

THE PRESENT VALUE OF A SERIES OF CASHFLOWS: CONVERGENCE IN A RANDOM ENVIRONMENT

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ABSTRACT

The present paper considers the present value, $Z(t)$, of a series of cashflows up to some time t . More specifically, the cashflows and the interest rate process will often be stochastic and not necessarily independent of one another or through time. We discuss under what circumstances $Z(t)$ will converge almost surely to some finite value as $t \rightarrow \infty$. This problem has previously been considered by DUFRESNE (1990) who provided a sufficient condition for almost sure convergence of $Z(t)$ (the Root Test) and then proceeded to consider some specific examples of such processes. Here, we develop Dufresne's work and show that the sufficient condition for convergence can be proved to hold for quite a general class of model which includes the growing number of Office Models with stochastic cashflows.

KEYWORDS

Stochastic discounting; cashflow models; almost sure convergence; office model.

1. INTRODUCTION

Suppose $\delta(t)$ is the constant force of interest during the period $[t-1, t)$, so that an investment of 1 at time $t-1$ will accumulate to $\exp \delta(t)$ at time t . The present value at time 0 of 1 due at time t is then

$$V(t) = \exp \left(- \sum_{s=1}^t \delta(s) \right) = \prod_{s=1}^t v_s$$

where $v_s = \exp(-\delta(s))$ is the discount factor for year s .

The present value of a series of cashflows $C(1)$ at time 1, $C(2)$ at time 2, ..., $C(t)$ at time t is therefore

$$Z(t) = \sum_{s=1}^t V(s) C(s) = \sum_{s=1}^t v_1 \dots v_s C(s)$$

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Such a process has been considered by DUFRESNE (1990) and AEBI *et al.* (1994) from the financial point of view and by VERVAAT (1979) and BRANDT (1986) from the mathematical point of view. All of these works consider the special case where $\{\delta(t)\}_{t=1}^{\infty}$ and $\{C(t)\}_{t=1}^{\infty}$ are independent and identically distributed and independent of one another. DUFRESNE (1990) considers the convergence of $Z(t)$ as t tends to infinity and its limiting distribution when the distributions of $\delta(t)$ and $C(t)$ are known. AEBI *et al.* (1994) show how Bootstrap methodology can be used to estimate the limiting distribution when a limited number of past observations of $\delta(t)$ and $C(t)$ are available.

DUFRESNE (1990) also considers more general models and provides sufficient conditions for almost sure convergence of $Z(t)$. In this paper we consider a number of specific examples for the process $Z(t)$ and it is demonstrated that Dufresne's conditions hold for quite a wide class of models.

In this paper we will restrict ourselves to discrete time models. However, the results described here also hold for the continuous time models for $\delta(t)$ described by PARKER (1993, 1994d) and NORBERG and MØLLER (1994).

DE SCHEPPER, TUENEN and GOOVAERTS (1994) consider the present value of annuities and of a perpetuity payable continuously. Using Laplace transforms they show that the perpetuity has an inverse Gamma distribution, matching the results of DUFRESNE (1990).

2. CONDITIONS FOR CONVERGENCE OF $Z(t)$

The principal result provided by DUFRESNE (1990) giving a sufficient condition for the almost sure convergence of $Z(t)$ is the Root Test:

Theorem 1 (Root Test, for example, see DUFRESNE, 1990)

If

$$\limsup_{t \rightarrow \infty} |V(t)C(t)|^{1/t} < 1 \text{ almost surely}$$

then $Z(t)$ converges almost surely to some finite limit as t tends to infinity.

Now, trivially, this is equivalent to the condition

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} \left\{ \log |C(t)| + \sum_{s=1}^t \log v_s \right\} < 0 \text{ almost surely} \\ \Leftrightarrow \limsup_{t \rightarrow \infty} \frac{1}{t} \left\{ \log |C(t)| - \sum_{s=1}^t \delta(s) \right\} < 0 \text{ almost surely} \end{aligned}$$

We therefore have the following

Corollary 2

If the force of interest process $\{\delta(t)\}_{t=1}^\infty$ is ergodic with $\frac{1}{t} \sum_{s=1}^t \delta(s) \rightarrow \delta = E[\delta(\cdot)]$ almost surely, and if

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |C(t)| = \rho \in [-\infty, +\infty)$$

where $\rho - \delta < 0$ then $Z(t)$ converges almost surely to some finite limit as t tends to infinity.

[The condition that $\rho < \delta$ means that the cashflow process, $C(t)$, must grow more slowly than the accumulation process $1/V(t)$. Consider the trivial case of a perpetuity where $C(t) = \exp \rho t$ and $V(t) = \exp -\delta t$ are both deterministic. Then $Z(t)$ converges if and only if $\rho < \delta$.]

Corollary 2 extends Dufresne’s subsequent development by allowing the process $C(t)$:

- to depend on the force of interest process;
- to be non-ergodic and, in particular, to include inflationary growth and growth in the underlying number of policies.

[A rigorous definition of ergodicity is given by KARLIN and TAYLOR (1975). If a process $X(t)$ is known to be ergodic then the following results hold:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n X(t) = \mu$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n [X(t) - \bar{X}_n]^2 = \sigma^2 \text{ where } \bar{X}_n = \frac{1}{n} \sum_{t=1}^n X(t)$$

$$\text{and if } I_{a,b}(x) = \begin{cases} 1 & \text{if } a < x \leq b \\ 0 & \text{otherwise} \end{cases}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n I_{a,b}[X(t)] = F(b) - F(a)$$

where μ and σ^2 are the unconditional mean and variance of $X(t)$ and $F(x)$ is the unconditional cumulative distribution function of $X(t)$.]

3. THE INTEREST RATE PROCESS

Before we concentrate on the cashflow process, it is worth discussing briefly some interest rate processes.

The ergodic condition for the force of interest process is not particularly onerous, and encompasses most of the widely used stochastic investment models.

- Independent and identically distributed returns: for example, WATERS (1978), DUFRESNE (1990), PAPACHRISTOU and WATERS (1991), PARKER (1993, 1994d) and AEBI *et al.* (1994) give but a few examples.
- Simple autoregressive models for the rate of return, such as the $AR(1)$ time series model, and the Ornstein-Uhlenbeck process: for example, DHAENE (1989, 1992), PARKER (1993, 1994a,b,c,d) and NORBERG and MØLLER (1994).
- Models with several asset classes: for example, WILKIE (1986, 1987, 1992, 1994, 1995).
- Models for the term structure of interest rates: for example, BOYLE (1978, 1980), BRENNAN and SCHWARZ (1979), ALBRECHT (1985), COX, INGERSOLL and ROSS (1985), BEECKMAN and SHIU (1988), HEATH, JARROW and MORTON (1990, 1992), SERCU (1991) and LONGSTAFF and SCHWARZ (1992).
- Autoregressive Conditionally Heteroscedastic (ARCH) models: for example, see WILKIE (1995, Appendix D).

Some non-ergodic models may still admit convergence but it is worth discussing some special cases which may be considered to be inappropriate.

First, DUFRESNE (1990, Proposition 3.2.1) shows that $Z(t)$ will converge if $\{v_s\}$ is stationary and ergodic and $Pr(v_s = 0) > 0$. This second condition is equivalent to $Pr(\delta(s) = \infty) > 0$ which does not seem appropriate.

Second, some authors (for example, DHAENE, 1989, 1992; DUFRESNE, 1990, and PARKER, 1993) consider non-stationary models for the force of interest process. One of two things may happen.

- The process may have positive drift, so that $\delta(t)$ tends to infinity almost surely (again unrealistic).
- The process may have no drift but unbounded variance. Examples of this are random walk models of the form $\delta(t+1) = \delta(t) + \varepsilon(t+1)$ where the ε are independent and identically distributed zero mean random variables, and, more generally, $ARIMA(p, d, q)$ models ($d \geq 1$). With such processes it is often easy to show (for example, see DUFRESNE, 1990, Proposition 4.4.4) that

$$\limsup_{t \rightarrow \infty} |Z(t)| = \infty.$$

Figures 1 and 2 demonstrate the problem. For many years $\delta(t)$ may remain positive (Figure 1). The process $Z(t)$ (Figure 2) may then give one the impression that it is converging and that it is safe to use $Z(50)$ or $Z(100)$, say, as an approximation to $Z(\infty)$. After a while, though, $\delta(t)$ takes a long excursion below zero and $Z(t)$ shoots off upwards.

4. THE CASHFLOW PROCESS

Corollary 2 provided us with a condition for the behaviour of $C(t)$ as t tends to infinity. We now consider this in more detail and provide the following lemma which will allow us to satisfy the conditions in Corollary 2.

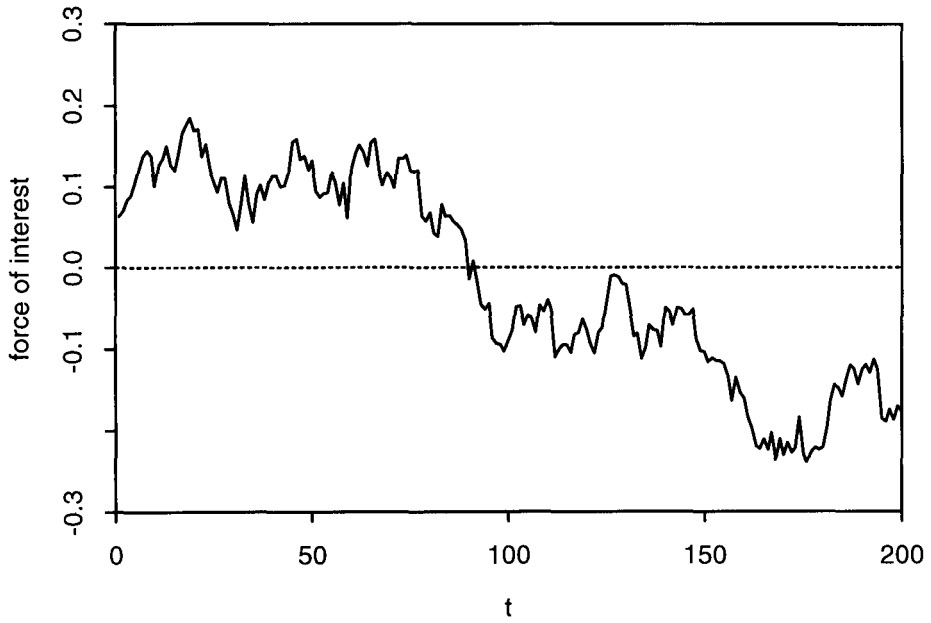


FIGURE 1. Sample path of a random walk interest rate process, $\delta(t)$.

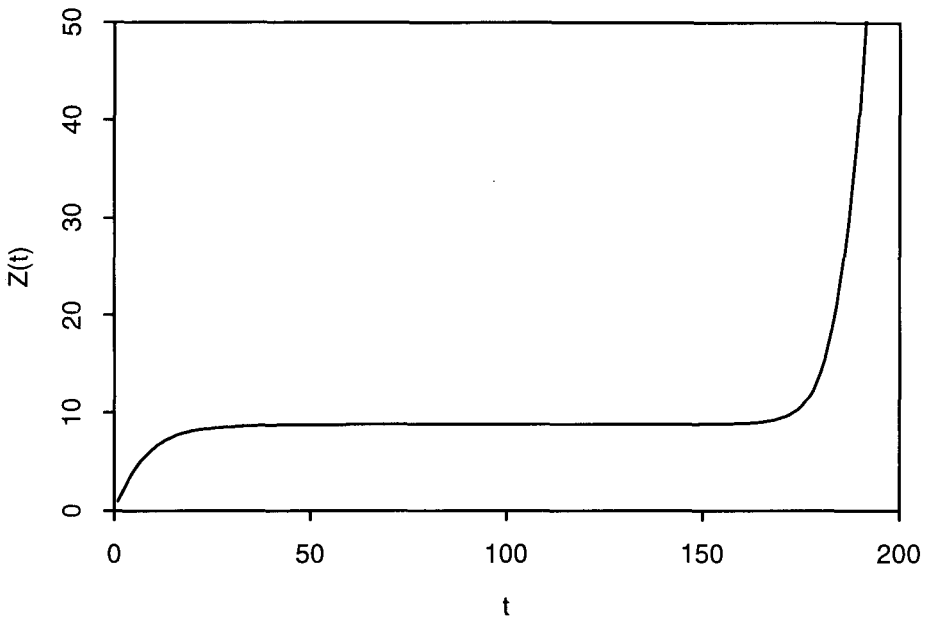


FIGURE 2. $Z(t)$ appears to converge initially but eventually diverges.

Lemma 3

Suppose that there exists a deterministic sequence $a(t)$ converging to zero as t tends to infinity such that

$$\sum_{t=1}^{\infty} \Pr \left(\frac{1}{t} \log |C(t)| \geq \rho + a(t) \right) < \infty.$$

Then

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |C(t)| \leq \rho \text{ almost surely.}$$

Proof

Let $E_t = \{t^{-1} \log |C(t)| \geq \rho + a(t)\}$. Then

$$\sum_{t=1}^{\infty} \Pr(E_t) < \infty$$

$\Rightarrow \Pr(\limsup_{t \rightarrow \infty} E_t) = 0$ by the first Borel-Cantelli Lemma

(for example, see WILLIAMS, 1991, Section 2.7)

$$\Rightarrow \limsup_{t \rightarrow \infty} \frac{1}{t} \log |C(t)| \leq \rho \text{ almost surely.}$$

Lemma 3 provides quite a weak condition on the cashflow process: cashflows need not be independent; and the tails of the cashflow distributions can be quite fat. In particular if we suppose that $\mu(t) = E |C(t)|$ then we are able to prove the following theorem which provides us with a relatively easy method for proving the convergence of $t^{-1} \log |C(t)|$.

Theorem 4

Let $\rho_0 = \inf \{ \rho : \limsup_{t \rightarrow \infty} e^{-\rho t} \mu(t) < \infty \}$. Then

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |C(t)| \leq \rho_0 \text{ almost surely.}$$

Proof See Appendix A.

This result covers many cases, some discussed previously by other authors:

- $C(t)$ independent and identically distributed (DUFRESNE, 1990; AEBI, *et al.*, 1994);
- $C(t)$ an ergodic stochastic process and hence with $\mu(t)$ constant, giving $\rho_0 = 0$ (DUFRESNE, 1990);
- closed funds (for which $\mu(t)$ tends to zero in finite time) (PAPACHRISTOU and WATERS, 1991; PARKER, 1993, 1994a,b,c,d; FREES, 1990);

— deterministic processes with exponentially bounded growth (for example, an office model with an assumed new business growth rate).

Theorem 4 gives a stronger condition for convergence than Lemma 3. For example, suppose $C(t)$ has a Log-Pareto distribution with drift: that is,

$$Pr [\log C(t) - \alpha t > x] = \frac{\text{constant}}{x^\delta} \text{ for } x > x_0 \text{ and } \delta > 0.$$

Then $E |C(t)| = \infty$ for all t so that $\{C(t)\}$ does not satisfy the conditions for Theorem 4. Nevertheless, $\{C(t)\}$ does satisfy the conditions for Lemma 3, implying that $\limsup_{t \rightarrow \infty} t^{-1} \log |C(t)|$ is still less than infinity. Theorem 4 does, however, provide us with a condition for convergence which is often easier to verify, as will be demonstrated in the next section.

5. A STOCHASTIC OFFICE MODEL

We now develop the last of these examples to include office models in which cashflows are stochastic. The office's portfolio is assumed to consist only of policies which do not participate in the profits of the company. The model described includes stochastic mortality, stochastic growth of new business volumes, stochastic inflation in the size of individual policies and conditionally independent and identically distributed policy sizes at a given time of inception.

The generality of the model, here, means that the notation may appear to be quite heavy going, but the reader should concentrate on:

- the total premium income at time t , $P(t)$;
- the total benefit outgo at time t , $B(t)$, which consists of benefits payable on death during the year $(t - 1, t]$, and on survival to duration t for $t = 1, 2, \dots$

Suppose that $\lambda = \{\lambda(t)\}_{t=-\infty}^{\infty}$, $\lambda(t) = (\lambda_1(t), \lambda_2(t))$, is the process with determines the volume of new business $(\lambda_1(t))$ and the individual policy size index $(\lambda_2(t))$.

Let $(x)_{t,j}$ represent the life corresponding to policy j taken out at time t . All policyholders are aged x at entry. Then, using standard notation, we have

$$\begin{aligned} {}_s p_x &= \text{probability that an individual now aged } x \text{ survives to age } x + s \\ &= 0 \text{ for all } s > \omega - x \text{ for some limiting age } \omega < \infty \\ \text{and } {}_{s-1} q_x &= \text{the probability that an individual now aged } x \text{ dies between ages} \\ &\quad x + s - 1 \text{ and } x + s \\ &= 0 \text{ for all } s > \omega - x \end{aligned}$$

First we consider the total premium income at time t . This is

$$\begin{aligned} P(t) &= \sum_{s=0}^{\infty} \sum_{j=1}^{N(t-s)} K_s^P A_{t-s}(j) I_{t-s,s}^S(j) \\ &= P_1(t) + P_2(t) \end{aligned}$$

$$\begin{aligned} \text{where } P_1(t) &= \sum_{s=t}^{\infty} \sum_{j=1}^{N(t-s)} K_s^P A_{t-s}(j) I_{t-s,s}^S(j) \\ &= \text{premium income in respect of policies issued at or before} \\ &\quad \text{time } 0 \end{aligned}$$

- $P_2(t) = \sum_{s=0}^{t-1} \sum_{j=1}^{N(t-s)} K_s^P A_{t-s}(j) I_{t-s,s}^S(j)$
 = premium income in respect of policies issued after time 0
 $N(s)$ = number of new policies taken out at time s
 $N(s) \mid \lambda_1(s) \sim Po(\lambda_1(s))$, ($s > 0$) and, given \mathcal{A} , $N(1)$, $N(2)$, ... are independent
 K_s^P = premium at duration s per unit of benefit
 $A_t(j)$ = number of units of benefit for policy j taken out at time t
 $F_t(x)$ = $Pr(A_t(j) \leq x)$
 with $A_t(1)$, $A_t(2)$, ... independent and identically distributed and independent of the $N(t)$
 and $A_t(j)$ and $A_s(k)$ are independent whenever $t \neq s$

$$I_{t-s,s}^S(j) = \begin{cases} 1 & \text{if } (x)_{t-s,j} \text{ is still alive at time } t \\ 0 & \text{otherwise} \end{cases}$$

By leaving K_s^P , K_s^S and K_s^D (defined below) as essentially arbitrary functions of duration, we maintain a high degree of flexibility. In particular, this covers a wide variety of policy types including annuities and assurances.

Given these assumptions we can then say that $P_2(t) \mid \mathcal{A}$ has the Compound Poisson distribution

$$P_2(t) \mid \mathcal{A} \sim CPo(A_p(t), F_t^P)$$

where $A_p(t) = \sum_{s=0}^{t-1} \lambda_1(t-s)_s p_x$

$$F_t^P(x) = \frac{1}{A_p(t)} \sum_{s=0}^{t-1} \lambda_1(t-s)_s p_x F_{t-s}(x/K_s^P)$$

If we assume that the history of the office is known up to and including time 0 then $P_1(t)$ is subject to much less uncertainty because the numbers and sizes of the existing policies are known. In any event $P_1(t)$ is equal to zero for all $t \geq \omega - x$.

Turning now to the benefits process, $B(t)$, we can proceed in a similar, but slightly more complex, way:

$$B(t) = \sum_{s=1}^{\infty} \sum_{j=1}^{N(t-s)} \{K_s^S A_{t-s}(j) I_{t-s,s}^S(j) + K_s^D A_{t-s}(j) I_{t-s,s}^D(j)\}$$

$$= B_1(t) + B_2(t)$$

where $B_1(t) = \sum_{s=t}^{\infty} \sum_{j=1}^{N(t-s)} \{K_s^S A_{t-s}(j) I_{t-s,s}^S(j) + K_s^D A_{t-s}(j) I_{t-s,s}^D(j)\}$

$$B_2(t) = \sum_{s=1}^{t-1} \sum_{j=1}^{N(t-s)} \{K_s^S A_{t-s}(j) I_{t-s,s}^S(j) + K_s^D A_{t-s}(j) I_{t-s,s}^D(j)\}$$

$$\begin{aligned}
 K_s^S &= \text{amount payable on survival to duration } s \text{ per unit benefit} \\
 K_s^D &= \text{amount payable on death during the year } (s - 1, s] \text{ per unit benefit} \\
 I_{t-s, s}^D(j) &= \begin{cases} 1 & \text{if } (x)_{t-s, j} \text{ dies during the year } (t - 1, t] \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

and note that

$$I_{t-s, s}^S(j) + I_{t-s, s}^D(j) = \begin{cases} 1 & \text{if } (x)_{t-s, j} \text{ is alive at } t - 1 \\ 0 & \text{otherwise} \end{cases}$$

As with the premium income we can then see that $B_2(t) \mid A$ has the Compound Poisson distribution

$$B_2(t) \mid A \sim CPo(A_B(t), F_t^B)$$

where $A_B(t) = \sum_{s=1}^{t-1} \lambda_1(t-s) {}_{s-1}p_x$

$$F_t^B(x) = \frac{1}{A_B(t)} \sum_{s=1}^{t-1} \lambda_1(t-s) \{ {}_s p_x F_{t-s}(x/K_s^S) + {}_{s-1} q_x F_{t-s}(x/K_s^D) \}$$

We can also say that $B_1(t) = 0$ for all $t > \omega - x$.

Now suppose that $m_1(t) = E[A_t(j)]$. Then for $t > \omega - x$

$$\begin{aligned}
 \mu(t) &= E(B(t)) \\
 &= E[E(B(t) \mid A)] \\
 &= E \left[\sum_{s=1}^{t-1} \lambda_1(t-s) m_1(t-s) ({}_s p_x K_s^S + {}_{s-1} q_x K_s^D) \right] \\
 &= \sum_{s=1}^{\omega-x} E[\lambda_1(t-s) m_1(t-s)] k_1(s)
 \end{aligned}$$

where $k_1(s) = {}_s p_x K_s^S + {}_{s-1} q_x K_s^D$

To make further progress we need to make further assumptions about the claim size distributions and in the model for new business growth.

Suppose then we assume that $F_t(x) = F(x/\lambda_2(t))$: that is, $A_t(j)/\lambda_2(t)$ and $A_s(k)/\lambda_2(s)$ are identically distributed when $s \neq t$ and where $\lambda_2(t)$ represents the benefit inflation index. Then $m_1(t) = \lambda_2(t) m_1$.

$$\Rightarrow \mu(t) = \sum_{s=1}^{\omega-x} m_1 k_1(s) E[\lambda_1(t-s) \lambda_2(t-s)]$$

Suppose also that

$$\begin{aligned}
 \lambda_1(t) &= \lambda_1 \exp [\rho_1 t + \sigma_1 W_1(t)] \\
 \lambda_2(t) &= \lambda_2 \exp [\rho_2 t + \sigma_2 W_2(t)]
 \end{aligned}$$

where $W_1(t)$ and $W_2(t)$ are independent standard Brownian motions. In particular, they have the properties for $i = 1, 2$:

- $W_i(0) = 0$;
 - if $t_1 < t_2$ then $W_i(t_2) - W_i(t_1) \sim N(0, t_2 - t_1)$;
 - if $t_1 < t_2 \leq t_3 < t_4$ then $W_i(t_2) - W_i(t_1)$ and $W_i(t_4) - W_i(t_3)$ are independent.
- For $t < 0$ $W_1(t)$ and $W_2(t)$ are known.

Then

$$\begin{aligned} E[\lambda_1(t)\lambda_2(t)] &= E[\lambda_1(t)]E[\lambda_2(t)] \\ &= \lambda_1\lambda_2 \exp\left[\left(\rho_1 + \rho_2 + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2\right)t\right] \\ &= \lambda_1\lambda_2 \exp[\rho t] \end{aligned}$$

where $\rho = \rho_1 + \rho_2 + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2$

We therefore have

$$\begin{aligned} \mu(t) &= \lambda_1\lambda_2 \sum_{s=1}^{\omega-x} m_1 k_1(s) \exp[\rho(t-s)] \\ &= \lambda_1\lambda_2 \exp(\rho t) \sum_{s=1}^{\omega-x} m_1 k_1(s) \exp(-\rho s) \\ &= \mu \exp(\rho t) \end{aligned}$$

where $\mu = \lambda_1\lambda_2 \sum_{s=1}^{\omega-x} m_1 k_1(s) \exp(-\rho s)$.

Hence, by Theorem 4 we deduce that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |B(t)| \leq \rho \text{ almost surely.}$$

We can prove similarly that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |P(t)| \leq \rho.$$

Hence, if $\rho - \delta < 0$ then, by Corollary 2

$$Z(t) = \sum_{s=1}^t V(s)C(s) = \sum_{s=1}^t V(s)P(s) - \sum_{s=1}^t V(s)B(s)$$

converges almost surely to some finite limit as t tends to infinity.

6. DISCUSSION

It is possible without great difficulty to relax many of the assumptions made in Section 5.

- Here we assumed that all new entrants will be of the same age in an effort to contain the already complex notation. Relaxing this to include a spread of ages will result in a sum of conditionally independent Compound Poisson processes (given \mathcal{A}) which is itself a Compound Poisson process.
- Similarly we could allow for more than one policy type, multiple state models (for example, permanent health insurance) and more than one risk group.
- Other forms of distribution for $N(t) \mid \lambda_1(t)$ would provide similar results. The Poisson assumption was made here for the convenience of its additive properties.
- Inclusion of expenses and reserves. (However, if we discount cashflows at the same rate of interest as that earned on the reserves, then the limiting value of $Z(t)$ will be unchanged.)

It should, therefore, follow that convergence can be shown to occur for a wide range of office models, beyond the already general case described here.

Suppose that $N(t) = 0$ for $t \leq 0$. Then, in the context of Section 5, $\lim_{t \rightarrow \infty} Z(t)$ represents the present value of profit on future new business. The present paper has shown that, subject to certain conditions, this quantity is well defined and exists almost surely. It is a quantity which is of genuine practical interest since it allows actuaries to assess the underlying value of a company.

It is unlikely that the limiting distribution or the moments of $Z(t)$ will be known under the majority of circumstances. (However, where the cashflows in different years are independent, the methods of PARKER, 1993, 1994a,b,c can be used to find the distribution of $Z(t)$ for $t < \infty$. This then provides, for large t , an approximation to the limiting distribution of $Z(t)$.) It will often, therefore, be necessary to carry out a Monte-Carlo study, simulating sample paths of $Z(t)$. The results described in this paper indicate that $Z(t)$ will converge to its limit at least as fast as the deterministic annuity function $\bar{a}_{\overline{t}|}$ with force of interest $\delta - \rho$ tends to its limiting value. This gives us a useful guide as to when the difference between $Z(t)$ and its limit falls within the maximum tolerable level of error.

Some idea of the limiting distribution of $Z(t)$ can be obtained by applying the results of PAPACHRISTOU and WATERS (1991) and FREES (1990) for large portfolios. The analogue here is that the distribution of $\lambda_1^{-1} Z(t)$ tends to that of $\lambda_1^{-1} E[Z(t) \mid \delta(s), \lambda_1(s), \lambda_2(s), = 1, 2, \dots, t]$ as λ_1 tends to infinity, and similarly the distribution of the limit of $Z(t)$ where this exists almost surely.

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APPENDIX A

We make use of the following Corollary to Lemma 3.

Corollary A.1

If there exists a deterministic sequence $a(t)$ converging to zero as t tends to infinity such that $\sum_{t=1}^{\infty} \exp[-\rho t - ta(t)] E|c(t)| < \infty$ then

$$\limsup_{t \rightarrow \infty} t^{-1} \log |C(t)| \leq \rho \text{ almost surely.}$$

Proof

As in Lemma 3 let

$$\begin{aligned} E_t &= \left\{ \frac{1}{t} \log |C(t)| \geq \rho + a(t) \right\} \\ &= \{ |C(t)| \geq \exp[\rho t + ta(t)] \} \end{aligned}$$

Now $E|C(t)| \geq c Pr\{|C(t)| \geq c\}$ for any $c > 0$ (for example, see WILLIAMS, 1991, Section 6.4). Hence

$$\begin{aligned} E|C(t)| \geq \exp[\rho t + ta(t)] Pr(E_t) \\ \Rightarrow \sum_{t=1}^{\infty} Pr(E_t) \leq \sum_{t=1}^{\infty} \exp[-\rho t - ta(t)] E|C(t)| < \infty. \end{aligned}$$

This is the condition in Lemma 3.

Proof of Theorem 4

Take any $\rho_2 > \rho_0$ and set $a(t) = 0$ in Corollary A.1. Choose any ρ_1 such that $\rho_0 < \rho_1 < \rho_2$ and let $k = \sup \{ \exp(-\rho_1 t) \mu(t) : t \geq 1 \}$. Since $\rho_1 > \rho_0$, k must be finite. Then

$$\begin{aligned} \sum_{t=1}^{\infty} \exp[-\rho_2 t - ta(t)] E|C(t)| &= \sum_{t=1}^{\infty} \exp[-\rho_1 t] E|C(t)| \exp[-(\rho_2 - \rho_1)t] \\ &< \sum_{t=1}^{\infty} k \exp[-(\rho_2 - \rho_1)t] < \infty. \end{aligned}$$

Hence $\limsup_{t \rightarrow \infty} t^{-1} \log |C(t)| \leq \rho_2$ almost surely.

This is true for all $\rho_2 > \rho_0$ so the result follows.

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