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A 2D model for the excitation of the linearly stable inertial modes of the Sun by turbulent convection

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Abstract. The newly discovered inertial modes in the Sun offer the opportunity to probe the solar convective zone down to the tachocline. While linear analysis predicts the frequencies and eigenfunctions of the modes, it gives no information about their excitation or their amplitudes. We present here a theoretical formalism for the stochastic excitation of the solar inertial modes by turbulent convection. The amplitudes predicted by our model are in complete agreement with observations, thus supporting the assumption that they are stochastically excited. Our work also uncovers a qualitative transition in the shape of the inertial mode spectrum, between $m \lesssim 5$ where the modes are clearly resolved in frequency, and $m \gtrsim 5$ where the modes overlap. This complicates the interpretation of the high-m data, and suggests that a model for the whole shape of the power spectrum is necessary to exploit the full seismic potential of solar inertial modes.

Keywords. waves – turbulence – Sun: oscillations – Sun: interior – Sun: helioseismology

1. Introduction

Internal rotation in the Sun allows for the development of inertial modes, which propagate through the Coriolis force. While theory has predicted their existence for a long time (Papaloizou and Pringle 1978), sectoral ($l = m$) Rossby modes have only recently been observed in solar data (Löptien et al. 2018; Liang et al. 2019). This was followed shortly thereafter by the report of additional families of inertial modes, due to the internal differential rotation (Gizon et al. 2021; Hanson et al. 2022).

These newly observed inertial modes offer the possibility to probe the dynamics and the structure of the solar convective envelope in a way that complements p-mode seismology. For example, they are expected to be much more sensitive on the superadiabatic temperature gradient or the turbulent viscosity throughout the convective zone. To exploit this potential, theoretical modelling of the solar inertial modes is necessary. So far, efforts have focused on linear analysis to model their eigenfrequencies and eigenfunctions (Gizon et al. 2020; Fournier et al. 2022; Bekki et al. 2022; Triana et al. 2022), which allows for the identification of the observed modes. However, this approach gives no information about the amplitude of the modes, that is to say about their excitation mechanism. Most of them are linearly stable, meaning that they are most likely stochastically excited by turbulent convection, like p-modes. Having a model for the excitation of the solar inertial modes would allow us not only to predict which modes are expected to be visible and identifiable, but also to put stronger constraints on the dynamics of the solar convective zone, rather than only its equilibrium structure.

We present here such a model, tailored for purely toroidal inertial modes – which the observed solar inertial modes are –, with the inclusion of latitudinal differential rotation and turbulent viscosity, and placing ourselves in the equatorial β-plane approximation. The model is similar to the commonly accepted picture for p-modes (e.g. Samadi and Goupil 2001), where turbulent emission, meaning the non-linear term in the momentum equation (Lighthill 1967), provides the necessary mechanical work to excite the modes. This presentation highlights the main characteristics of the model, as well as the results that we obtained when applying it to the solar case.

2. Synthetic power spectrum

We focus here on quasi-toroidal vorticity modes, for which the horizontal part of the wave equation can be separated from its radial part. This allows us to adopt a 2D setting, and study the excitation of vorticity waves in a 2D shear flow representing the solar differential rotation. We place ourselves in the equatorial β-plane approximation, that is transforming the latitude λ and longitude ϕ into Cartesian coordinates $x \equiv R\phi$ and $y \equiv R \sin\lambda$, where R is the radius of the spherical shell under consideration. The incompressibility of the flow allows us to describe it by means of a stream function Ψ, related to the velocity $\mathbf{u}$ by

$$\mathbf{u} = \boldsymbol{\nabla} \times (\Psi \mathbf{e_z}) \ , \tag{1}$$

where $\mathbf{e_z}$ is the unit vector normal to the surface. After some algebra, we get a linear wave equation with a source term (Philidet and Gizon 2023)

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\Delta\Psi_{\mathrm{osc}} + (\beta - U'')\frac{\partial \Psi_{\mathrm{osc}}}{\partial x} - \nu_{\mathrm{turb}}\Delta^2\Psi_{\mathrm{osc}} = \frac{\partial \Psi_{\mathrm{turb}}}{\partial x}\frac{\partial \Delta\Psi_{\mathrm{turb}}}{\partial y} - \frac{\partial \Psi_{\mathrm{turb}}}{\partial y}\frac{\partial \Delta\Psi_{\mathrm{turb}}}{\partial x} \ , \tag{2}$$

where U'' is the second derivative of $U(y)$ (which is the velocity due to differential rotation), Δ is the Laplacian operator, and ν_{turb} the turbulent viscosity. This was written under the assumption that we could separate the field into an oscillation part Ψ_{osc} and a turbulence part Ψ_{turb}. We can then go in Fourier domain in time t and azimuth x, so that the wave equation becomes a 1D ordinary differential equation (ODE).

Because the oscillation and turbulence parts are decoupled, the latter can be treated as a source term, which we will consider as an input to the model, and will be taken from observation of the solar surface. Then Eq. (2) becomes a linear ODE with a known source term, whose solution can therefore be obtained by convolving the source term with the Green function $G(y_1, y_2)$ associated to the linear left-hand side. Eventually, one obtains the following expression for the expected power spectra in terms of latitudinal velocity, azimuthal velocity and vorticity respectively (Philidet and Gizon 2023)

$$\left\langle |\widehat{u}_{x,\mathrm{osc}}(y_{\mathrm{obs}})|^2 \right\rangle = \int_{-R}^{R} \mathrm{d}y_s \ \left| \frac{\partial G}{\partial y_{\mathrm{obs}}} \right|^2 \mathcal{I}(y_s) \ , \tag{3}$$

$$\left\langle |\widehat{u}_{y,\mathrm{osc}}(y_{\mathrm{obs}})|^2 \right\rangle = \int_{-R}^{R} \mathrm{d}y_s \ |k_x G(y_{\mathrm{obs}}, y_s)|^2 \, \mathcal{I}(y_s) \ , \tag{4}$$

$$\left\langle \left|\widehat{\zeta}_{\mathrm{osc}}(y_{\mathrm{obs}})\right|^2 \right\rangle = \int_{-R}^{R} \mathrm{d}y_s \ \left| \left(k_x^2 G - \frac{\partial^2 G}{\partial y_{\mathrm{obs}}^2} \right) \right|^2 \mathcal{I}(y_s) \ , \tag{5}$$

where y_{obs} is the latitudinal coordinate at which the power spectrum is evaluated, and $\langle \, . \, \rangle$ denotes an ensemble average. The function $\mathcal{I}(y_s)$ denotes the source covariance, and

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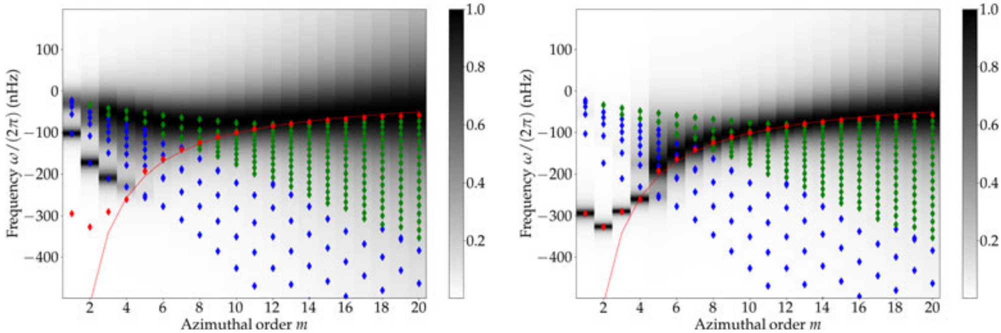

Figure 1. Equatorial power spectrum in the m-ω plane, in terms of azimuthal velocity (**left**) and latitudinal velocity (**right**). Each vertical slice is normalised separately such that the maximum is unity. The diamonds show the real part of the eigenfrequencies of the linear homogeneous problem. The colour code refers to the mode categories: in particular, the red diamonds represent the sectoral Rossby modes. The solid red line shows the theoretical dispersion relation for sectoral Rossby modes.

is defined by

$$\mathcal{I}(y_s) \equiv \int \mathrm{d}Y \ \left\langle \widehat{S}(y_s)\ \widehat{S}^*(y_s + Y) \right\rangle \ , \tag{6}$$

where $\widehat{S}$ is the Fourier transform in t and x of the right-hand side of Eq. (2).

The only assumption required to write Eqs. (3) to (5) is that the correlation scale of the source is much smaller than the typical variation scale of the Green function. Since most of the turbulent energy comes from the granulation scales at $\ell \sim 120$ and the inertial modes under consideration are low degree modes, this assumption is valid. Some additional algebra is required to specify the source covariance; we eventually find (Philidet and Gizon 2023)

$$\mathcal{I} = \frac{1}{4\pi^3} \int \mathrm{d}\omega'\ \mathrm{d}^2\mathbf{k}'\ k_x^3 k_x' k_y'^2\ |\mathbf{k}' + \mathbf{k}/2|^2$$
$$\times\, \mathcal{E}_\Psi \left(\omega' - \frac{\omega}{2}, \mathbf{k}' - \frac{\mathbf{k}}{2}\right) \mathcal{E}_\Psi^* \left(\omega' + \frac{\omega}{2}, \mathbf{k}' + \frac{\mathbf{k}}{2}\right) \ . \tag{7}$$

The function $\mathcal{E}_\Psi$ represents the turbulent stream function spectrum, and is defined by

$$\mathcal{E}_\Psi(\omega, \mathbf{k}) \equiv \int \mathrm{d}\tau\ \mathrm{d}^2\mathbf{x}\ \langle \Psi_{\mathrm{turb}}(T, \mathbf{X}) \Psi_{\mathrm{turb}}(T + \tau, \mathbf{X} + \mathbf{x}) \rangle\ \ e^{\mathrm{j}(\omega\tau - \mathbf{k}\cdot\mathbf{x})} \ . \tag{8}$$

The two ingredients needed to apply Eqs. (3) to (5) to the solar case are the turbulent spectrum $\mathcal{E}_\Psi$ and the Green function G. The former is extracted from observations, and the latter is computed numerically using spectral methods (see Philidet and Gizon 2023, for more details). Once the differential rotation profile is set, the turbulent viscosity is the only adjustable parameter.

3. Results

Figure 1 highlights how the shape of the inertial mode power spectrum predicted by our model, and as it would be observed at the equator, depends on azimuthal order m. Naturally, the region of excess power (black regions) align with the linear eigenfrequencies of the left-hand side of the wave equation. A striking feature, however, is that the spectrum seems to undergo a qualitative transition at around $m \sim 5$. Below, the regions of excess power are very thin, with clearly resolved and identifiable modes, especially in

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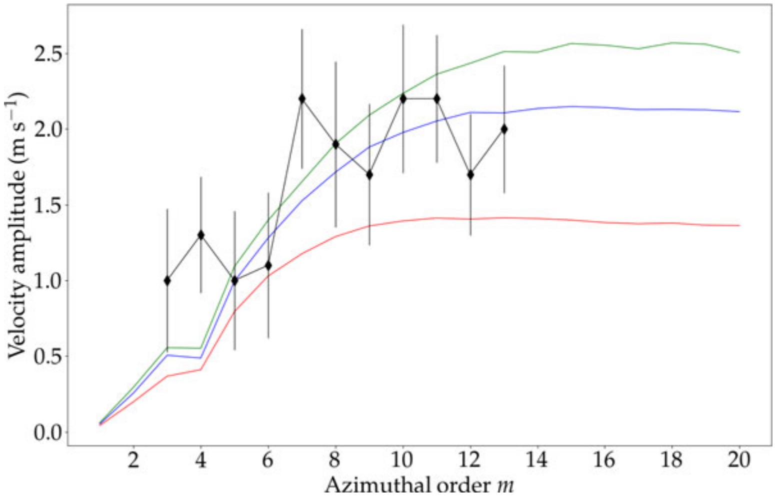

Figure 2. Prediction for the Rossby mode amplitudes (coloured solid lines) compared to observations (black line with error bars), as a function of azimuthal order m. The colour code refers to the value of the turbulent Reynolds number: $\mathrm{Re}_{\mathrm{turb}} = 300$ (red), 700 (blue), and 1000 (green).

the latitudinal velocity spectrum, which is where the sectoral Rossby modes will be most visible. Above $m \sim 5$, the regions of excess power start blending with each other, forming a continuum of overlapping modes where modes cannot be resolved and identification proves more delicate.

From these equatorial power spectra for each m, we can predict the amplitude of the Rossby mode, like so

$$A = \left(\int_{\omega_a}^{\omega_b} \mathrm{d}\omega \; P(\omega) \right)^{1/2} , \tag{9}$$

where the boundaries ω_a and ω_b enclose the whole peak in the power spectrum. We show the result in Fig. 2. The solid lines represent our model predictions, with three different values of the turbulent viscosity ν_{turb}, or alternatively of the turbulent Reynolds number $\mathrm{Re}_{\mathrm{turb}} \equiv \overline{U} R / \nu_{\mathrm{turb}}$ ($\overline{U} = 244$ m/s being the velocity due to differential rotation at the equator, and R the solar radius). Also shown in the plot are the observed solar Rossby modes amplitudes, in black. There is not only qualitative, but also quantitative agreement between the two, with Rossby mode amplitudes starting out at a fraction of m/s for very low order modes, then increasing with m, until they reach a plateau at around $m \sim 10$, with modes reaching an amplitude between 1.5 and 2.5 m/s depending on $\mathrm{Re}_{\mathrm{turb}}$.

4. Discussion and conclusion

The general qualitative and quantitative agreement between the Rossby mode amplitudes predicted by our model and the solar observations is striking, and supports the hypothesis that the linearly-stable solar inertial modes are indeed stochastically excited by turbulent convection. The mechanism is similar to the one traditionally invoked to explain the observed the observed amplitudes of acoustic modes in low-mass stars (Goldreich and Keeley 1977; Samadi and Goupil 2001), where the small-scale turbulent motions in the convective zone exert mechanical work on (and therefore inject energy into) the modes. A key difference is that while p-modes are mainly excited by vertical turbulent motions, inertial modes are much more sensitive to toroidal turbulent motions.

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The qualitative transition that we find at $m \sim 5$ between clearly resolved structures for low orders and overlapping modes for higher orders is an indication that the interpretation of the data in the latter case may be much more complex than in the former. In particular, it would suggest that in order to exploit the full probing potential of inertial modes, it is necessary to model not only their individual, discrete eigenfrequencies (as well as their respective eigenfunctions), but also to model the whole shape of the power spectral density, a task for which linear analysis is not sufficient.

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