BULL. AUSTRAL. MATH. SOC. 12E05, 14A25
VOL. 60 (1999) [67–72]

A NOTE ON COHOMOLOGY OVER NON ALGEBRAICALLY CLOSED FIELDS

J.M. GAMBOA

We characterise algebraically closed fields as those for which the first cohomology group $H^1(k^n, O_n)$ of the sheaf O_n of regular functions over k^n vanishes for all positive intergers n.

INTRODUCTION

It is very well known (see for example [**1**]) that the first cohomology group $H^1(k^n, O_n)$ of the sheaf O_n of regular functions over k^n vanishes if k is an algebraically closed field and n is a natural number. The goal of this short note is to prove that this is no longer true if k is not algebraically closed and $n > 1$. This was first observed by Tognoli in [**3**] in the case that k is a real closed field, and we shall show that in fact, the vanishing of this first cohomology group characterises algebraically closed fields. The key point to show this is the existence of irreducible polynomials $f \in k[X_1, \dots, X_n]$ whose set of k-zeros,

$$Z_k(f) = \{x \in k^n : f(x) = 0\}$$

is a reducible algebraic subset of k^n. To be precise, we shall prove the following.

PROPOSITION 1. *Let k be a field. The following statements are equivalent:*

(1) *k is algebraically closed.*
(2) *For each positive integer n and for every irreducible polynomial $f \in k[X_1, \dots, X_n]$, the set $Z_k(f)$ of k-zeroes of f is irreducible in the Zariski topology of k^n.*
(3) *There exists an integer $n > 1$ such that $Z_k(f)$ is irreducible in the Zariski topology of k^n for every irreducible polynomial $f \in k[X_1, \dots, X_n]$.*

The second ingredient is the existence of "optimal" denominators for regular functions, which is the content of

Received 17th December, 1998
Partially supported by DGICYT PB95-0354.

https://doi.org/10.1017/S0004972700033335 Published online by Cambridge University Press

PROPOSITION 2. *Let n be a positive integer, k a non algebraically closed field, U a Zariski open subset of k^n and $g \in O_n(U)$. Then there exists a representative p/q of g on U, $p, q \in k[X_1, \dots, X_n]$, such that every representative p'/q' of g on U, satisfies the inclusion $Z_k(q) \subseteq Z_k(q')$.*

This way one obtains the announced result:

THEOREM 1. *Let k be a non algebraically closed field and $n > 1$ an integer. Then $H^1(k^n, O_n) \neq \emptyset$.*

PROOFS OF THE RESULTS

PROOF OF PROPOSITION 1: The implication (1) $\Rightarrow$ (2) is a particular case of the classical Hilbert Nullstellensatz, whilst (2) $\Rightarrow$ (3) is obvious. For (3) $\Rightarrow$ (1), suppose that k is not algebraically closed, and let $n > 1$ be an integer. We distinguish first the case $k = \mathbf{F}_2$ is the field with two elements. Then the polynomial

$$f(X) = X_1^2 + X_1X_2(X_2+1) + X_2^2(X_2+1)^2$$

is irreducible in $\mathbf{F}_2[X_1, \dots, X_n]$ but $Z_{\mathbf{F}_2}(f) = \{(0,0)\} \times \mathbf{F}_2^{n-2} \cup \{(0,1)\} \times \mathbf{F}_2^{n-2}$. In fact, if f were reducible, we could write $f = (X_1 + g_1)\cdot(X_1 + g_2)$ for some polynomials g_1 and $g_2 \in A = \mathbf{F}_2[X_2, \dots, X_n]$. Hence,

$$g_1 + g_2 = X_2(X_2+1) \text{ and } g_1 \cdot g_2 = X_2^2(X_2+1)^2.$$

Consequently, $(g_1+g_2)^2 = g_1 \cdot g_2$. In other words, $\varphi = g_1/g_2$ belongs to the quotient field of A, and it is integral over A, since $\varphi^2+\varphi+1 = 0$. Thus $\varphi \in A$ because this ring being an U.F.D. is integrally closed. Then, $-1 = \varphi(\varphi+1)$, and by counting degrees we conclude that $\varphi \in \mathbf{F}_2$, and so $-1 = \varphi(\varphi+1) = 0$, a contradiction.

In what follows we can assume that $k \neq \mathbf{F}_2$. Let us take an element $w \notin k$ which is algebraic over k, and $u \in k$ with $u(u-1) \neq 0$. If $\varphi(T) = \operatorname{Irr}(w;k)$ is the minimal polynomial of w over k, written as

$$\varphi(T) = T^p + a_1T^{p-1} + \cdots + a_p,$$

the polynomial

$$f(X_1, \dots, X_n) = X_1^p + \sum_{j=1}^{p} \frac{a_j}{u^j(u-1)^j} X_1^{p-j} X_2^j (X_2-1)^j$$

is irreducible in $k[X_1, \dots, X_n]$ but $Z_k(f) = \{(0,0)\} \times k^{n-2} \cup \{(0,1)\} \times k^{n-2}$. In fact, if f were reducible, there would exist monic polynomials $g, h \in k[X_1, \dots, X_n]$ with respect to the variable X_1, such that $f = g \cdot h$. After evaluation at $X_2 = u$, we obtain

$$\varphi(X_1) = G(X_1)H(X_1),$$

https://doi.org/10.1017/S0004972700033335 Published online by Cambridge University Press

where $G(X_1) = g(X_1, u, X_3, \dots, X_n)$, and $H(X_1) = h(X_1, u, X_3, \dots, X_n)$. Both G and H are monic with respect to X_1 and so non constant, and this contradicts the irreducibility of φ in $k[T]$. On the other hand, each point b with coordinates $b = (b_1, \dots, b_n) \in Z_k(f)$ satisfies $b_2(b_2 - 1) = 0$, since otherwise

$$v = \frac{b_1 u(u-1)}{b_2(b_2-1)} \in k$$

would be a root of φ. Hence $b_1^p = 0$, and so $b_1 = 0$; $b_2 = 0$ or $b_2 = 1$, which provides the desired decomposition of the zero set of f. $\square$

REMARKS.

(1) In the case $k = \mathbf{R}$ is the field of real numbers, we can choose in the proposition above $w = \sqrt{-1}$ and so $\varphi(T) = T^2 + 1$, and $u = -1$. This way, one gets the irreducible polynomial $f \in \mathbf{R}[X_1, \dots, X_n]$ defined as

$$f(X_1, \dots, X_n) = X_1^2 + X_2^2(X_2 - 1)^2.$$

(2) In fact we have proved in Proposition 1, if k is not algebraically closed and $n > 1$, the existence of an irreducible polynomial $f \in k[X_1, \dots, X_n]$ whose zero set can be decomposed as a disjoint union of algebraic subsets

$$Z_k(f) = \{(0,0)\} \times k^{n-2} \cup \{(0,1)\} \times k^{n-2}.$$

Before proving Proposition 2 we need the following.

CONSTRUCTION

Let k be a non algebraically closed field. There exists a family $\{P_n : n \geqslant 1\}$ of polynomials in n variables with coefficients in k, such that for every positive integer $m \geqslant 1$ and every set $f_1, \dots, f_n$ of polynomials in the polynomial ring $k[X_1, \dots, X_m]$, the sets

$$\{x \in k^m : f_1(x) = \dots = f_n(x) = 0\} = \{x \in k^m : P_n(f_1(x), \dots, f_n(x)) = 0\}$$

coincide.

We construct the P_n's recursively, with $P_1(Z_1) = Z_1$. Let us assume that $n > 1$ and $P_1, \dots, P_{n-1}$ are constructed, and choose an element $w \notin k$ which is algebraic over k. Let $Z' = (Z_1, \dots, Z_{n-1})$,

$$\varphi(T) = T^p + a_1 T^{p-1} + \dots + a_p = \operatorname{Irr}(w; k),$$

and define

$$P_n(Z_1, \dots, Z_n) = Z_n^p + \sum_{j=1}^{p} a_j Z_n^{p-j} P_{n-1}^j(Z').$$

Clearly, if $x \in k^m$ and $f_1(x) = \cdots = f_n(x) = 0$, then, by the inductive hypothesis, $P_{n-1}(y) = 0,\ y = \big(f_1(x), \dots, f_{n-1}(x)\big)$ and so,

$$P_n\big(f_1(x), \dots, f_n(x)\big) = f_n^p(x) + \sum_{j=1}^{p} a_j f_n^{p-j}(x) P_{n-1}^j(y) = 0.$$

Conversely, if $P_n\big(f_1(x), \dots, f_n(x)\big) = 0$, then $P_{n-1}(y) = 0$, since otherwise $v = f_n(x)/P_{n-1}(y) \in k$ would be a root of φ. Hence, by the inductive hypothesis, $f_1(x) = \cdots = f_{n-1}(x) = 0$, and also $f_n^p(x) = 0$. ∎

REMARK. In the case $k = \mathbf{R}$ is the field of real numbers, we can choose in the construction above $w = \sqrt{-1}$ and so $\varphi(T) = T^2 + 1$. This way

$$P_2(Z_1, Z_2) = Z_2^2 + Z_1^2;\ P_3(Z_1, Z_2, Z_3) = Z_3^2 + \big(Z_2^2 + Z_1^2\big)^2,$$

and so on.

PROOF OF PROPOSITION 2: By the noetherianess of $k\big[X_1, \dots, X_n\big]$, there exists a finite set $p_1/q_1, \dots, p_r/q_r$ of representatives of g in U such that $Z(Q)$ contains $Z_k(q_1) \cap \ldots \cap Z_k(q_r)$ for each representative P/Q of g in U. So, it suffices to find another representative p/q of g in U such that

$$Z_k(q) = Z_k(q_1) \cap \ldots \cap Z_k(q_r).$$

Of course, it is enough to handle the case $r = 2$. With the notations in the last construction, consider

$$P_2\big(q_1(X),\ q_2(X)\big) = q_2(X)^p + \sum_{j=1}^{p} a_j q_2^{p-j}(X) q_1^j(X).$$

Since both p_1/q_1, p_2/q_2 are representatives of g in U, we obtain for each $j = 1, \dots, p-1$, the equalities

$$g = \frac{p_1}{q_1} = \frac{p_1 a_j q_2^{p-j} q_1^{j-1}}{a_j q_2^{p-1} q_1^j} = \frac{p_2 q_2^{p-1}}{q_2^p},$$

and so, after addition, $g = p/q$ on U, where

$$p = p_2 q_2^{p-1} + \sum_{j=1}^{p-1} p_1 a_j q_2^{p-j} q_1^{j-1} \text{ and } q = q_2^p + \sum_{j=1}^{p-1} a_j q_2^{p-j} q_1^j = P_2(q_1, q_2).$$

This is the required representative of g on U, since

$$Z_k(q) = Z_k\big(P_2(q_1, q_2)\big) = Z_k(q_1) \cap Z_k(q_2). \qquad \square$$

We are ready to give the

PROOF OF THEOREM 1: As we remarked, there exists an irreducible polynomial $f \in k[X_1, \ldots, X_n]$, whose zero set $Z_k(f)$ can be decomposed as $Z_k(f) = M_1 \cup M_2$, where

$$M_1 = \{(0,0)\} \times k^{n-2} \text{ and } M_2 = \{(0,1)\} \times k^{n-2}.$$

Since the intersection $M_1 \cap M_2$ is empty, the family

$$\mathbf{V} = \{U_1 = k^n - M_1;\ U_2 = k^n - M_2\},$$

is an open covering of k^n with respect to the Zariski topology. The Cech cohomology group $H^1(\mathbf{V}; O_n)$ of k^n with respect to the covering $\mathbf{V}$ is embedded in $H^1(k^n, O_n)$, see [**2**], and so it suffices to check that $H^1(\mathbf{V}; O_n) \neq 0$. Define on $U_{12} = U_1 \cap U_2 = k^n - Z_k(f)$ the cocycle $\Psi = 1/f$. It is enough to prove that it is not a coboundary with respect to $\mathbf{V}$. Otherwise, there would exist regular functions $f_i \in O_n(U_i)$, $i = 1, 2$, such that $\Psi = f_1 - f_2$ on U_{12}. Since each f_i is regular, it follows from Proposition 2 that it can be written as $f_i = p_i/q_i$, in such a way that the intersection $Z_k(q_i) \cap U_i$ is empty. Thus, since also $M_1 \cap M_2$ is empty, it follows that

$$M_1 \subseteq k^n - Z_k(q_2) \text{ and } M_2 \subseteq k^n - Z_k(q_1). \tag{1}$$

On the other hand, the equality $(p_1/q_1) - (p_2/q_2) = 1/f$ on the dense subset U_{12} of k^n (with respect to the Zariski topology), implies, by the continuity of regular functions that, as polynomials in $k[X_1, \ldots, X_n]$,

$$(p_1 q_2 - p_2 q_1) f = q_1 q_2.$$

Since f is irreducible, this implies that either q_1 or q_2 is a multiple of f and so either

$$Z_k(f) \subseteq Z_k(q_1), \text{ or } Z_k(f) \subseteq Z_k(q_2),$$

which contradicts (1). $\square$

REFERENCES

[**1**] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52** (Springer-Verlag, Berlin, Heidelberg, New York, 1977).

[**2**] J.P. Serre, 'Faisceaux algebriques cohérents', *Ann. of Math. Stud.* **61** (1955), 197–258.

[3] A. Tognoli, 'Some basic facts in algebraic geometry on a non algebraically closed field', *Ann. Scuola Normal Sup. Pisa Cl. Sci.* **3** (1976), 341–359.

Facultad de Matemáticas
Universidad Complutense de Madrid
28040 Madrid
España
e-mail: jmgamboa@eucmax.sim.ucm.es