Bull. Aust. Math. Soc. **84** (2011), 255–260
doi:10.1017/S0004972711002243

WEAKLY UNIFORM RANK TWO VECTOR BUNDLES ON MULTIPROJECTIVE SPACES

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(Received 13 November 2010)

Abstract

Here we classify the weakly uniform rank two vector bundles on multiprojective spaces. Moreover, we show that every rank $r > 2$ weakly uniform vector bundle with splitting type $a_{1,1} = \cdots = a_{r,s} = 0$ is trivial and every rank $r > 2$ uniform vector bundle with splitting type $a_1 > \cdots > a_r$ splits.

2010 *Mathematics subject classification*: primary 14F05; secondary 14J60.

Keywords and phrases: uniform vector bundles, splitting type, multiprojective spaces.

1. Introduction

We denote by $\mathbb{P}^n$ the n-dimensional projective space over an algebraic field of characteristic zero. A rank r vector bundle E on $\mathbb{P}^n$ is said to be uniform if there is a sequence of integers $(a_1, \ldots, a_r)$ with $a_1 \geq \cdots \geq a_r$ such that for every line L on $\mathbb{P}^n$, $E_{|L} \cong \bigoplus_{i=1}^r \mathcal{O}(a_i)$. The sequence $(a_1, \ldots, a_r)$ is called the splitting type of E.

The classification of these bundles is known in many cases: rank $E \leq n$ for $n \geq 2$ (see [5, 9, 11]); rank $E = n + 1$ for $n = 2$ and $n = 3$ (see [4, 6]); rank $E = 5$ for $n = 3$ (see [1]). Nevertheless, there are uniform vector bundles (of rank $2n$) which are not homogeneous (see [3]).

In [2] the authors gave the notion of weakly uniform bundles on $\mathbb{P}^1 \times \mathbb{P}^1$. For the study of rank two weakly uniform vector bundles on $(\mathbb{P}^1)^s$, see [2, 7, 10].

Here we are interested in vector bundles on multiprojective spaces. Fix integers $s \geq 2$ and $n_i \geq 1$. Let $X := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$ be a multiprojective space. Let

$$u_i : X \to \mathbb{P}^{n_i}$$

be the projection on the ith factor. For all $1 < i < j$ let

$$u_{ij} : X \to \mathbb{P}^{n_i} \times \mathbb{P}^{n_j}$$

denote the projection onto the product of the ith factor and the jth factor. Set $\mathcal{O} := \mathcal{O}_X$. For all integers $b_1, \ldots, b_s$ set $\mathcal{O}(b_1, \ldots, b_s) := \bigotimes_{i=1}^s u_i^*(\mathcal{O}_{\mathbb{P}^{n_i}}(b_i))$. We recall that every line bundle on X is isomorphic to a unique line bundle $\mathcal{O}(b_1, \ldots, b_s)$. Set $X_i := \prod_{j \neq i} \mathbb{P}^{n_j}$. Let

$$\pi_i : X \to X_i$$

be the projection. Hence, $\pi_i^{-1}(P) \cong \mathbb{P}^{n_i}$ for each $P \in X_i$. Let E be a rank r vector bundle on X. We say that E is *weakly uniform* with splitting type $(a_{h,i})$, $1 \le h \le r$, $1 \le i \le s$, if for all $i \in \{1, \ldots, s\}$, every $P \in X_i$ and every line $D \subseteq \pi_i^{-1}(P)$ the vector bundle $E|D$ on $D \cong \mathbb{P}^1$ has splitting type $a_{1,i} \ge \cdots \ge a_{r,i}$. A weakly uniform vector bundle E on X is called *uniform* if there is a line bundle $(a_1, \ldots, a_s)$ such that the splitting types of $E(a_1, \ldots, a_s)$ with respect to all π_i are the same. In this case a splitting type of E is the splitting type $c_1 \ge \cdots \ge c_r$, $r := \operatorname{rank}(E)$, of $E(a_1, \ldots, a_s)$. Notice that the r-tuple $(c_1, \ldots, c_r)$ of integers is not uniquely determined by E, but that the $(s-1)$-tuple $(c_1 - c_2, \ldots, c_{s-1} - c_s)$ depends only on E. Indeed, a rank r weakly uniform vector bundle E of splitting type $(a_{h,i})$, $1 \le h \le r$, $1 \le i \le s$, is uniform if and only if there are $s-1$ integers d_j, $2 \le j \le s$, such that $a_{h,i} = a_{h,1} + d_i$ for all $i \in \{2, \ldots, s\}$. If E is uniform, then the r-tuples $(a_{1,1} + y, \ldots, a_{r,1} + y)$, $y \in \mathbb{Z}$, are exactly the splitting types of E. If E is uniform, it is usually better to consider $E(0, a_{1,2} - a_{1,1}, \ldots, a_{1,s} - a_{1,1})$ instead of E, because all the splitting types of $E(0, a_{1,2} - a_{1,1}, \ldots, a_{1,s} - a_{1,1})$ as a weakly uniform vector bundle are the same.

In this paper we prove the following result.

THEOREM 1.1. *Let E be a rank two vector bundle on X. Then E is weakly uniform if and only if there are $L \in \operatorname{Pic}(X)$, indices $1 \le i < j \le s$ and a rank two weakly uniform vector bundle G on $\mathbb{P}^{n_i} \times \mathbb{P}^{n_j}$ such that $E \otimes L \cong u_{ij}^*(G)$. The vector bundle E splits if either $n_i \ge 3$ or $n_j \ge 3$. If $1 \le n_1 \le 2$, $1 \le n_2 \le 2$ and $(n_1, n_2) \ne (1, 1)$, then E splits unless there is $h \in \{1, 2\}$ such that $n_h = 2$ and $E \otimes L \cong u_h^*(T\mathbb{P}^2)$ for some $L \in \operatorname{Pic}(X)$.*

Moreover, we discuss the case of higher rank. We show that every rank $r > 2$ weakly uniform vector bundle with splitting type $a_{1,1} = \cdots = a_{r,s} = 0$ is trivial and every rank $r > 2$ uniform vector bundle with splitting type $a_1 > \cdots > a_r$ splits. Our methods do not allow us to attack other splitting types.

2. Weakly uniform rank two vector bundles

In order to prove Theorem 1.1 we need a few lemmas. We first consider the case $s = 2$.

LEMMA 2.1. *Assume $s = 2$, $n_1 = 1$ and $n_2 = 2$. Let E be a rank two vector bundle on $\mathbb{P}^1 \times \mathbb{P}^2$. The vector bundle E is weakly uniform if and only if either E splits as the direct sum of two line bundles or there is a line bundle L on $\mathbb{P}^1 \times \mathbb{P}^2$ such that $E \cong L \otimes \pi_2^*(T\mathbb{P}^2)$.*

PROOF. Since the 'if' part is obvious, it is sufficient to prove the 'only if' part. Let $(a_{h,i})$, $1 \le h \le 2$, $1 \le i \le s$, be the splitting type of E. Up to a twist by a line bundle we may assume $a_{1,1} = a_{1,2} = 0$. By rigidity or looking at the Chern classes $c_i(E|\{Q\} \times \mathbb{P}^2)$, $i = 1, 2$, it is easy to see that if one of these two cases occurs for some Q, then it occurs for all Q. First assume $a_{2,2} = 0$. Since the trivial line bundle on $\mathbb{P}^1$ is spanned, the theorem of changing basis implies that $F := \pi_{2*}(E)$ is a rank two vector bundle on $\mathbb{P}^2$ and that the natural map $\pi_2^*(F) \to E$ is an isomorphism [8, p. 11]. Since E is weakly uniform, F is uniform. The classification of all rank two uniform vector bundles on $\mathbb{P}^2$ shows that either F splits or it is isomorphic to a twist of $T\mathbb{P}^2$ (see [5]), concluding the proof in the case $a_{2,2} = 0$. Similarly, if $a_{2,1} = 0$, there is a rank two vector bundle G on $\mathbb{P}^1$ such that $\pi_1^*(G) \cong E$. Since every vector bundle on $\mathbb{P}^1$ splits, we have that E splits also. Now we may assume $a_{2,2} < 0$ and $a_{2,1} < 0$. Since $a_{2,2} < 0$, the base-change theorem gives that $\pi_{2*}(E)$ is a line bundle, say of degree b_2, and that the natural map $\pi_2^*\pi_{2*}(E) \to E$ has locally free cokernel [8, p. 11]. Thus, in this case E fits in an exact sequence

$$0 \to \mathcal{O}(0, b_2) \to E \to \mathcal{O}(a_{2,1}, -b_2 - a_{2,2}) \to 0. \tag{2.1}$$

The term $a_{2,1}$ in the last line bundle of (2.1) comes from $c_1(E)$. If (2.1) splits, then we are done. Since $a_{2,1} \leq 1$, Künneth's formula gives $H^1(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}(-a_{2,1}, 2b_2 + a_{2,2})) = 0$. Hence (2.1) splits. □

LEMMA 2.2. *Assume $s = 2$, $n_1 = 1$ and $n_2 \geq 3$. Then every rank two weakly uniform vector bundle on X is the direct sum of two line bundles.*

PROOF. We copy the proof of Lemma 2.1. Every rank two uniform vector bundle on $\mathbb{P}^m$, $m \geq 3$, splits. Hence E splits even in the case $a_{2,2} = 0$. □

LEMMA 2.3. *Assume $s = 2$ and $n_1 = n_2 = 2$. Let E be a rank two indecomposable weakly uniform vector bundle on X. Then either $E \cong u_1^*(T\mathbb{P}^2)(u, v)$ or $E \cong u_2^*(T\mathbb{P}^2)(u, v)$.*

PROOF. Let $(a_{h,i})$ be the splitting type of E. Up to a twist by a line bundle we may assume $a_{1,1} = a_{1,2} = 0$. As in the proof of Lemma 2.1, the theorem of changing basis gives that either $E \cong u_1^*(T\mathbb{P}^2(-2))$ or E splits if $a_{2,1} = 0$ and that $E \cong u_2^*(T\mathbb{P}^2(-2))$ or E splits if $a_{2,2} = 0$. If $a_{2,1} < 0$ and $a_{2,2} < 0$, then we apply π_{2*} and get an exact sequence (2.1). Here Künneth's formula gives that (2.1) splits, without using any information on the integer $a_{2,2}$. □

LEMMA 2.4. *Assume $s = 2$, $n_1 \geq 3$ and $n_2 = 2$. Let E be a rank two weakly uniform vector bundle on X. Then either E splits or $E \cong u_2^*(T\mathbb{P}^2)(u, v)$ for some integers u, v.*

PROOF. Let (a_{hi}) be the splitting type of E. Up to a twist by a line bundle we may assume $a_{1,1} = a_{1,2} = 0$. As in the proof of Lemma 2.1, the theorem of changing basis gives that $E \cong u_1^*(T\mathbb{P}^2(-2))$ or E splits if $a_{2,1} = 0$ and that E splits in the case $a_{1,2} < 0$, because (2.1) splits by Künneth's formula. □

LEMMA 2.5. *Assume $s = 2$, $n_1 \geq 3$ and $n_2 \geq 3$. Let E be a rank two weakly uniform vector bundle on X. Then E splits.*

https://doi.org/10.1017/S0004972711002243 Published online by Cambridge University Press

PROOF. Let (a_{hi}) be the splitting type of E. Up to a twist by a line bundle we may assume $a_{1,1} = a_{1,2} = 0$. If $a_{2,2} = 0$, then base change gives $E \cong u_2^*(F)$ for some uniform vector bundle on $\mathbb{P}^2$. Thus, we may assume $a_{2,2} < 0$. We have again the extension (2.1). Here again (2.1) splits by Künneth's formula. □

Now we are ready to prove the main theorem.

PROOF OF THEOREM 1.1. First assume $s = 2$. Theorem 1.1 says nothing in the case $n_1 = n_2 = 1$ for which a full classification is not known ([2] shows that moduli arise). Lemmas 2.1, 2.2, 2.3, 2.4 and 2.5 cover all cases with $s = 2$. Hence we may assume $s \geq 3$ and use induction on s. If $n_i = 1$ for all i, then we may apply [2, Theorem 4]. For arbitrary n_i the proof of [2, Theorem 4] works verbatim, but for the reader's sake we repeat that proof. Let (a_{hi}) be the splitting type of E. Up to a twist by a line bundle we may assume $a_{1i} = 0$ for all i. If $a_{2i} = 0$ for some i, then the base-change theorem gives $E \cong \pi_i^*(F)$ for some weakly uniform vector bundle F on X_i. If $s = 3$, then we are done. In the general case we reduce to the case $s' := s - 1$. Thus, to complete the proof it is sufficient either to obtain a contradiction or to get that E splits under the additional condition that $a_{2i} < 0$ for all i and $s \geq 3$. Applying the base-change theorem to π_{1*} we get that E fits in the following extension:

$$0 \to \mathcal{O}(0, c_2, \ldots, c_s) \to E \to \mathcal{O}(a_{1,2}, d_2, \ldots, d_s) \to 0. \tag{2.2}$$

Since $-a_{1,2} \geq 0$, Künneth's formula shows that (2.2) splits unless $n_i = 1$ for all $i \geq 2$. Using π_{2*} instead of π_{1*} we get that E splits unless $n_1 = 1$. □

3. Higher rank weakly uniform vector bundles

Now we consider higher rank weakly uniform vector bundles.

PROPOSITION 3.1. *Let E be a rank r weakly uniform vector bundle on X with splitting type $(0, \ldots, 0)$. Then E is trivial.*

PROOF. The case $s = 1$ is true by [8, Theorem 3.2.1]. Hence we may assume $s \geq 2$ and use induction on s. By the inductive assumption, $E|\pi_1^{-1}(P)$ is trivial for each $P \in \mathbb{P}^{n_1}$. By the base-change theorem, $F := \pi_{1*}(E)$ is a rank r vector bundle on X_1 and the natural map $\pi_1^*(F) \to E$ is an isomorphism. This isomorphism implies that F is uniform of splitting type $(0, \ldots, 0)$. Hence, the inductive assumption gives that F is trivial. Thus E is trivial. □

In order to study uniform vector bundles with $a_1 > \cdots > a_r$ we need the following lemmas.

LEMMA 3.2. *Fix an integer $r \geq 2$ and a rank r vector bundle on X. Assume the existence of an integer $i \in \{1, \ldots, s\}$ such that $E|\pi_i^{-1}(P)$ is the direct sum of line bundles for all $P \in X_i$. If $n_i = 1$ assume that the splitting type of $E|\pi_i^{-1}(P)$ is the same for all $P \in X_i$. Let $(a_1, \ldots, a_r) = (b_1^{m_1}, \ldots, b_k^{m_k})$, $b_1 > \cdots > b_k$, $m_1 + \cdots + m_k = r$, be the splitting type of $E|\pi^{-1}(P)$ for any $P \in X_i$. Then there are k vector*

https://doi.org/10.1017/S0004972711002243 Published online by Cambridge University Press

bundles $F_1, \ldots, F_k$ *on* X_i *and* k *vector bundles* $E_1, \ldots, E_k$ *on* X *such that* $\mathrm{rank}(F_i) = m_i$, $E_k = E$, E_{i-1} *is a subbundle of* E_i *and* $E_i/E_{i-1} \cong \pi_i^*(F_i)(-b_i)$ *(with the convention* $E_0 = 0$*).*

PROOF. Notice that even in the case $n_i \geq 2$ the splitting type of $E|\pi^{-1}(P)$ does not depend on the choice of $P \in X_i$ (for example, use Chern classes or local rigidity of direct sums of line bundles). Thus, $E|\pi_i^{-1}(P) \cong \bigoplus_{j=1}^k \mathcal{O}_{\pi_i^{-1}(P)}(b_j)^{\oplus m_j}$ for all $P \in X_i$.

Set $F_1 := \pi_{i*}(E(0, \ldots, -b_1, \ldots, 0))$. By the base-change theorem, F_1 is a rank m_1 vector bundle on X_i and the natural map $\rho : \pi_i^*(F_1)(0, \ldots, b_1, \ldots) \to E$ is a vector bundle embedding, that is, either ρ is an isomorphism (case $r = m_1$) or $\mathrm{Coker}(\rho)$ is a rank $r - m_1$ vector bundle on X. If $m_1 = r$, then $k = 1$ and we are done. Now assume $k \geq 2$, that is, $m_1 < r$. Fix any $P \in X_i$. By definition, $\mathrm{Coker}(\rho)$ fits in an exact sequence of vector bundles on X:

$$0 \to \pi_i^*(F_1)(0, \ldots, b_1, \ldots, 0) \to E \to \mathrm{Coker}(\rho) \to 0 \tag{3.1}$$

and the restriction to $\pi_i^{-1}(P)$ of the injective map of (3.1) induces an embedding of vector bundles $j_P : \mathcal{O}_{\pi_i^{-1}(P)}(b_1)^{\oplus m_1} \to \bigoplus_{j=1}^k \mathcal{O}_{\pi_i^{-1}(P)}(b_j)^{\oplus m_j}$. Since $b_1 > b_j$ for all $j > 1$, we get

$$\mathrm{Coker}(j_P) \cong \bigoplus_{j=2}^{k} \mathcal{O}_{\pi_i^{-1}(P)}(b_j)^{\oplus m_j}.$$

We apply to $\mathrm{Coker}(\rho)$ the inductive assumption on k. □

LEMMA 3.3. *Assume* $s = 2$ *and* $n_1 \geq 2$, $n_2 \geq 3$. *Fix an integer* r *such that* $3 \leq r \leq n_2$ *and a rank* r *uniform vector bundle* E *with splitting type* $a_1 > \cdots > a_r$. *Then* E *is isomorphic to a direct sum of* r *line bundles.*

PROOF. Since $r \geq 3$, we have $a_r \leq a_1 - 2$. Thus, the classification of uniform vector bundles on $\mathbb{P}^{n_2}$ with rank $r \leq n_2$ gives $E|\pi_1^{-1}(P) \cong \bigoplus_{i=1}^r \mathcal{O}_{\pi_1^{-1}(P)}(a_i)$ for all $P \in \mathbb{P}^{n_1}$. Apply Lemma 3.2 with respect to the integers $i = 1$ and $k = r$ and let F_i, E_i, $1 \leq i \leq r$, be the vector bundles given by the lemma. Since $E_r = E$, it is sufficient to prove that each E_i is a direct sum of i line bundles. Since $\mathrm{rank}(E_i) = i$, the latter assertion is obvious if $i = 1$. Fix an integer i such that $1 \leq i < r$ and assume that E_i is isomorphic to a direct sum of i line bundles. Lemma 3.2 gives an extension

$$0 \to E_i \to E_{i+1} \to L \to 0$$

with L a line bundle on $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$. Since $n_1 \geq 2$ and $n_2 \geq 2$, Künneth's formula gives that any extension of two line bundles on $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ splits. Thus E_{i+1} is a direct sum of $i + 1$ line bundles. □

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PROPOSITION 3.4. *Fix an integer $r \geq 3$ and a rank r uniform vector bundle on X with splitting type $a_1 > \cdots > a_r$. Assume $s \geq 2$, $n_2 \geq r$ and $n_i \geq 2$ for all $i \neq 2$. Then E is isomorphic to a direct sum of r line bundles.*

PROOF. The case $s = 2$ is Lemma 3.3. Thus we may assume $s \geq 3$ and that the proposition is true for $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{s-1}}$. By the inductive assumption, $E|u_s^{-1}(P) \cong \bigoplus_{i=1}^r \mathcal{O}_{u_s^{-1}(P)}(a_i, \ldots, a_i)$ for all $P \in \mathbb{P}^{n_s}$. As in the proof of Lemma 3.2, taking instead of π_i the projection $u_i : X \to \mathbb{P}^{n_i}$, we get line bundles L_i, $1 \leq i \leq r$, of $\mathbb{P}^{n_s}$ (that is, line bundles $u_i^*(L) \cong \mathcal{O}(0, \ldots, 0, c_i, 0, \ldots, 0)$ on X) and subbundles $E_1 \subset E_2 \subset \cdots E_r = E$ such that $E_i/E_{i-1} \cong \mathcal{O}_X(a_{i-1}, \ldots, a_{i-1}, c_i)$ (with the convention $E_0 = 0$). It is sufficient to prove that each E_i is isomorphic to a direct sum of i line bundles. Since this is obvious for $i = 1$, we may use induction on i. Fix an integer $i \in \{2, \ldots, r\}$. Our assumption on X implies that the extension of any two line bundles splits. Hence, $E_i \cong E_{i-1} \oplus \mathcal{O}_X(a_{i-1}, \ldots, a_{i-1}, c_i)$. □

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