freeness much the same as we did in the last result. Again we use $c_1, \ldots, c_n$ as a "basis" in the following sense: if c is a simple closed curve on F, then its "coordinates" will be the numbers $i(c_1, c), \ldots, i(c_n, c)$. The above lemma shows that the action of a Dehn twist C on these coordinates is much the same as the action of a Dehn twist on the homology coordinates considered above. In fact if T_i is the transvection whose ith row is

$$(i(c_i, c_j), \ldots, i(c_i, c_{i-1}), 1, i(c_i, c_{i+1}), \ldots, i(c_i, c_n)),$$

all other entries being the same as the corresponding entries of the identity matrix, then we have

$$C_i(c)_j = i(C_i(c), c_j) \geq |i(c_i, c)i(c_i, c_j)| - |i(c, c_j)|$$
$$\geq \min_{k \neq 0} \{|(T_i^k)^t(c_*)_j|\},$$

where c_* denotes the coordinates of c and the j subscript indicates the jth coordinate of c_*. Thus for any freely reduced word α in the letters $C_1, \ldots, C_n$ the coordinates of $\alpha(c)$

are bounded above by the coordinates of $\alpha_\#(c_*)$ where $\alpha_\#$ is the corresponding product of T_i's. The following result now follows from this observation and Theorem 2.6.

THEOREM 2.8. *For simple closed curves* $c_1, \ldots, c_n$ *we define* $\lambda_{ijk} = |i(c_i, c_j)i(c_k, c_j)|/|i(c_i, c_k)|$ *for all distinct i, j, k. Then the group* $\langle C_1, \ldots, C_n \rangle$ *is free of rank n if either*

(i) $\lambda_{ijk} \geq 6$ *for all distinct i, j, k*;

or

(ii) $n = 3$, $\lambda_{ijk} \geq 2$ *for all distinct i, j, k and* $|i(c_i, c_j)| \geq 7$ *for all distinct i, j*;

or

(iii) $n = 3$, $\lambda_{132} \geq 33$, $\lambda_{123} \geq 3{\cdot}6$, $\lambda_{213} \geq 4{\cdot}2$, $|i(c_1, c_2)|^2 \geq 15$, $|i(c_1, c_3)|^2 \geq 200$, $|i(c_2, c_3)|^2 \geq 240$;

or

(iv) $n = 3$, $|i(c_1, c_2)/c|$, $|i(c_2, c_3)/c|$, $|i(c_1, c_3)/c| \geq 200$, *where* $c = |i(c_1, c_2)i(c_1, c_3)|$;

or

(v) $n = 3$, $\lambda_{ijk} \geq 2$ *for all i, j, k*, $|i(c_i, c_j)| \geq 60$ *for all* $i \neq j$ *with the possible exception of* $|i(c_1, c_2)| < 12$.

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