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Nagoya Math. J.
Vol. 79 (1980), 145–149

TOPOLOGICALLY STABLE HOMEOMORPHISMS OF THE CIRCLE

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Introduction

In this paper, we study topologically stable homeomorphisms of the circle. Our results are the following:

THEOREM 1. *A homeomorphism of the circle is topologically stable if and only if it is topologically conjugate to some Morse-Smale diffeomorphism.*

THEOREM 2. *There exists a homeomorphism of the circle which has the pseudo-orbit-tracing-property but is not topologically stable.*

In [1], Bowen introduced the concept of the pseudo-oribt-tracing-property and essentially showed that expansive homeomorphisms with this property are topologically stable. Recently in [2], Morimoto has proved that the topological stability implies the pseudo-orbit-tracing-property. Theorem 2 above shows that expansiveness condition is necessary in Bowen's result.

The author would like to thank Professor A. Morimoto who posed this problem in his lectures at the University of Tokyo.

Notations and Definitions

Let X be a compact metric space with metric d. For continuous maps h_1, h_2 from X to itself, we set

$$\bar{d}(h_1, h_2) = \sup_{x \in X} d(h_1(x), h_2(x)) .$$

With this metric, the set of all continuous maps from X to itself is a metric space. Let f and g be homeomorphisms of X.

DEFINITION 1. We say that g is *topologically conjugate* (resp. *semi-conjugate*) to f by h, if h is a homeomorphism (resp. a continuous map)

Received March 29, 1979.

from X onto itself satisfying $h \circ g = f \circ h$.

If g is topologically conjugate (resp. semi-conjugate) to f by h, g^n is also topologically conjugate (resp. semi-conjugate) to f^n by h for any integer n.

DEFINITION 2. We call f *topologically stable* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that every homeomorphism f' with $\bar{d}(f, f') < \delta$ is topologically semi-conjugate to f by some h satisfying $\bar{d}(h, \mathrm{id}_X) < \varepsilon$.

DEFINITION 3. A sequence $\{x_n\}_{n \in Z}$ of points in X is called a *δ-pseudo orbit* of f if $d(f(x_n), x_{n+1}) < \delta$ for any integer n, and is said to be *ε-traced* by a point x of X if $d(f^n(x), x_n) < \varepsilon$ for any integer n.

DEFINITION 4. We say that f has the *pseudo-orbit-tracing-property* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit of f is ε-traced by some point.

Consider S^1 as $\boldsymbol{R}/\boldsymbol{Z}$ and let d denote the standard metric on S^1. The set of all fixed points of f and the set of all periodic points of f are denoted by $\mathrm{Fix}(f)$ and $\mathrm{Per}(f)$, respectively.

DEFINITION 5. A diffeomorphism f of S^1 is called *Morse-Smale* if $\mathrm{Per}(f)$ is non-empty and every element of $\mathrm{Per}(f)$ is hyperbolic, i.e., the differential of f^n at x is different from ± 1 for every periodic point x of f with period n.

DEFINITION 6. Let f be an orientation preserving homeomorphism of S^1. A fixed point x of f is said to be *topologically hyperbolic* if x is isolated in $\mathrm{Fix}(f)$ and $f(t) - t$ changes its sign at $t = x$. A periodic point x of any homeomorphism g of S^1 is called *topologically hyperbolic* if x is a topologically hyperbolic fixed point of g^{2n}, where n is a period of x.

Proof of Theorem 1

By a theorem of Nitecki [3], every Morse-Smale diffeomorphism is topologically stable. Since the topological stability is invariant under the topological conjugacy, every homeomorphism which is topologically conjugate to some Morse-Smale diffeomorphism is topologically stable.

It is easy to see that a homeomorphism f of S^1 is topologically conjugate to some Morse-Smale diffeomorphism if and only if it satisfies the following two conditions:

(a) $\mathrm{Per}(f)$ is non-empty and finite.

(b) Every element of $\mathrm{Per}(f)$ is topologically hyperbolic.

Hence, to prove Theorem 1, it suffices to show that every topologically stable homeomorphism of S^1 satisfies the above conditions (a) and (b). First we prove the following

LEMMA 1. *Suppose f and g are orientation preserving homeomorphisms of S^1 and g is topologically semi-conjugate to f by some h. Then if* $\mathrm{Per}(g)$ *is non-empty, so is* $\mathrm{Per}(f)$, *and there exists a constant C depending only on f such that if h satisfies $\bar{d}(h, \mathrm{id}_{S^1}) \leq C$, the cardinality of* $\mathrm{Per}(g)$ *is not less than that of* $\mathrm{Per}(f)$.

Proof. Since for every homeomorphism f' of S^1 there exists an integer n such that $\mathrm{Per}(f') = \mathrm{Fix}(f'^n)$, it is enough to prove Lemma 1 replacing $\mathrm{Per}(\ )$ by $\mathrm{Fix}(\ )$.

The proof of the first part is immediate. To prove the second part, we take a positive constant C satisfying the following two conditions:

(1) $C \leq 1/8$.

(2) If I is a closed interval in S^1 of length not greater than $4C$, then the length of $f(I)$ is not greater than $1/4$.

Suppose g is topologically semi-conjugate to f by a map h with $\bar{d}(h, \mathrm{id}_{S^1}) \leq C$ and take a fixed point x of f. Since $h^{-1}(x)$ is a non-empty closed g-invariant subset of S^1 contained in $[x - C, x + C]$, we can define $\sup h^{-1}(x)$ and $\inf h^{-1}(x)$ without confusion. Put $B = [\inf h^{-1}(x), \sup h^{-1}(x)]$. Then we have

$$(\text{length of } B) \leq 2C$$

and

$$\begin{aligned}(\text{length of } g(B)) &\leq (\text{length of } h \circ g(B)) + 2\bar{d}(h, \mathrm{id}_{S^1}) \\ &\leq (\text{length of } f \circ h(B)) + 2C \\ &\leq \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}.\end{aligned}$$

For the last inequality, we used the condition (2) and the estimation:

$$(\text{length of } h(B)) \leq (\text{length of } B) + 2\bar{d}(h, \mathrm{id}_{S^1}) \leq 4C.$$

Hence

$$(\text{length of } B) + (\text{length of } g(B)) \leq 2C + \tfrac{1}{2} \leq \tfrac{3}{4}.$$

By the invariance of $h^{-1}(x)$, both end points of $g(B)$ are in B and those of B in $g(B)$. Because the total length of S^1 is one, the above estimation shows that B is g-invariant. Since g is orientation preserving, $\sup h^{-1}(x)$ is a fixed point of g. Hence we have an injection from $\mathrm{Fix}(f)$

to Fix (g), which maps x to $\sup h^{-1}(x)$. This completes the proof of Lemma 1.

Now suppose that f is topologically stable. Since every homeomorphism of S^1 can be approximated by a diffeomorphism, by a theorem of Peixoto ([4]; p. 51), f is approximated by a Morse-Smale diffeomorphism. Thus there exists a Morse-Smale diffeomorphism which is topologically semi-conjugate to f. Therefore, by Lemma 1, Per (f) is non-empty and finite. (If f is orientation reversing, apply Lemma 1 to f^2.)

Next, take a periodic point x of f. If x is not topologically hyperbolic, we can eliminate this periodic point by a small perturbation, this contradicts Lemma 1. Therefore f satisfies the conditions (a) and (b), this completes the proof of Theorem 1.

Proof of Theorem 2

First consider a homeomorphism f_0 of $[0, 1]$ defined by

$$f_0(t) = \begin{cases} \frac{1}{2}t\,, & 0 \leq t \leq \frac{1}{4}\,, \\ \frac{3}{2}t - \frac{1}{4}\,, & \frac{1}{4} \leq t \leq \frac{3}{4}\,, \\ \frac{1}{2}t + \frac{1}{2}\,, & \frac{3}{4} \leq t \leq 1\,. \end{cases}$$

Let $p_n = 1/2^n$, $p_{-n} = 1 - 1/2^n$, $q_n = (1/2^{n+1})(1+1/2)$ and $q_{-n} = 1-(1/2^{n+1})(1+1/2)$ for a positive integer n. Then the desired homeomorphism f of $S^1 = [0, 1]/\sim$ is given as follows:

$$f(0) = 0\,,$$

$$f(x) = \begin{cases} p_{n+1} + (1/2^{n+1})f_0(2^{n+1}(x - p_{n+1}))\,, & p_{n+1} \leq x \leq p_n\,, \\ p_{-n} + (1/2^{n+1})f_0(2^{n+1}(x - p_{-n}))\,, & p_{-n} \leq x \leq p_{-n-1}\,, \end{cases}$$

$$n = 1, 2, \cdots.$$

Since Fix (f) is an infinite set, Theorem 1 implies that f is not topologically stable. So we have only to show that f has the pseudo-orbit-tracing-property.

LEMMA 2. *For a real number k, let L_k be the linear map from $\boldsymbol{R}$ to itself defined by $L_k(x) = kx$. Suppose $\{x_n\}$ is a δ-pseudo-orbit of L_k with $x_0 \in [-M, M]$.*

(i) *If $0 < k < 1$ and $\delta \leq (1-k)M$, then $x_n \in [-M, M]$ for every $n \geq 0$.*

(ii) *If $k > 1$ and $\delta \leq (k-1)M$, then $x_n \in [-M, M]$ for every $n \leq 0$.*

The proof is immediate and is omitted.

Fix an arbitrary positive number ε and choose a positive integer n satisfying $1/2^n < \varepsilon$. Let $I = [p_{-n}, p_n]$, $J = [p_{n+1}, p_{-n-1}]$ and $J' = [p_{n+2}, p_{-n-2}]$. Then $J \subset J'$ and $I \cup J = S^1$. By the definition of f, there exists a homeomorphism $\tilde{f}$ of S^1, which is topologically conjugate to some Morse-Smale diffeomorphism and satisfies $\tilde{f}|_{J'} = f|_{J'}$.

Now we take a constant δ satisfying the following two conditions:

(1) Every δ-pseudo-orbit of $\tilde{f}$ can be $1/2^{n+2}$-traced by some point.

(2) $\delta \leq 1/2^{n+4}$.

Suppose $\{x_n\}$ is a δ-pseudo-orbit of f. Then we can show that $\{x_n\}$ is ε-traced by some point as follows:

Case 1. For every integer n, x_n is in I.

It is evident that $0 \in S^1$ ε-traces this sequence with respect to f.

Case 2. There exists m such that $x_m \notin I$.

By Lemma 2 together with the condition (2), this sequence does not jump over the intervals $[p_n - 1/2^{n+3}, p_n + 1/2^{n+3}]$ and $[p_{-n} - 1/2^{n+3}, p_{-n} + 1/2^{n+3}]$ in the positive direction and the intervals $[q_n - 1/2^{n+3}, q_n + 1/2^{n+3}]$ and $[q_{-n} - 1/2^{n+3}, q_{-n} + 1/2^{n+3}]$ in the negative direction. Therefore this sequence always stays in J, and is a δ-pseudo-orbit of $\tilde{f}$. By the condition (1), there exists a point x of S^1 which $1/2^{n+2}$-traces this sequence with respect to $\tilde{f}$. In particular x is in the $1/2^{n+2}$-neighborhood of x_0, hence in J'. It follows that x ε-traces this sequence with respect to f.

Thus we have shown that every δ-pseudo-orbit of f can be ε-traced by some point. This completes the proof of Theorem 2.

References

[1] R. Bowen, ω-limit sets for Axiom A diffeomorphisms, J. Diff. Eq., **18** (1975), 333–339.
[2] A. Morimoto, Stochastic stable diffeomorphisms and Takens' conjecture, to appear.
[3] Z. Nitecki, On semi-stability for diffeomorphisms, Inv. Math., **14** (1971), 81–122.
[4] Z. Nitecki, Differentiable dynamics, M.I.T. Press (1971).

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