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A CHARACTERISTIC PROPERTY OF $PSL_2(7)$

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Abstract

In this note we characterize $PSL_2(7)$ by conditions on the order of the group and the orders of its elements.

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In [7] we have characterized $PSL(5)$ by conditions only on the order of the group and the orders of its elements. That is, the characteristic property of $PSL_2(5)$ is:

(1) the order of the group contains at least three different prime factors;

(2) the order of every non-identity element in the group is a prime.

S. Adnan has characterized $PSL_2(7)$ using the simplicity of the group and properties of its maximal subgroups (see [1] and [2]). In this note we continue the discussion of [7], and using the conclusions of [3] and [5] we present a characterization of $PSL_2(7)$ by conditions only on the order of the group and the orders of its elements.

THEOREM. *Let G be a finite group satisfying the following conditions*:

(1) $|G|$ *contains at least three different prime factors, that is*, $|\pi(G)| \geqslant 3$;

(2) *the order of every non-identity element in G is either a power of* 2 *or a prime different from* 5. *Then G is isomorphic to* $PSL_2(7)$.

PROOF. By condition (2) we see that G is a group in which every element has prime power order. From [3] Theorem 1 if G is solvable, then $|\pi(G)| \leqslant 2$. Thus by condition (1) we conclude that G is non-solvable. Furthermore, by a theorem of P.

Hall ([3] Theorem 4) G has the normal series

$$G \geqslant N > P \geqslant 1,$$

where P is the largest solvable normal subgroup of G and is a p-group and N/P is the unique minimal normal subgroup of G/P and is a simple group of composite order. Since every element of N/P has prime power order, [5] Theorem 16 implies that N/P is isomorphic to one of the following groups: $PSL_2(q)$, $q = 5, 7, 8, 9, 17$, $PSL_3(4)$, $Sz(8)$ or $Sz(32)$. Because $PSL_2(q)$, $q = 5, 9$, $PSL_3(4)$, $Sz(8)$ and $Sz(32)$ all contain elements of order 5 and $PSL_2(q)$, $q = 8, 17$ all contain elements of order 9, N/P must be isomorphic to $PSL_2(7)$. Indeed $PSL_2(7)$ contains only elements of order 2^2 and elements of prime order not equal to 5.

Suppose at first that P is not a 2-group. As G does not contain any elements of order $2p$ ($p \neq 2$) a Sylow 2-subgroup S_2 of N acts fixed-point-freely on P. From [4] Theorem 7.24 we conclude that S_2 is a cyclic group or a generalized quaternion group. But $PSL_2(7)$ and hence N has Sylow 2-subgroups which are dihedral groups of order 8. This contradiction shows that P is a 2-group. Let S_7 be a Sylow 7-subgroup of N. Then $C_N(S_7) = S_7$ and hence $N_N(S_7)$ is a group of order 21. Now $P.N_N(S_7)$ is a solvable group in which every element has prime power order. It follows from [3] Theorem 1 that $P = 1$. Thus $N \lhd G$ and as $C_G(N) = 1$ we conclude that G is a subgroup of $\mathrm{Aut}(N)$. By [6] we see that $|\mathrm{Aut}(N)| = 2 \cdot |PSL_2(7)| = 2^4 \cdot 3 \cdot 7$. Hence G can only be N or $\mathrm{Aut}(N)$. If $|G| = 2^4 \cdot 3 \cdot 7$, then $|N_G(S_7)| = 2 \cdot 3 \cdot 7$ and hence G contains an element of order 6 contrary to condition (2). It follows that $G \cong PSL_2(7)$.

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