Hardy's inequality for averages

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The basic inequality

The prolific output of G. H. Hardy included a number of inequalities, each known, in its own context, simply as 'Hardy's inequality'. Here we give an account of one of them, together with some applications and generalisations. It relates to averages.

We introduce some notation that enables a neat statement of the results. A sequence (x_n), finite or infinite, will be denoted simply by x. Given x, let y be the sequence of its averages:

$$y_n = \tfrac{1}{n}\left(x_1 + x_2 + \ldots + x_n\right). \tag{1}$$

This can be seen as a matrix transformation: $y = Cx$, where C is the (infinite) lower-triangular 'Cesàro matrix' defined by:

$$c_{n,k} = \begin{cases} \frac{1}{n} & \text{for } k \leqslant n, \\ 0 & \text{for } k > n. \end{cases}$$

Observe next that convergence of $\sum_{n=1}^{\infty} x_n$ does not imply convergence of $\sum_{n=1}^{\infty} y_n$. To see this, we only need to take $x = (1, 0, 0, \ldots)$: then $y_n = \frac{1}{n}$ for each n.

The situation is different if we consider $\sum_n x_n^2$. To discuss this, we use the notation $\|x\| = \sqrt{\sum_n x_n^2}$. For finite sequences, this is simply the length of x, regarded as a vector. More generally, it is called the 'Euclidean norm' of x. We denote by ℓ_2 the set of infinite sequences x for which $\|x\|$ is finite (equivalently, $\sum_n x_n^2$ is convergent), and by ℓ_2^N the set of finite sequences of length N, equipped with this norm. For now, the only property we need is the triangle inequality $\|x + y\| \leqslant \|x\| + \|y\|$.

Now let x be given and y defined by (1). Note that in the special case where (x_n) is decreasing and positive, then $y_n \geqslant x_n$ for all n, so both $\|x\|$ and $\|y - x\|$ are not greater than $\|y\|$. This sets the scene for our main result, no longer restricted to decreasing sequences.

Theorem 1: Let x be an element of ℓ_2^N or ℓ_2. Then

$$\|Cx - x\| \leqslant \|x\|, \tag{2}$$

$$\|Cx\| \leqslant 2\|x\|. \tag{3}$$

Written out fully, with $Cx = y$, the two statements are

$$\sum_n (y_n - x_n)^2 \leqslant \sum_n x_n^2 \qquad \text{and} \qquad \sum_n y_n^2 \leqslant 4 \sum_n x_n^2.$$

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Note that (3) follows at once from (2), since $\|y\| \leqslant \|y - x\| + \|x\|$. Actually, Hardy stated (3) but not (2), and (2) remains distinctly less well known. In fact, Hardy proved a more general statement than (3); we return to this below.

We now give the proof, which the author finds rather elegant. In essence, it follows the original one in [1, pp. 239-241].

Proof of Theorem 1: We prove (2) for x and y in ℓ_2^N. The statement for infinite sequences then follows on taking limits as $N \to \infty$. For $n \geqslant 2$, we have $x_n = ny_n - (n-1)y_{n-1}$, which we can rewrite as

$$x_n - y_n = (n-1)\left(y_n - y_{n-1}\right).$$

This is also true for $n = 1$ if we define y_0 to be 0. For any a, b, we have $2a(a-b) \geqslant a^2 - b^2$ (this equates to $a^2 + b^2 \geqslant 2ab$). So $2y_n\left(y_n - y_{n-1}\right) \geqslant y_n^2 - y_{n-1}^2$, hence

$$2y_n\left(x_n - y_n\right) = 2(n-1)y_n\left(y_n - y_{n-1}\right) \geqslant (n-1)\left(y_n^2 - y_{n-1}^2\right),$$

equivalently

$$2x_ny_n - y_n^2 \geqslant ny_n^2 - (n-1)y_{n-1}^2.$$

Add these inequalities for $1 \leqslant n \leqslant N$: by cancellation, we obtain

$$2\sum_{n=1}^{N} x_ny_n - \sum_{n=1}^{N} y_n^2 \geqslant Ny_N^2.$$

We actually only use the fact that this is non-negative, so that

$$\sum_{n=1}^{N} y_n^2 \leqslant 2\sum_{n=1}^{N} x_ny_n. \tag{4}$$

Hence

$$\sum_{n=1}^{N} (y_n - x_n)^2 = \sum_{n=1}^{N} \left(y_n^2 - 2x_ny_n + x_n^2\right) \leqslant \sum_{n=1}^{N} x_n^2.$$

Another way to state Theorem 1 is in terms of norms of matrices. The norm of a matrix A, again denoted by $\|A\|$, is defined to be the least M such that $\|Ax\| \leqslant M\|x\|$ for all x in the domain (ℓ_2^N or ℓ_2 as appropriate). So (2) equates to $\|C - I\| \leqslant 1$, where I is the identity matrix, and (3) equates to $\|C\| \leqslant 2$.

Best constants

We now show that in (2) and (3), regarded as statements for infinite sequences or for sequences of any finite length, the respective constants 1 and 2 are optimal, so that (as operators on ℓ_2), we have $\|C - I\| = 1$ and $\|C\| = 2$.

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For (2), this is easily shown. Let e_n be the sequence with 1 in place n and 0 elsewhere. Then $\|e_n\| = 1$ and component n of $Ce_n - e_n$ is $\frac{1}{n} - 1$, so $\|Ce_n - e_n\| \geqslant 1 - \frac{1}{n}$.

For (3), we again follow Hardy. Fix N, and let

$$x_n = \begin{cases} \frac{1}{\sqrt{n}} & \text{for } n \leqslant N, \\ 0 & \text{for } n > N. \end{cases}$$

Then $\sum_{n=1}^{N} x_n^2 = \sum_{n=1}^{N} \frac{1}{n}$: denote this (as usual) by H_N. By integral estimation, for $n \leqslant N$,

$$\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} \frac{1}{\sqrt{j}} > \int_1^n \frac{1}{\sqrt{t}}\,dt = 2\left(\sqrt{n} - 1\right),$$

so

$$y_n > \frac{2}{\sqrt{n}}\left(1 - \frac{1}{\sqrt{n}}\right),$$

hence

$$y_n^2 > \frac{4}{n}\left(1 - \frac{1}{\sqrt{n}}\right)^2 > \frac{4}{n}\left(1 - \frac{2}{\sqrt{n}}\right).$$

So

$$\sum_{n=1}^{N} y_n^2 > 4H_N - 8S,$$

where $S = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, and hence

$$\frac{\sum_{n=1}^{N} y_n^2}{\sum_{n=1}^{N} x_n^2} > 4 - \frac{8S}{H_N},$$

which tends to 4 as $N \to \infty$.

(For a *fixed* N, determination of the best constants in (2) and (3) is not easy, and there are no pleasant answers.)

Hardy's theorem for general p

As mentioned earlier, Hardy actually proved a more general result than (3). For any $p \geqslant 1$, define $\|x\|_p = \left(\sum_n |x_n|^p\right)^{1/p}$, so our $\|x\|$ is $\|x\|_2$. For a matrix A, define $\|A\|_p$ to be the least M such that $\|Ax\|_p \leqslant M\|x\|_p$ for all x. For $p > 1$, define the *conjugate index* p^* by $p^* = p/(p - 1)$, so that $\frac{1}{p} + \frac{1}{p^*} = 1$. Hardy's theorem is: $\|C\|_p = p^*$ for all $p > 1$. The case $p = 2$ is (3).

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We indicate briefly how the proof of Theorem 1 can be modified to prove this. It is enough to consider non-negative x_n. The inequality $a^2 - b^2 \leqslant 2a(b - a)$ is replaced by $a^p - b^p \leqslant pa^{p-1}(a - b)$ for positive a, b: this is a straightforward consequence of the mean-value theorem. Following the previous steps, we then find that (4) becomes

$$(p - 1)\sum_{n=1}^{N} y_n^p \leqslant p\sum_{n=1}^{N} x_n y_n^{p-1}.$$

Nothing like (2) can be deduced from this. In fact, the evaluation of $\|C - I\|_p$ for other p is quite tricky, and has only recently been achieved in [2]. However, a neat application of Hölder's inequality now gives

$$(p - 1)\sum_{n=1}^{N} y_n^p \leqslant p\left(\sum_{n=1}^{N} x_n^p\right)^{1/p}\left(\sum_{n=1}^{N} y_n^p\right)^{1/p^*},$$

hence $\|y\|_p \leqslant p^* \|x\|_p$.

For most purposes, the case $p = 2$ is quite enough, but later on, in Theorem 3, we will see an application where the statement for general p leads to a better result.

The *continuous* case. A 'continuous' version of Hardy's theorem applies to functions instead of sequences. For a function $x(t)$ on $(0, \infty)$, let $\|x\|_p = \left(\int_0^\infty |x(t)|^p\,dt\right)^{1/p}$ if this is finite. (This may require attention to convergence of $\int_\delta^X |x(t)|^p\,dt$ both as $\delta \to 0$ and as $X \to \infty$.) Let

$$y(t) = \frac{1}{t}\int_0^t x(u)\,du.$$

Then again we have $\|y\|_p \leqslant p^* \|x\|_p$. The proof [1, p. 242] is a recognisable variant of the proof for the discrete case, with finite differences replaced by integration by parts. For the case $p = 2$, it transpires that (2) actually holds with equality: $\|y - x\|_2 = \|x\|_2$ (see [3]).

The dual: Copson's inequality

Denote by A^T the transpose of a matrix A (finite or infinite). For the Cesàro matrix C, we have $C^T x = z$, where

$$z_n = \sum_{k=n}^{\infty} \frac{x_k}{k}.$$

For matrix operators on ℓ_2, it is well known that $\|A^T\| = \|A\|$ (we return to this shortly). If we assume this, then we can deduce at once from Theorem 1 that $\|C^T - I\| \leqslant 1$ and $\|C^T\| \leqslant 2$. However, a direct proof along the lines of Theorem 1 is actually slightly simpler than the proof of Theorem 1 itself, so we give it here for any readers with the appetite for it. We go straight to the infinite case.

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Theorem 2: Let x be a sequence in ℓ_2. Then $C^T x$ is in ℓ_2 and

$$\|C^T x - x\| \leqslant \|x\|, \qquad (5)$$

$$\|C^T x\| \leqslant 2\|x\|. \qquad (6)$$

Proof: Let $C^T x = z$. Note first that, by the Cauchy–Schwarz inequality,

$$z_{n+1}^2 \leqslant \left(\sum_{k=n+1}^{\infty} \frac{1}{k^2}\right)\left(\sum_{k=n+1}^{\infty} x_k^2\right) \leqslant \frac{1}{n} \sum_{k=n+1}^{\infty} x_k^2, \qquad (7)$$

since

$$\sum_{k=n+1}^{\infty} \frac{1}{k^2} < \sum_{k=n+1}^{\infty} \frac{1}{(k-1)k} = \frac{1}{n}.$$

Now $x_n = n(z_n - z_{n+1})$ and $2z_n(z_n - z_{n+1}) \geqslant z_n^2 - z_{n+1}^2$, hence

$$2x_n z_n \geqslant n(z_n^2 - z_{n+1}^2) = z_n^2 + (n-1)z_n^2 - nz_{n+1}^2.$$

Adding for $1 \leqslant n \leqslant N$, we obtain

$$2\sum_{n=1}^{N} x_n z_n \geqslant \sum_{n=1}^{N} z_n^2 - Nz_{N+1}^2.$$

With (7), we deduce

$$\sum_{n=1}^{N} (z_n - x_n)^2 \leqslant \sum_{n=1}^{N} x_n^2 + Nz_{N+1}^2 \leqslant \sum_{n=1}^{\infty} x_n^2.$$

This applies for all N, so (5) follows.

The statement for general $p \geqslant 1$ is $\|C^T x\|_p \leqslant p\|x\|_p$. Furthermore, for non-negative x_n, the reverse inequality applies when $0 < p < 1$. These two statements together comprise Copson's inequality. Of course, they require a little more work, except for the case $p = 1$, which is very easy: in this case, we have the pleasing identity

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \frac{x_k}{k} = \sum_{n=1}^{\infty} \frac{x_k}{k} \sum_{n=1}^{k} 1 = \sum_{k=1}^{\infty} x_k.$$

A slick alternative proof, and some equalities underlying the inequalities

For readers familiar with basic Hilbert space theory, we outline a very slick alternative proof of Theorems 1 and 2 which was given in [4] (other readers can move on to the next section). In fact, the method delivers a stronger statement. Very briefly, the facts needed are as follows. We use the notation $\langle x, y\rangle$ for the inner product $\sum_{n=1}^{\infty} x_n y_n$. Then $\langle x, x\rangle = \|x\|^2$, and the Cauchy–Schwarz inequality says $|\langle x, y\rangle| \leqslant \|x\| \cdot \|y\|$. Consequently, $\|x\| = \sup\{|\langle x, y\rangle| : \|y\| = 1\}$, and for a matrix A, we have

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$\|A\| = \sup\{|\langle Ax, y\rangle| : \|x\| = \|y\| = 1\}$. The transpose A^T satisfies $\langle Ax, y\rangle = \langle x, A^T y\rangle$, hence $\|A^T\| = \|A\|$. Further,

$$\langle AA^T x, x\rangle \leqslant \langle A^T x, A^T x\rangle = \|A^T x\|^2.$$

Since $\langle AA^T x, x\rangle \leqslant \|AA^T\| \cdot \|x\|^2$ it follows that $\|AA^T\| = \|A^T\|^2 = \|A\|^2$.

For the Cesàro matrix C, it is quite easy to verify that CC^T is the matrix having $1/\max(j, k)$ in place (j, k), and hence that $C + C^T = CC^T + D$, where D is the diagonal matrix with entries $\left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$. So

$$(C - I)\left(C^T - 1\right) = CC^T - C - C^T + I = I - D. \tag{8}$$

Now $I - D$ is also diagonal, with entries $1 - \frac{1}{n}$, so $\|I - D\| = 1$. Hence $\|C - I\| = \|C^T - I\| = 1$. However, (8) actually tells us more than this: it gives

$$\|C^T x - x\|^2 = \langle (I - D)x, x\rangle = \sum_{n=1}^{\infty}\left(1 - \frac{1}{n}\right)x_n^2. \tag{9}$$

This is an *equality* that clearly implies the inequality (5).

Rather similar reasoning delivers a neat relation between $\|Cx\|$ and $\|C^T x\|$ (as we have seen, both are bounded by $2\|x\|$, so a direct comparison between them is of interest). Again write $Cx = y$ and $C^T x = z$. One can show that $C^T \Delta C = CC^T$, where Δ is the diagonal matrix with nth component $n/(n + 1)$. Now $\langle CC^T x, x\rangle = \|C^T x\|^2 = \sum_{n=1}^{\infty} z_n^2$, while

$$\langle C^T \Delta Cx, x\rangle = \langle \Delta Cx, Cx\rangle = \langle \Delta y, y\rangle = \sum_{n=1}^{\infty} \frac{n}{n + 1} y_n^2,$$

so

$$\sum_{n=1}^{\infty} \frac{n}{n + 1} y_n^2 = \sum_{n=1}^{\infty} z_n^2. \tag{10}$$

Since $n/(n + 1) \geqslant \frac{1}{2}$ for all $n \geqslant 1$, it follows that $\|C^T x\| \leqslant \|Cx\| \leqslant \sqrt{2}\|C^T x\|$.

There is also an equality that underlies the inequality (2): if $Cx = y$, then

$$\sum_{n=2}^{\infty} \frac{n}{n - 1}(y_n - x_n)^2 = \sum_{n=1}^{\infty} x_n^2. \tag{11}$$

The proof is not by matrix identities, and entails rather more work: it is presented in [5]. It needs to be emphasised that both (10) and (11) apply strictly to *infinite* sequences. In fact, if curtailed to $\mathbb{R}^n$, we have $Cx = x$, where $x = (1, 1, \ldots, 1)$, so $Cx - x = 0$. By contrast, in infinite dimensions, $C(1, \ldots 1, 0, \ldots)$ has continuing terms n/k for $k > n$; the reader may care to verify that (11) indeed holds in this case.

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As well as implying (2), identity (11) also implies the reverse inequality $\|Cx - x\| \geqslant \frac{1}{\sqrt{2}}\|x\|$. The factor $\frac{1}{\sqrt{2}}$ occurs for $x = (1, -1, 0, \dots)$: then $Cx - x = (0, 1, 0, \dots)$.

Carleman's inequality

Roughly speaking, Carleman's inequality replaces the arithmetic means in Hardy's inequality by geometric means, but the terms do not need to be squared. The exact statement is as follows.

Theorem 3: Let $x_n > 0$ for $1 \leqslant n \leqslant N$, and let $G_n = (x_1 x_2 \dots x_n)^{1/n}$. Then

$$\sum_{n=1}^{N} G_n \leqslant e \sum_{n=1}^{N} x_n. \tag{12}$$

First, we show how this can be deduced from Hardy's inequality. To obtain the correct constant e, we need the inequality for general p, not just $p = 2$.

Proof 1: Take $p > 1$. Let $z_n = x_n^{1/p}$, so $x_n = z_n^p$. Let $Z_n = \sum_{j=1}^{n} z_j$. By the inequality of the means, $(z_1 z_2 \dots z_n)^{1/n} \leqslant Z_n / n$, hence

$$G_n = (z_1 z_2 \dots z_n)^{p/n} \leqslant \left(\frac{Z_n}{n}\right)^p.$$

So by Hardy's inequality for general p,

$$\sum_{n=1}^{N} G_n \leqslant \sum_{n=1}^{N} \left(\frac{Z_n}{n}\right)^p \leqslant \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{N} z_n^p = \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{N} x_n.$$

Now $\left(\frac{p}{p-1}\right)^p \to e$ as $p \to \infty$, so (12) follows. (If we only considered $p = 2$, we would obtain the constant 4 instead of e.)

Again, it is possible to give quite a short self-contained proof along the lines of Theorem 1 itself, as follows.

Proof 2: We have $\sum_{j=1}^{n} \log x_j = n \log G_n$, so for $n \geqslant 2$,

$$\log x_n = n \log G_n - (n-1) \log G_{n-1} = \log G_n + (n-1)\big(\log G_n - \log G_{n-1}\big).$$

This also holds for $n = 1$, since $x_1 = G_1$. We apply the elementary inequality $\log b - \log a \geqslant \frac{b-a}{b}$, which is easily seen from the integral $\int_a^b \frac{1}{t} dt$. This gives

$$\log x_n \geqslant \log G_n + (n-1) \frac{G_n - G_{n-1}}{G_n},$$

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hence

$$G_n(\log x_n - \log G_n + 1) \geqslant nG_n - (n-1)G_{n-1}.$$

Adding these inequalities for $1 \leqslant n \leqslant N$, we obtain

$$\sum_{n=1}^{N} G_n(\log x_n - \log G_n + 1) \geqslant NG_N \geqslant 0.$$

Since $\log x \leqslant x - 1$ for $x > 0$,

$$\log x_n - \log G_n + 1 = \log \frac{ex_n}{G_n} \leqslant \frac{ex_n}{G_n} - 1,$$

so

$$0 \leqslant \sum_{n=1}^{N} G_n\left(\frac{ex_n}{G_n} - 1\right) = e\sum_{n=1}^{N} x_n - \sum_{n=1}^{N} G_n.$$

Numerous other proofs have appeared. One such can be seen in [1, pp. 249-250]: it is rather more elaborate. Given the resemblance to the proof of Hardy's inequality, it is a little surprising that nothing like proof 2 is given there.

Is e the best constant? Yes, again in the sense of a constant that applies for all N. To show this, take $x_n = \frac{1}{n}$ for $1 \leqslant n < N$, so $\sum_{n=1}^{N} x_n = H_n$ while $G_n = \frac{1}{(n!)^{1/n}}$. By integral estimation,

$$\sum_{r=1}^{n-1} \log r \leqslant \int_1^n \log x \, dx = n \log n - n + 1,$$

hence $n! \leqslant \frac{n^{n+1}}{e^{n-1}}$ and

$$G_n \geqslant \frac{1}{(en)^{1/n}} \frac{e}{n}.$$

For given $\varepsilon > 0$, this implies that $G_n \geqslant (1 - \varepsilon)\frac{e}{n}$ for large enough n. The conclusion follows in a routine way.

Some generalisations

Arguably, a test of a good theorem is that it stimulates further theorems. Hardy's inequality certainly passes this test: there is a massive literature presenting various generalisations of it. Here we describe just two of them.

A *summability matrix* is a lower-triangular matrix with non-negative entries and row sums equal to 1. Of course, the Cesàro matrix is an example. So, in a trivial way, is the identity matrix. Bennett [6, Theorem 1.14] established the following attractive generalisation of Hardy's inequality.

Suppose that $A = (a_{n,k})$ is a summability matrix. Then

(i) if $a_{n,1} \leqslant a_{n,2} \leqslant \ldots \leqslant a_{n,n}$ for each n, then $\|A\|_p \leqslant p^*$;

(ii) if $a_{n,1} \geqslant a_{n,2} \geqslant \ldots \geqslant a_{n,n}$ for each n, then $\|A\|_p \geqslant p^*$.

Now let $w = (w_n)$ be a sequence of positive numbers, and write $W_n = \sum_{k=1}^{n} w_k$. The *weighted mean* matrix A_w is the summability matrix $(a_{n,k})$ defined by

$$a_{n,k} = \begin{cases} \frac{w_k}{W_n} & \text{for } k \leqslant n, \\ 0 & \text{for } > n, \end{cases}$$

so that $y = A_w x$ equates to

$$y_n = \frac{1}{W_n} \sum_{k=1}^{n} w_k x_k.$$

The Cesàro matrix is the case $w_n = 1$ for all n.

A simple modification of the proof of Theorem 1, which can be seen in [9], gives a weighted version of Hardy's inequality: $\sum_{n=1}^{\infty} w_n y_n^p \leqslant (p^*)^p \sum_{n=1}^{\infty} w_n x_n^p$ for non-negative x_n. Another generalisation was proved by J. Cartlidge. It was published in his doctoral thesis [7], which is not easily accessible; proofs, again along the lines of Theorem 1, can be seen in [8] or [9]. The statement is as follows. Let

$$S(w) = \sup_{n \geqslant 1} \left(\frac{W_{n+1}}{w_{n+1}} - \frac{W_n}{w_n} \right).$$

If $p \geqslant 1$ and $S(w) < p$, then

$$\|A_w\|_p \leqslant \frac{p}{p - S(w)}.$$

This reproduces Hardy's inequality, because if $w_n = 1$, then $S(w) = 1$. For the case $p = 2$, our (2) can be extended as follows: if $S(w) \leqslant 1$, then $\|A_w - I\|_2 \leqslant 1$.

Example: Let $w_n = n$, so that $W_n = \frac{1}{2}n(n + 1)$ and $a_{n,k} = \dfrac{2k}{n(n + 1)}$. (This is the 'gamma matrix of order 2'.) Then $W_n / w_n = \frac{1}{2}(n + 1)$, hence $S(w) = \frac{1}{2}$ and $\|A_w\|_p \leqslant \dfrac{2p}{2p - 1}$ for all $p \geqslant 1$.

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There are corresponding generalisations of Carleman's theorem to *weighted* geometric means, proved by suitable modification of either of the earlier methods. Let

$$G_n(w) = \left(\prod_{k=1}^{n} x_k^{w_k}\right)^{1/W_n}.$$

Then for all $N \geqslant 1$,

$$\sum_{n=1}^{N} G_n(w) \leqslant e^{S(w)} \sum_{n=1}^{N} x_n.$$

Also, $\sum_{n=1}^{N} w_n G_n(w) \leqslant e \sum_{n=1}^{N} w_n x_n$. Again, see [8] or [9].

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