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Strongly exceptional Legendrian connected sum of two Hopf links

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Abstract. In this paper, we give a complete coarse classification of strongly exceptional Legendrian realizations of the connected sum of two Hopf links in contact 3-spheres. This is the first classification result about exceptional Legendrian representatives for connected sums of link families.

1 Introduction

A Legendrian link in an overtwisted contact 3-manifold is *exceptional* (a.k.a. *non-loose*) if its complement is tight. There have been several classifications for exceptional Legendrian knots and links in overtwisted contact 3-spheres, including unknots [6], [5], torus knots [12], [16], [9], and Hopf links [11]. The connected bindings of open book decompositions in certain overtwisted contact 3-manifolds have been partially classified in [7]. While there has been very little progress in the classification of Legendrian links with two or more components in either tight or overtwisted contact 3-spheres, a few papers, [1], [2], [3], [11], have tackled the problem.

In this paper, we study the classification of Legendrian realizations of the connected sum of two Hopf links up to coarse equivalence in any contact 3-sphere. This is one of the first families of the connected sum of links for which a classification is known. Two Legendrian realizations $K_0 \cup K_1 \cup K_2$ and $K_0' \cup K_1' \cup K_2'$ of the connected sum of two Hopf links in some contact 3-sphere S^3 are coarsely equivalent if there is a contactomorphism of S^3 sending $K_0 \cup K_1 \cup K_2$ to $K_0' \cup K_1' \cup K_2'$ as an ordered, oriented link.

Let $A_3 = K_0 \cup K_1 \cup K_2 \subset S^3$ be the oriented connected sum of two Hopf links, where K_0 is the central component. It is shown in Figure 1. The orientations of the components are also indicated. We think of K_1 and K_2 as two oriented meridians of K_0.

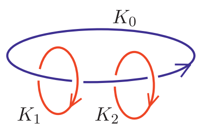

Figure 1: The link $A_3 = K_0 \cup K_1 \cup K_2$ in S^3.

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We consider the Legendrian realizations of A_3 in all contact 3-spheres. For $i = 0, 1, 2$, denote the Thurston-Bennequin invariant of K_i by t_i, and the rotation number of K_i by r_i.

Let (M, ξ) be a contact 3-manifold and $[T]$ an isotopy class of embedded tori in M. The *Giroux torsion* of (M, ξ) is the supremum of $n \in \mathbb{N}_0$ for which there is a contact embedding of

$$(T^2 \times [0,1], \ker(\sin(n\pi z)dx + \cos(n\pi z)dy))$$

into (M, ξ), with $T^2 \times \{z\}$ being in the class $[T]$.

An exceptional Legendrian link in an overtwisted contact 3-manifold is called *strongly exceptional* if its complement has zero Giroux torsion. This paper focuses on the classification of strongly exceptional Legendrian realizations of the A_3 link in contact 3-spheres up to coarse equivalence. We use the notation ξ_{st} to refer to the standard tight contact structure on S^3. The countably many overtwisted contact structures on S^3 are determined by their d_3-invariants in $\mathbb{Z} + \frac{1}{2}$ [11, Section 2]. If the d_3-invariant of an overtwisted contact 3-sphere is d, then we denote this contact 3-sphere by (S^3, ξ_d). Note that the d_3-invariant of ξ_{st} is $-\frac{1}{2}$.

We enumerate all the strongly exceptional Legendrian A_3 links up to coarse equivalence.

Theorem 1.1 *Suppose $t_1 < 0$ and $t_2 < 0$. Then, the number of strongly exceptional Legendrian A_3 links is*

$$\begin{cases} 2t_1t_2 - 2t_1 - 2t_2 + 2, & \text{if } t_0 \geq 2, \\ t_1t_2 - 2t_1 - 2t_2 + 2, & \text{if } t_0 = 1, \\ -2t_1 - 2t_2 + 2, & \text{if } t_0 = 0, \\ -t_0t_1t_2, & \text{if } t_0 \leq -1. \end{cases}$$

Moreover, if $t_0 \leq -1$, then the $-t_0t_1t_2$ Legendrian A_3 links are in the standard tight contact 3-sphere (S^3, ξ_{st}).

Theorem 1.2 *Suppose $t_1 = t_2 = 1$. Then, the number of strongly exceptional Legendrian A_3 links is*

$$\begin{cases} 8, & \text{if } t_0 \geq 6, \\ 7, & \text{if } t_0 = 5, \\ 6, & \text{if } t_0 = 4, \\ 4 - t_0, & \text{if } t_0 \leq 3. \end{cases}$$

Theorem 1.3 *Suppose $t_1 > 1$ and $t_2 = 1$. Then, the number of strongly exceptional Legendrian A_3 links is*

$$\begin{cases} 12, & \text{if } t_0 \geq 5 \text{ and } t_1 = 2, \\ 10, & \text{if } t_0 = 4 \text{ and } t_1 = 2, \\ 8, & \text{if } t_0 = 3 \text{ and } t_1 = 2, \\ 16, & \text{if } t_0 \geq 5 \text{ and } t_1 \geq 3, \\ 14, & \text{if } t_0 = 4 \text{ and } t_1 \geq 3, \\ 12, & \text{if } t_0 = 3 \text{ and } t_1 \geq 4, \\ 11, & \text{if } t_0 = t_1 = 3, \\ 6 - 2t_0, & \text{if } t_0 \leq 2. \end{cases}$$

Theorem 1.4 *Suppose $t_1 > 1$ and $t_2 > 1$. Then, the number of strongly exceptional Legendrian A_3 links is*

$$\begin{cases} 18, & \text{if } t_0 \geq 4 \text{ and } t_1 = t_2 = 2, \\ 14, & \text{if } t_0 = 3 \text{ and } t_1 = t_2 = 2, \\ 10, & \text{if } t_0 = 2 \text{ and } t_1 = t_2 = 2, \\ 24, & \text{if } t_0 \geq 4, t_1 \geq 3 \text{ and } t_2 = 2, \\ 20, & \text{if } t_0 = 3, t_1 \geq 3 \text{ and } t_2 = 2, \\ 16, & \text{if } t_0 = 2, t_1 \geq 3 \text{ and } t_2 = 2, \\ 32, & \text{if } t_0 \geq 4, t_1 \geq 3 \text{ and } t_2 \geq 3, \\ 28, & \text{if } t_0 = 3, t_1 \geq 3 \text{ and } t_2 \geq 3, \\ 24, & \text{if } t_0 = 2, t_1 \geq 3 \text{ and } t_2 \geq 3, \\ 8 - 4t_0, & \text{if } t_0 \leq 1. \end{cases}$$

Theorem 1.5 *Suppose $t_1 < 0$ and $t_2 = 1$. Then, the number of strongly exceptional Legendrian A_3 links is*

$$\begin{cases} 4 - 4t_1, & \text{if } t_0 \geq 4, \\ 4 - 3t_1, & \text{if } t_0 = 3, \\ 4 - 2t_1, & \text{if } t_0 = 2, \\ t_0 t_1 - 2t_1, & \text{if } t_0 \leq 1. \end{cases}$$

Theorem 1.6 *Suppose $t_1 < 0$ and $t_2 > 1$. Then, the number of strongly exceptional Legendrian A_3 links is*

$$\begin{cases} 6 - 6t_1, & \text{if } t_0 \geq 3, t_2 = 2, \\ 6 - 4t_1, & \text{if } t_0 = 2, t_2 = 2, \\ 6 - 2t_1, & \text{if } t_0 = 1, t_2 = 2, \\ 8 - 8t_1, & \text{if } t_0 \geq 3, t_2 \geq 3, \\ 8 - 6t_1, & \text{if } t_0 = 2, t_2 \geq 3, \\ 8 - 4t_1, & \text{if } t_0 = 1, t_2 \geq 4, \\ 8 - 3t_1, & \text{if } t_0 = 1, t_2 = 3, \\ 2t_0 t_1 - 2t_1, & \text{if } t_0 \leq 0. \end{cases}$$

Theorem 1.7 *Suppose $t_1 = 0$. Then, the number of strongly exceptional Legendrian A_3 links is*

$$\begin{cases} 2 - 2t_2, & \text{if } t_2 \leq 0, \\ 4, & \text{if } t_2 = 1, \\ 6, & \text{if } t_2 = 2, \\ 8, & \text{if } t_2 \geq 3. \end{cases}$$

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By exchange of the roles of K_1 and K_2 as necessary, we have covered all cases. Therefore, we have completely classified strongly exceptional Legendrian A_3 links. The reader can look up the explicit rotation numbers and corresponding d_3-invariants in Lemmas 4.3–4.6, 4.8–4.28, 4.30–4.40, 4.43–4.46 of Section 4. In particular, we have the following:

Theorem 1.8 *The strongly exceptional Legendrian A_3 links are determined up to coarse equivalence by their Thurston-Bennequin invariants and rotation numbers.*

Remark 1.9 Strongly exceptional Legendrian A_3 links exist only in overtwisted contact 3-spheres with $d_3 = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$.

Remark 1.10 Suppose $t_1, t_2 \neq 0$. If $t_0 + \lceil -\frac{1}{t_1} \rceil + \lceil -\frac{1}{t_2} \rceil \geq 2$, then any strongly exceptional Legendrian A_3 link can be destabilized at the component K_0 to another strongly exceptional one. If $t_0 + \lceil -\frac{1}{t_1} \rceil + \lceil -\frac{1}{t_2} \rceil < 1$, then any strongly exceptional Legendrian A_3 link can be destabilized at the component K_0 to a strongly exceptional Legendrian link with $t_0 + \lceil -\frac{1}{t_1} \rceil + \lceil -\frac{1}{t_2} \rceil = 1$. In the cases either $t_1 = 0$ or $t_2 = 0$, any strongly exceptional Legendrian A_3 link can be destabilized at the component K_0 to another strongly exceptional one. Furthermore, if $t_1 = 0$, then any strongly exceptional Legendrian A_3 link can be destabilized at the component K_2 to another strongly exceptional one unless $t_2 = 0$. However, a positive (or negative) stabilization at the component K_0 (and K_2 in the case $t_1 = 0$) of a strongly exceptional Legendrian A_3 link is strongly exceptional if and only if the resulted rotation numbers are indeed the rotation numbers of a strongly exceptional Legendrian A_3 link. Therefore, one can read out the mountain ranges of K_0 (and K_2 in the case $t_1 = 0$) through the Thurston-Bennequin invariants, rotation numbers, and d_3-invariants shown in Section 4. Section 5 explains how strongly exceptional Legendrian representatives relate to each other. Detailed analysis of the (de)stabilizations, as well as detailed analysis of the mountain range of K_2 for the links in Theorem 1.7, will be presented in Section 5.

The following is the structure of this paper. Section 2 presents upper bounds for appropriate tight contact structures on $\Sigma \times S^1$. In Section 3, we discuss various methods to realize the strongly exceptional Legendrian A_3 links. Section 4 focuses on the realization of the strongly exceptional Legendrian A_3 links, including the calculation of their rotation numbers and the d_3-invariants of their ambient contact S^3. In Section 5, we explore the stabilizations among the strongly exceptional Legendrian A_3 links. Finally, the last section provides a detailed computation as a sample, showcasing the calculation of rotation numbers and d_3-invariants.

2 Tight contact structures on $\Sigma \times S^1$

For $i = 0, 1, 2$, let $N(K_i)$ be the standard neighborhood of K_i in a contact 3-sphere. The Seifert longitude and meridian of K_i are denoted by λ_i and μ_i, respectively. The exterior of the link $A_3 = K_0 \cup K_1 \cup K_2$, $\overline{S^3 \backslash (N(K_0) \cup N(K_1) \cup N(K_2))}$, is diffeomorphic to $\Sigma \times S^1$, where Σ is a pair of pants. Suppose $\partial\Sigma = c_0 \cup c_1 \cup c_2$ as shown in Figure 2. Let h denote the S^1 factor – namely, the vertical circle. Then, $\lambda_0 = c_0$, $\lambda_1 = \lambda_2 = h$, $\mu_0 = h$, $\mu_1 = -c_1$, $\mu_2 = -c_2$. Suppose $\partial(\Sigma \times S^1) = T_0 \cup T_1 \cup T_2$, where $T_i = c_i \times S^1$. Then, the dividing set of T_0 has slope t_0 (i.e., has the homology $[c_0] + t_0[h]$), and the dividing

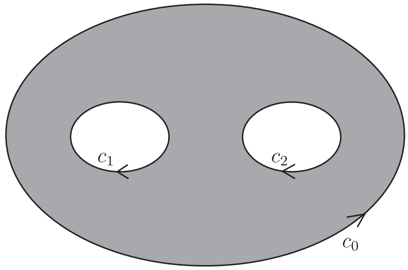

Figure 2: A pair of pants Σ.

set of T_i has slope $-\frac{1}{t_i}$ (i.e., has the homology $-t_i[c_i] + [h]$, for $i = 1, 2$). Furthermore, each boundary torus of the exterior of a Legendrian A_3 link is minimal convex. Namely, its dividing set consists of exactly two parallel simple closed curves.

Following [18], we say that a tight contact structure ξ on $\Sigma \times S^1$ with minimal convex boundary is *appropriate* if there is no contact embedding of

$$(T^2 \times [0,1], \ker(\sin(\pi z)dx + \cos(\pi z)dy))$$

into (M, ξ), where $T^2 \times \{0\}$ is isotopic to a boundary component of $\Sigma \times S^1$. A Legendrian representation of the A_3 link in an overtwisted contact 3-sphere is strongly exceptional if and only if its exterior is an appropriate tight contact $\Sigma \times S^1$.

In this section, we study the appropriate tight contact structures on $\Sigma \times S^1$ with minimal convex boundary. The boundary slopes are $s_0 = s(T_0) = t_0$, $s_1 = s(T_1) = -\frac{1}{t_1}$, and $s_2 = s(T_2) = -\frac{1}{t_2}$, where t_0, t_1, t_2 are integers.

Lemma 2.1 [13] *Let T^2 be a convex surface in a contact 3-manifold with $\#\Gamma_{T^2} = 2$ and slope s. If a bypass D is attached to T^2 from the front (the back, resp.) along a Legendrian ruling curve of slope $r \neq s$, then the resulting convex surface $\tilde{T}^2$ will have $\#\Gamma_{\tilde{T}^2} = 2$ and the slope s' which is obtained as follows: Take the arc $[r, s] \subset \partial\mathbb{H}^2$ obtained by starting from r and moving counterclockwise (clockwise, resp.) until we hit s, where $\mathbb{H}^2$ is the Poincare disk shown in Figure 3. On this arc, let s' be the point that is closest to r and has an edge from s' to s.*

Every vertical circle in a contact $\Sigma \times S^1$ has a canonical framing that arises from the product structure. Let γ be a Legendrian circle that lies in the vertical direction. The twisting number $t(\gamma)$ of γ measures the amount by which the contact framing of γ deviates from the canonical framing. If $t(\gamma) = 0$, then we say that γ is a 0-*twisting vertical Legendrian circle.*

Lemma 2.2 *Suppose ξ is an appropriate tight contact structure on $\Sigma \times S^1$ with boundary slopes $s_0 = t_0$, $s_i = -\frac{1}{t_i}$ for $i = 1, 2$. If $t_1, t_2 \neq 0$ and $t_0 + \lceil -\frac{1}{t_1} \rceil + \lceil -\frac{1}{t_2} \rceil \geq 2$, then ξ has a 0-twisting vertical Legendrian circle.*

Proof We assume the Legendrian rulings on T_1 and T_2 to have infinite slopes. Consider a convex vertical annulus A such that the boundary consists of a Legendrian ruling on T_1 and a Legendrian ruling on T_2. The dividing set of A intersects T_i,

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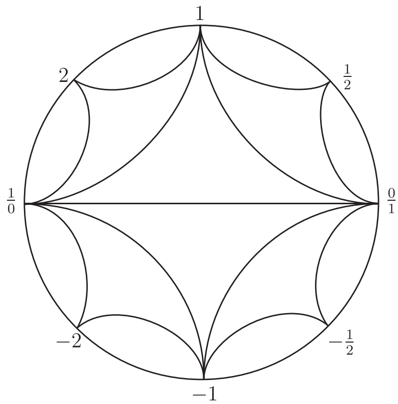

Figure 3: Farey graph on the Poincare disk $\mathbb{H}^2$.

$i = 1, 2$, in exactly $2|t_i|$ points. If every dividing curve of A is boundary parallel, then there exists a 0-twisting vertical Legendrian circle in A. So we assume that there exist dividing arcs on A, which connect the two boundary components of A. If there is a boundary parallel dividing curve on A, then we perform a bypass attachment (attached from the back of T_i) to eliminate it.

(1) Suppose $t_1 < 0$ and $t_2 < 0$. By Lemma 2.1, we can obtain a submanifold $\tilde{\Sigma} \times S^1$ of $\Sigma \times S^1$ whose boundary is $T_0 \cup \tilde{T}_1 \cup \tilde{T}_2$, where both $\tilde{T}_1$ and $\tilde{T}_2$ have slopes $-\frac{1}{t_3}$ for some integer $t_3 \in [\max\{t_1, t_2\}, -1]$. Moreover, each dividing curve on $\tilde{A} = A \cap (\tilde{\Sigma} \times S^1)$ connects the two boundary components. Let N be a neighborhood of $\tilde{T}_1 \cup \tilde{T}_2 \cup \tilde{A}$, and $\partial N = \tilde{T}_1 \cup \tilde{T}_2 \cup \tilde{T}$. Then, by edge-rounding, $\tilde{T}$ has slope $\frac{1}{t_3} + \frac{1}{t_3} + \frac{1}{-t_3} = \frac{1}{t_3}$ (as seen form T_0). Therefore, the thickened torus $\tilde{\Sigma} \times S^1 \backslash N$ has boundary slopes t_0 and $\frac{1}{t_3}$. Since $t_0 \geq 0 > \frac{1}{t_3}$, there must exist a 0-twisting vertical Legendrian circle in this thickened torus, and hence in $\Sigma \times S^1$.

(2) Suppose $t_1 = 1$ and $t_2 = 1$. It follows from [14, Lemma 5.1].

(3) Suppose $t_1 > 1$ and $t_2 = 1$. By Lemma 2.1, we can obtain a submanifold $\tilde{\Sigma} \times S^1$ of $\Sigma \times S^1$ whose boundary is $T_0 \cup \tilde{T}_1 \cup T_2$, where $\tilde{T}_1$ has slope 0. Moreover, each dividing curve on $\tilde{A} = A \cap (\tilde{\Sigma} \times S^1)$ connects the two boundary components. Let N be a neighborhood of $\tilde{T}_1 \cup T_2 \cup \tilde{A}$, and $\partial N = \tilde{T}_1 \cup T_2 \cup \tilde{T}$. Then, by edge-rounding, $\tilde{T}$ has slope $0 + 1 + 1 = 2$ (as seen form T_0). Therefore, the thickened torus $\tilde{\Sigma} \times S^1 \backslash N$ has boundary slopes t_0 and 2. Since $t_0 \geq 3 > 2$, there must exist a 0-twisting vertical Legendrian circle in this thickened torus, and hence in $\Sigma \times S^1$.

(4) Suppose $t_1 > 1$ and $t_2 > 1$. We divide this case into two subcases:

(i) There exist boundary parallel dividing curves on A. By Lemma 2.1, we can obtain a submanifold $\tilde{\Sigma} \times S^1$ of $\Sigma \times S^1$ whose boundary is $T_0 \cup \tilde{T}_1 \cup \tilde{T}_2$, where both $\tilde{T}_1$ and $\tilde{T}_2$ have slopes 0. Moreover, each dividing curve on $\tilde{A} = A \cap (\tilde{\Sigma} \times S^1)$ connects the two boundary components. Let N be a neighborhood of $\tilde{T}_1 \cup \tilde{T}_2 \cup \tilde{A}$, and $\partial N = \tilde{T}_1 \cup \tilde{T}_2 \cup \tilde{T}$. Then, by edge-rounding, $\tilde{T}$ has slope $0 + 0 + 1 = 1$ (as seen form T_0). Therefore, the thickened torus $\tilde{\Sigma} \times S^1 \backslash N$ has boundary slopes t_0 and 1. Since $t_0 \geq 2 > 1$, there

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must exist a 0-twisting vertical Legendrian circle in this thickened torus, and hence in $\Sigma \times S^1$.

(ii) There exists no boundary parallel dividing curve on A. Then $t_1 = t_2$ and all dividing curves on A connect the two boundary components of A. Let N be a neighborhood of $T_1 \cup T_2 \cup \tilde{A}$, and $\partial N = T_1 \cup T_2 \cup \tilde{T}$. Then, by edge-rounding, $\tilde{T}$ has slope $\frac{1}{t_1} + \frac{1}{t_1} + \frac{1}{t_1} = \frac{3}{t_1}$ (as seen form T_0). Therefore, the thickened torus $\Sigma \times S^1 \backslash N$ has boundary slopes t_0 and $\frac{3}{t_1}$. Since $t_0 \geq 2 > \frac{3}{t_1}$, there must exist a 0-twisting vertical Legendrian circle in this thickened torus, and hence in $\Sigma \times S^1$.

(5) Suppose $t_1 < 0$ and $t_2 = 1$. There are boundary parallel dividing curves on A. By Lemma 2.1, we can obtain a submanifold $\tilde{\Sigma} \times S^1$ of $\Sigma \times S^1$ whose boundary is $T_0 \cup \tilde{T}_1 \cup T_2$, where both $\tilde{T}_1$ have slopes 1. Moreover, each dividing curve on $\tilde{A} = A \cap (\tilde{\Sigma} \times S^1)$ connects the two boundary components. Let N be a neighborhood of $\tilde{T}_1 \cup T_2 \cup \tilde{A}$, and $\partial N = \tilde{T}_1 \cup T_2 \cup \tilde{T}$. Then, by edge-rounding, $\tilde{T}$ has slope $1 + (-1) + 1 = 1$ (as seen form T_0). Therefore, the thickened torus $\tilde{\Sigma} \times S^1 \backslash N$ has boundary slopes t_0 and 1. Since $t_0 \geq 2 > 1$, there must exist a 0-twisting vertical Legendrian circle in this thickened torus, and hence in $\Sigma \times S^1$.

(6) Suppose $t_1 < 0$ and $t_2 > 1$. We divide this case into two subcases.

(i) If there exist boundary parallel dividing curves on A whose boundary points belong to $A \cap T_2$, we can use Lemma 2.1 to obtain a submanifold $\tilde{\Sigma} \times S^1$ of $\Sigma \times S^1$ whose boundary is $T_0 \cup \tilde{T}_1 \cup \tilde{T}_2$, where $\tilde{T}_1$ has slope 1 and $\tilde{T}_2$ has slope 0. Furthermore, each dividing curve on $\tilde{A} = A \cap (\tilde{\Sigma} \times S^1)$ connects the two boundary components. Let N be a neighborhood of $\tilde{T}_1 \cup \tilde{T}_2 \cup \tilde{A}$, and $\partial N = \tilde{T}_1 \cup \tilde{T}_2 \cup \tilde{T}$. By performing edge-rounding, $\tilde{T}$ will have slope $-1 + 0 + 1 = 0$ (as seen form T_0). Therefore, the thickened torus $\tilde{\Sigma} \times S^1 \backslash N$ has boundary slopes t_0 and 1. Since $t_0 \geq 1 > 0$, there must exist a 0-twisting vertical Legendrian circle in this thickened torus, and hence in $\Sigma \times S^1$.

(ii) If there are no boundary parallel dividing curves on A whose boundary points belong to $A \cap T_2$, we can use Lemma 2.1 to obtain a submanifold $\tilde{\Sigma} \times S^1$ of $\Sigma \times S^1$ whose boundary is $T_0 \cup \tilde{T}_1 \cup T_2$, where $\tilde{T}_1$ has slope $\frac{1}{t_2}$. Furthermore, each dividing curve on $\tilde{A} = A \cap (\tilde{\Sigma} \times S^1)$ connects the two boundary components. Let N be a neighborhood of $\tilde{T}_1 \cup T_2 \cup \tilde{A}$, and $\partial N = \tilde{T}_1 \cup T_2 \cup \tilde{T}$. By performing edge-rounding, $\tilde{T}$ will have slope $-\frac{1}{t_2} + \frac{1}{t_2} + \frac{1}{t_2} = \frac{1}{t_2}$ (as seen form T_0). Therefore, the thickened torus $\tilde{\Sigma} \times S^1 \backslash N$ has boundary slopes t_0 and 1. Since $t_0 \geq 1 > \frac{1}{t_2}$, there must exist a 0-twisting vertical Legendrian circle in this thickened torus, and hence in $\Sigma \times S^1$. ∎

Lemma 2.3 *If ξ is a tight contact structure on $\Sigma \times S^1$ with boundary slopes $s_0 = t_0$, $s_i = -\frac{1}{t_i}$ for $i = 1, 2$, has a 0-twisting vertical Legendrian circle, where $t_1, t_2 \neq 0$. Then, it admits a factorization $\Sigma \times S^1 = L'_0 \cup L'_1 \cup L'_2 \cup \Sigma' \times S^1$, where L'_i are disjoint thickened tori with minimal twisting and minimal convex boundary $\partial L'_i = T_i - T'_i$, and all the components of $\partial \Sigma' \times S^1 = T'_0 \cup T'_1 \cup T'_2$ have boundary slopes ∞.*

Proof The proof is similar to that of [14, Lemma 5.1, Part 1]. ∎

Let ξ be a contact structure on $\Sigma \times S^1$ with boundary slopes $s_0 = t_0$, $s_i = -\frac{1}{t_i}$ for $i = 1, 2$, where $t_1, t_2 \neq 0$. Assume it admits a factorization $\Sigma \times S^1 = L'_0 \cup L'_1 \cup L'_2 \cup \Sigma' \times S^1$, where L'_i are disjoint thickened tori with minimal twisting and minimal convex boundary $\partial L'_i = T_i - T'_i$, and all the components of $\partial \Sigma' \times S^1 = T'_0 \cup T'_1 \cup T'_2$

have boundary slopes ∞. Then, in the thickened torus L_i', $i = 1, 2$, there exists a basic slice B_i' with one boundary component T_i' and another boundary slope $\lceil -\frac{1}{t_i} \rceil$. This is because $\lceil -\frac{1}{t_i} \rceil$ is counterclockwise of $-\frac{1}{t_i}$ and clockwise of ∞ in the Farey graph shown in Figure 3. Let C_i', $i \in \{1, 2\}$, be the continued fraction block in L_i' that contains B_i'. The basic slices in C_i' can be shuffled. Namely, any basic slice in C_i' can be shuffled to be B_i'.

Lemma 2.4 *(1) Suppose $t_0 + \lceil -\frac{1}{t_1} \rceil + \lceil -\frac{1}{t_2} \rceil = 3$. If the signs of L_0', B_1', and B_2' are the same, then the restriction of ξ to $L_0' \cup B_1' \cup B_2' \cup \Sigma' \times S^1$ remains unchanged if we change the three signs simultaneously.*

(2) Suppose $t_0 + \lceil -\frac{1}{t_1} \rceil + \lceil -\frac{1}{t_2} \rceil \le 2$. If the signs of L_0', B_1', and B_2' are the same, then ξ is overtwisted.

Proof The restriction of ξ on $L_0' \cup B_1' \cup B_2' \cup \Sigma' \times S^1$ has boundary slopes t_0, $\lceil -\frac{1}{t_1} \rceil$ and $\lceil -\frac{1}{t_2} \rceil$. So the lemma follows by applying [14, Lemma 5.1] to $L_0' \cup B_1' \cup B_2' \cup \Sigma' \times S^1$. ■

Lemma 2.5 [9] *There is a unique appropriate tight contact structure on $\Sigma \times S^1$ whose three boundary slopes are all ∞ up to isotopy (not fixing the boundary point-wise, but preserving it set-wise).*

Lemma 2.6 *Let ξ be a contact structure on $\Sigma \times S^1$. Assume that each T_i is minimal convex with dividing curves of finite slope t_0, $-\frac{1}{t_1}$ and $-\frac{1}{t_2}$. If ξ has 0-twisting vertical Legendrian circles and $t_0 + \lceil -\frac{1}{t_1} \rceil + \lceil -\frac{1}{t_2} \rceil \le 1$, then ξ is not appropriate tight.*

Proof As there is a 0-twisting vertical Legendrian circle, there exists a minimal convex torus T_i', parallel to T_i, with slope $\lceil -\frac{1}{t_i} \rceil$, $i = 1, 2$. Consider a convex annulus $\tilde{A}$ with a boundary consisting of a Legendrian ruling on T_1' and a Legendrian ruling on T_2'. Let N be a neighborhood of $T_1' \cup T_2' \cup \tilde{A}$, and $\partial N = T_1' \cup T_2' \cup \tilde{T}$. Then, through edge-rounding, $\tilde{T}$ has slope $-\lceil -\frac{1}{t_1} \rceil - \lceil -\frac{1}{t_2} \rceil + 1$ (as seen form T_0). We obtain a thickened torus with boundary slopes t_0 and $-\lceil -\frac{1}{t_1} \rceil - \lceil -\frac{1}{t_2} \rceil + 1$, and a boundary parallel convex torus with slope ∞. Thus, from $t_0 \le -\lceil -\frac{1}{t_1} \rceil - \lceil -\frac{1}{t_2} \rceil + 1$, it follows that the Giroux torsion of this thickened torus is at least 1. Hence, the Lemma holds. ■

Lemma 2.7 *Let ξ be an appropriate tight contact structure on $\Sigma \times S^1$. Assume that each T_i is minimal convex with dividing curves of finite slope t_0, $-\frac{1}{t_1}$ and $-\frac{1}{t_2}$. Suppose ξ has no 0-twisting vertical Legendrian circle. Then, there exist collar neighborhoods L_i'' of T_i for $i = 1, 2$ satisfying that $\Sigma \times S^1 = \Sigma'' \times S^1 \cup L_1'' \cup L_2''$, and the boundary slopes of $\Sigma'' \times S^1$ are t_0, $\lceil -\frac{1}{t_1} \rceil$, and $\lceil -\frac{1}{t_2} \rceil$.*

Proof We modify the Legendrian rulings on T_0 and T_i to have infinite slopes. Consider a convex vertical annulus A whose boundary consists of Legendrian rulings on T_0 and T_i. The dividing set of A intersects T_0 in exactly 2 points. The dividing set of A intersects T_i, $i = 1, 2$, in exactly $2|t_i|$ points. As ξ has no 0-twisting vertical Legendrian circle, there exist dividing arcs on A that connect the two boundary components of A. If there is a boundary parallel dividing curve on A, then its endpoints must belong to $A \cap T_i$ for some $i = 1, 2$. We perform a bypass (attached from the back of T_i) to eliminate it. Applying Lemma 2.1, we obtain a thickened torus L_i'' for $i = 1, 2$

that satisfies $\Sigma \times S^1 = \Sigma'' \times S^1 \cup L_1'' \cup L_2''$, and the boundary slopes of $\Sigma'' \times S^1$ are t_0, $\lceil -\frac{1}{t_1} \rceil$, and $\lceil -\frac{1}{t_2} \rceil$. ∎

Now we present upper bounds for appropriate tight contact structures on $\Sigma \times S^1$.

2.1 $t_1 < 0$ and $t_2 < 0$.

Lemma 2.8 *Suppose $t_1 < 0$ and $t_2 < 0$. Then, there are at most*

$$\begin{cases} 2t_1t_2 - 2t_1 - 2t_2 + 2, & \text{if } t_0 \geq 2, \\ t_1t_2 - 2t_1 - 2t_2 + 2, & \text{if } t_0 = 1, \\ -2t_1 - 2t_2 + 2, & \text{if } t_0 = 0, \\ -t_0t_1t_2, & \text{if } t_0 \leq -1 \end{cases}$$

appropriate tight contact structures on $\Sigma \times S^1$ with the given boundary slopes.

Proof By Lemma 2.2, if $t_0 \geq 0$, then the tight contact structures on $\Sigma \times S^1$ always exist 0-twisting vertical Legendrian circles.

If an appropriate contact structure ξ on $\Sigma \times S^1$ has a 0-twisting vertical Legendrian circle, then Lemma 2.3 tells us that $\Sigma \times S^1$ can be factored into $L_0' \cup L_1' \cup L_2' \cup \Sigma' \times S^1$, where the boundary slopes of $\Sigma' \times S^1$ are all ∞, the boundary slopes of L_0' are ∞ and t_0, and the boundary slopes of L_i' are ∞ and $-\frac{1}{t_i}$ for $i = 1, 2$. Moreover, There are 2 minimally twisting tight contact structures on L_0'.

If $t_i < 0$, $i = 1, 2$, we have

$$\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -t_i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -t_i + 1 \end{bmatrix}.$$

The thickened torus L_i is a continued fraction block with $-t_i$ basic slices and therefore admits $-t_i + 1$ minimally twisting tight contact structures.

By applying Lemma 2.5, we can conclude that there are at most $2t_1t_2 - 2t_1 - 2t_2 + 2$ appropriate tight contact structures on $\Sigma \times S^1$ if $t_0 \geq 2$. If $t_0 = 1$ and there are basic slices in L_i' which have the same signs as that of L_0' for $i = 1, 2$, then after shuffling, we can assume that L_0', B_1' and B_2' have the same signs. According to Lemma 2.4, a tight contact structure that has positive basic slices in L_i' for $i = 0, 1, 2$ are isotopic to a tight contact structure which is obtained by changing a positive basic slice in L_i' for $i = 0, 1, 2$ to a negative basic slice. Therefore, there are at most $t_1t_2 - 2t_1 - 2t_2 + 2$ appropriate tight contact structures on $\Sigma \times S^1$ if $t_0 = 1$. If $t_0 = 0$, then by Lemma 2.4, a contact structure which has positive basic slices in L_i' for $i = 0, 1, 2$ is overtwisted. Thus, there are at most $-2t_1 - 2t_2 + 2$ appropriate tight contact structures on $\Sigma \times S^1$ if $t_0 = 0$.

Suppose $t_0 \leq -1$. By Lemma 2.6, there are no appropriate tight contact structures having a 0-twisting vertical Legendrian circle. We consider the appropriate tight contact structures without a 0-twisting vertical Legendrian circle. By Lemma 2.7, we can factorize $\Sigma \times S^1 = \Sigma'' \times S^1 \cup L_1'' \cup L_2''$, where the boundary slopes of $\Sigma'' \times S^1$ are t_0, 1 and 1, and the boundary slopes of L_i'' are 1 and $-\frac{1}{t_i}$ for $i = 1, 2$. Since $t_0 < 0$, by [14, Lemma 5.1], there are exactly $-t_0$ tight contact structures on $\Sigma'' \times S^1$ without any 0-twisting vertical Legendrian circle. By [13, Theorem 2.2], there are $-t_i$ minimally twisting tight contact structures on L_i'' for $i = 1, 2$. Therefore, there are at most $-t_0t_1t_2$

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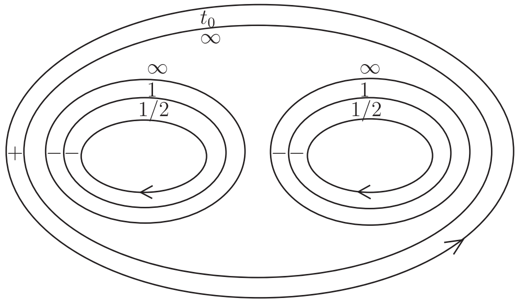

Figure 4: A pair of pants Σ, where $t_0 = 0$, $t_1 = t_2 = -2$.

tight contact structures on $\Sigma \times S^1$ without any 0-twisting vertical Legendrian curve and with boundary slopes $s_0 = t_0$, $s_i = -\frac{1}{t_i}$ for $i = 1, 2$. ∎

To denote the $2t_1t_2 - 2t_1 - 2t_2 + 2$ contact structures on $\Sigma \times S^1$ with 0-twisting vertical Legendrian circle, we use the decorations $(\pm)(\underbrace{\pm\cdots\pm}_{-t_1})(\underbrace{\pm\cdots\pm}_{-t_2})$. See Figure 4 for an example. The sign in the first bracket corresponds to the sign of the basic slice L'_0, while the signs in the second and the third brackets correspond to the signs of the basic slices in L'_1 and L'_2, respectively. We order the basic slices in L'_1 and L'_2 from the innermost boundary to the outmost boundary. As both L'_1 and L'_2 are continued fraction blocks, the signs in the second and the third brackets can be shuffled. For example, the decorations $(+)(+--)(--)$ and $(+)(--+)(--)$ denote the same contact structures.

2.2 $t_1 > 0$ and $t_2 > 0$.

Lemma 2.9 *Suppose $t_1 = t_2 = 1$. Then, there are exactly*

$$\begin{cases} 8, & \text{if } t_0 \geq 6, \\ 7, & \text{if } t_0 = 5, \\ 6, & \text{if } t_0 = 4, \\ 4 - t_0, & \text{if } t_0 \leq 3 \end{cases}$$

appropriate tight contact structures on $\Sigma \times S^1$ with the given boundary slopes.

Proof The boundary slopes of $\Sigma \times S^1$ are t_0, -1 and -1. If $t_0 \leq 3$, according to [14, Lemma 5.1], there are exactly $4 - t_0$ appropriate tight contact structures on $\Sigma \times S^1$ without a 0-twisting vertical Legendrian circle. By Lemma 2.6, there are no appropriate tight contact structures on $\Sigma \times S^1$ with a 0-twisting vertical Legendrian circle. If $t_0 \geq 4$, then any tight contact structure on $\Sigma \times S^1$ has a 0-twisting vertical Legendrian circle. By applying [14, Lemma 5.1] again, we can conclude that when $t_0 = 4$, there are exactly 6 appropriate tight contact structures on $\Sigma \times S^1$. When $t_0 = 5$, there are exactly 7 appropriate tight contact structures on $\Sigma \times S^1$. When $t_0 \geq 6$, there are exactly 8 appropriate tight contact structures on $\Sigma \times S^1$. ∎

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We use the decorations $(\pm)(\pm)(\pm)$ to denote the 8 contact structures on $\Sigma \times S^1$ with a 0-twisting vertical Legendrian circle.

Lemma 2.10 *Suppose $t_1 > 1$ and $t_2 = 1$. Then, there are at most*

$$\begin{cases} 12, & \text{if } t_0 \geq 5 \text{ and } t_1 = 2, \\ 10, & \text{if } t_0 = 4 \text{ and } t_1 = 2, \\ 8, & \text{if } t_0 = 3 \text{ and } t_1 = 2, \\ 16, & \text{if } t_0 \geq 5 \text{ and } t_1 \geq 3, \\ 14, & \text{if } t_0 = 4 \text{ and } t_1 \geq 3, \\ 12, & \text{if } t_0 = 3 \text{ and } t_1 \geq 4, \\ 11, & \text{if } t_0 = t_1 = 3, \\ 6 - 2t_0, & \text{if } t_0 \leq 2 \end{cases}$$

appropriate tight contact structures on $\Sigma \times S^1$ with the given boundary slopes.

Proof The boundary slopes of $\Sigma \times S^1$ are $s_0 = t_0$, $s_1 = -\frac{1}{t_1}$, and $s_2 = -1$. If $t_0 \geq 3$, then the tight contact structures on $\Sigma \times S^1$ always exist 0-twisting vertical Legendrian circles.

If $t_1 > 1$, we have

$$\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} t_1 \\ -1 \end{bmatrix} = \begin{bmatrix} t_1 - 1 \\ -2t_1 + 1 \end{bmatrix},$$

$$\frac{-2t_1 + 1}{t_1 - 1} = [-3, \underbrace{-2, \cdots, -2}_{t_1 - 2}].$$

If $t_1 = 2$, then L_1' is a continued fraction block with two basic slices with slopes $-\frac{1}{2}$, 0, and ∞, and thus admits exactly 3 tight contact structures. If $t_1 \geq 3$, then L_1' consists of two continued fraction blocks, each of which has one basic slice. The slopes are $-\frac{1}{t_1}$, 0, and ∞. Therefore, it admits exactly 4 tight contact structures.

If $t_0 \geq 5$ and $t_1 = 2$, then there are at most $2 \times 3 \times 2 = 12$ tight contact structures. The number of such contact structures depends on the signs of the basic slices in L_i' for $i = 0, 1, 2$. If $t_0 = 4$ and $t_1 = 2$, then there are at most 10 tight contact structures by deleting 2 duplications. If $t_0 \leq 3$ and $t_1 = 2$, then there are at most 8 tight contact structures by deleting 4 overtwisted cases.

If $t_0 \geq 5$ and $t_1 \geq 3$, then there are at most $2 \times 4 \times 2 = 16$ tight contact structures. The number of such contact structures depends depend on the signs of the basic slices in L_i' for $i = 0, 1, 2$. If $t_0 = 4$ and $t_1 \geq 3$, then there are at most 14 tight contact structures by deleting 2 duplications. If $t_0 \leq 3$ and $t_1 \geq 3$, then there are at most 12 tight contact structures by deleting 4 overtwisted cases.

Suppose $t_0 \leq 2$. By Lemma 2.6, there are no appropriate tight contact structures with a 0-twisting vertical Legendrian circle. We consider the appropriate tight contact structures without a 0-twisting vertical Legendrian circle. By Lemma 2.7, we can factorize $\Sigma \times S^1 = \Sigma'' \times S^1 \cup L_1''$, where the boundary slopes of $\Sigma'' \times S^1$ are t_0, 0 and -1, and the boundary slopes of L_1'' are 0 and $-\frac{1}{t_1}$. Since $t_0 < 3$, according to [14, Lemma 5.1], there are exactly $3 - t_0$ tight contact structures on $\Sigma'' \times S^1$ without a 0-twisting vertical Legendrian circle. There are 2 minimally twisting tight contact structures on L_1''. Therefore, there are at most $6 - 2t_0$ appropriate tight contact structures on $\Sigma \times S^1$ without a 0-twisting vertical Legendrian circle and with boundary slopes $s_0 = t_0$, $s_i = -\frac{1}{t_i}$ for $i = 1, 2$.

If $t_1 = 2$, then we denote the 12 contact structures on $\Sigma \times S^1$ with a 0-twisting vertical Legendrian circle using the decorations $(\pm)(\pm\pm)(\pm)$. For $t_1 \geq 3$, we use the decorations $(\pm)((\pm)(\pm))(\pm)$ to denote the 16 contact structures on $\Sigma \times S^1$ with a 0-twisting vertical Legendrian circle. In the latter case, $((\pm)(\pm))$ refers to the two signed basic slices in L_1' that do not form a continued fraction block.

If $t_0 = t_1 = 3$ and $t_2 = 1$, we claim the two decorations $(+)((-)(+))(+)$ and $(-)((+)(-))(-)$ denote the same contact structure on $\Sigma \times S^1$. As before, there is a convex vertical annulus A such that ∂A consists of a Legendrian ruling on T_0 and a Legendrian ruling on T_2, and the dividing set on A run from one boundary component to the other. If we cut $L_0' \cup L_1' \cup L_2' \cup \Sigma' \times S^1$ along A, we will obtain a thickened torus admitting a factorization into two basic slices with slopes $-\frac{1}{3}, 0$ and $0, -1$, and opposite signs. Here, the slope -1 is obtained by $-s_0 - s_2 + 1 = -3 - (-1) + 1$. The three slopes can be transformed into $\frac{1}{3}$, $\frac{1}{2}$, and 1 as follows:

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So these two basic slices form a continued fraction block and can be interchanged. Similar to the argument in [14, Page 135], this leads to an exchange between $(+)((-)(+))(+)$ and $(-)((+)(-))(-)$ while preserving the isotopy classes of contact structures. ■

Lemma 2.11 *Suppose $t_1 > 1$ and $t_2 > 1$. Then, there are at most*

$$\begin{cases} 18, & \text{if } t_0 \geq 4 \text{ and } t_1 = t_2 = 2, \\ 14, & \text{if } t_0 = 3 \text{ and } t_1 = t_2 = 2, \\ 10, & \text{if } t_0 = 2 \text{ and } t_1 = t_2 = 2, \\ 24, & \text{if } t_0 \geq 4 \text{ and } t_1 \geq 3, t_2 = 2, \\ 20, & \text{if } t_0 = 3 \text{ and } t_1 \geq 3, t_2 = 2, \\ 16, & \text{if } t_0 = 2 \text{ and } t_1 \geq 3, t_2 = 2, \\ 32, & \text{if } t_0 \geq 4 \text{ and } t_1 \geq 3, t_2 \geq 3, \\ 28, & \text{if } t_0 = 3 \text{ and } t_1 \geq 3, t_2 \geq 3, \\ 24, & \text{if } t_0 = 2 \text{ and } t_1 \geq 3, t_2 \geq 3, \\ 8 - 4t_0, & \text{if } t_0 \leq 1 \end{cases}$$

appropriate tight contact structures on $\Sigma \times S^1$ with the given boundary slopes.

Proof If $t_0 \geq 2$, then the tight contact structures on $\Sigma \times S^1$ always exist a 0-twisting vertical Legendrian circles.

If $t_0 \geq 4$ and $t_1 = t_2 = 2$, then there are at most $2 \times 3 \times 3 = 18$ tight contact structures. If $t_0 \geq 4$, $t_1 \geq 3$, and $t_2 = 2$, then there are at most $2 \times 4 \times 3 = 24$ tight contact structures. If $t_0 \geq 4$, $t_1 \geq 3$, and $t_2 \geq 3$, then there are at most $2 \times 4 \times 4 = 32$ tight contact structures. The number of such contact structures depends on the signs of the basic

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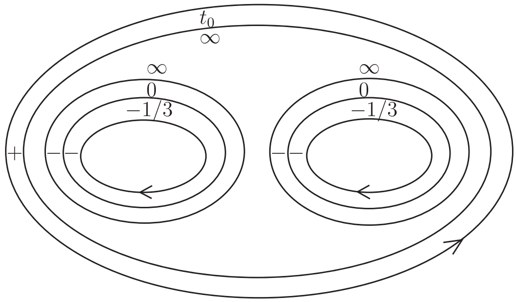

Figure 5: A pair of pants Σ, where $t_0 = 0$, $t_1 = t_2 = 3$.

slices in L'_i for $i = 0, 1, 2$. For the other cases, the upper bound can be obtained by deleting the duplications or the overtwisted contact structures.

Suppose $t_0 \leq 1$. By Lemma 2.6, there are no appropriate tight contact structures with a 0-twisting vertical Legendrian circle. We consider the appropriate tight contact structures without a 0-twisting vertical Legendrian circle. By Lemma 2.7, we can factorize $\Sigma \times S^1 = \Sigma'' \times S^1 \cup L''_1 \cup L''_2$, where the boundary slopes of $\Sigma'' \times S^1$ are t_0, 0 and 0, and the boundary slopes of L''_i are 0 and $-\frac{1}{t_i}$. Since $t_0 \leq 1$, according to [14, Lemma 5.1], there are exactly $2 - t_0$ tight contact structures on $\Sigma'' \times S^1$ without a 0-twisting vertical Legendrian circle. There are 2 minimally twisting tight contact structures on L''_i. Therefore, there are at most $8 - 4t_0$ appropriate tight contact structures on $\Sigma \times S^1$ without a 0-twisting vertical Legendrian circle and with boundary slopes $s_0 = t_0$, $s_i = -\frac{1}{t_i}$ for $i = 1, 2$. ■

If $t_1 = t_2 = 2$, then the 18 contact structures on $\Sigma \times S^1$ with a 0-twisting vertical Legendrian circle are denoted using the decorations $(\pm)(\pm\pm)(\pm\pm)$. For $t_1 \geq 3$ and $t_2 = 2$, we use the decorations $(\pm)((\pm)(\pm))(\pm\pm)$ to represent the 24 contact structures on $\Sigma \times S^1$ with a 0-twisting vertical Legendrian circle. When $t_1 \geq 3$ and $t_2 \geq 3$, we use the decorations $(\pm)((\pm)(\pm))((\pm)(\pm))$ to signify the 32 contact structures on $\Sigma \times S^1$ with a 0-twisting vertical Legendrian circle. See Figure 5 for an example.

2.3 $t_1 < 0$ and $t_2 > 0$.

Lemma 2.12 *Suppose $t_1 < 0$ and $t_2 = 1$. Then, there are at most*

$$\begin{cases} 4 - 4t_1, & \text{if } t_0 \geq 4, \\ 4 - 3t_1, & \text{if } t_0 = 3, \\ 4 - 2t_1, & \text{if } t_0 = 2, \\ t_0 t_1 - 2t_1, & \text{if } t_0 \leq 1 \end{cases}$$

appropriate tight contact structures on $\Sigma \times S^1$ with the given boundary slopes.

Proof The boundary slopes of $\Sigma \times S^1$ are $s_0 = t_0$, $s_1 = -\frac{1}{t_1} > 0$, and $s_2 = -1$.

If $t_0 \geq 2$, then the tight contact structures on $\Sigma \times S^1$ always contain a 0-twisting vertical Legendrian circle.

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If $t_0 \geq 4$, $t_1 < 0$, and $t_2 = 1$, then there are at most $2 \times (1 - t_1) \times 2 = 4(1 - t_1)$ tight contact structures. They depend on the signs of the basic slices in L'_i for $i = 0, 1, 2$. For the other cases, the upper bound can be obtained by deleting the duplication or the overtwisted contact structures.

Suppose $t_0 \leq 1$. By Lemma 2.6, there are no appropriate tight contact structures with a 0-twisting vertical Legendrian circle. We consider the appropriate tight contact structures without a 0-twisting vertical Legendrian circle. By Lemma 2.7, we can factorize $\Sigma \times S^1 = \Sigma'' \times S^1 \cup L''_1$, where the boundary slopes of $\Sigma'' \times S^1$ are t_0, 0 and 1, and the boundary slopes of L''_1 are 0 and $-\frac{1}{t_1}$. Since $t_0 \leq 1$, according to [14, Lemma 5.1], there are exactly $2 - t_0$ tight contact structures on $\Sigma'' \times S^1$ without a 0-twisting vertical Legendrian circle. There are $-t_1$ minimally twisting tight contact structures on L''_1. Therefore, there are at most $-2t_1 + t_0 t_1$ tight contact structures on $\Sigma \times S^1$ without a 0-twisting vertical Legendrian circle and with boundary slopes $s_0 = t_0$, $s_i = -\frac{1}{t_i}$ for $i = 1, 2$. ∎

We use the decorations $(\pm)(\underbrace{\pm \cdots \pm}_{-t_1})(\pm)$ to denote the $4 - 4t_1$ contact structures on $\Sigma \times S^1$ with a 0-twisting vertical Legendrian circle.

Lemma 2.13 *Suppose $t_1 < 0$ and $t_2 > 1$. Then, there are at most*

$$\begin{cases} 6 - 6t_1, & \text{if } t_0 \geq 3, t_2 = 2, \\ 6 - 4t_1, & \text{if } t_0 = 2, t_2 = 2, \\ 6 - 2t_1, & \text{if } t_0 = 1, t_2 = 2, \\ 8 - 8t_1, & \text{if } t_0 \geq 3, t_2 \geq 3, \\ 8 - 6t_1, & \text{if } t_0 = 2, t_2 \geq 3, \\ 8 - 4t_1, & \text{if } t_0 = 1, t_2 \geq 4, \\ 8 - 3t_1, & \text{if } t_0 = 1, t_2 = 3, \\ 2t_0 t_1 - 2t_1, & \text{if } t_0 \leq 0, t_2 \geq 3 \end{cases}$$

appropriate tight contact structures on $\Sigma \times S^1$ with the given boundary slopes.

Proof The boundary slopes of $\Sigma \times S^1$ are $s_0 = t_0$, $s_1 = -\frac{1}{t_1} > 0$, and $s_2 = -\frac{1}{t_2} \in (-1, 0)$.

If $t_0 \geq 1$, then the tight contact structures on $\Sigma \times S^1$ always contain a 0-twisting vertical Legendrian circle.

If $t_0 \geq 3$, $t_1 < 0$, and $t_2 = 2$, then there are at most $2 \times (1 - t_1) \times 3 = 6(1 - t_1)$ appropriate tight contact structures. If $t_0 \geq 3$, $t_1 < 0$, and $t_2 \geq 3$, then there are at most $2 \times (1 - t_1) \times 4 = 8(1 - t_1)$ appropriate tight contact structures. The number of such contact structures depends on the signs of the basic slices in L'_i for $i = 0, 1, 2$. For the other cases, the upper bound can be obtained by deleting the duplication or the overtwisted contact structures.

Suppose $t_0 \leq 0$. By Lemma 2.6, there are no appropriate tight contact structures with a 0-twisting vertical Legendrian circle. We consider the appropriate tight contact structures without a 0-twisting vertical Legendrian circle. By Lemma 2.7, we can factorize $\Sigma \times S^1 = \Sigma'' \times S^1 \cup L''_1 \cup L''_2$, where the boundary slopes of $\Sigma'' \times S^1$ are t_0, 1 and 0, the boundary slopes of L''_1 are 1 and $-\frac{1}{t_1}$, and the boundary slopes of L''_2 are 0 and $-\frac{1}{t_2}$. Since $t_0 \leq 0$, according to [14, Lemma 5.1], there are exactly $1 - t_0$ tight contact structures on $\Sigma'' \times S^1$ without a 0-twisting vertical Legendrian circle. There are

$-t_1$ minimally twisting tight contact structures on L_1''. There are 2 minimally twisting tight contact structures on L_2''. Therefore, there are at most $-2t_1 + 2t_0 t_1$ appropriate tight contact structures on $\Sigma \times S^1$ without a 0-twisting vertical Legendrian circle and with boundary slopes $s_0 = t_0$, $s_i = -\frac{1}{t_i}$ for $i = 1, 2$.

When $t_2 = 2$, the $6 - 6t_1$ contact structures on $\Sigma \times S^1$ with a 0-twisting vertical Legendrian circle are denoted using the decorations $(\pm)(\underbrace{\pm \cdots \pm}_{-t_1})(\pm\pm)$. For $t_2 \geq 3$, we use the decorations $(\pm)(\underbrace{\pm \cdots \pm}_{-t_1})((\pm)(\pm))$ to represent the $8 - 8t_1$ contact structures on $\Sigma \times S^1$ with a 0-twisting vertical Legendrian circle.

If $t_0 = 1$, $t_1 < 0$, and $t_2 = 3$, we claim the two decorations

$$(+)(\underbrace{+ \cdots +}_{l}\underbrace{- \cdots -}_{k})((-)(+)) \text{ and } (-)(\underbrace{- \cdots -}_{k+1}\underbrace{+ \cdots +}_{l-1})((+)(-)),$$

where $l \geq 1, k \geq 0, k + l = -t_1$, denote the same contact structure on $\Sigma \times S^1$. We consider $L_0' \cup B_1' \cup L_2' \cup \Sigma' \times S^1$ in $\Sigma \times S^1$ with the first decoration, where B_1' is the inner most basic slice in L_1' with two boundary slopes ∞ and 1. We can assume the sign of B_1' is positive since L_1' is a continued fraction block containing at least one positive basic slice. As before, there is a convex vertical annulus A such that ∂A consists of a Legendrian ruling on T_0 and a Legendrian ruling on the boundary component of B_1' with slope 1, and the dividing set on A run from one boundary component to the other. If we cut $L_0' \cup B_1' \cup L_2' \cup \Sigma' \times S^1$ along A, we will obtain a thickened torus admitting a factorization into two basic slices with slopes $-\frac{1}{3}$, 0 and 0, -1, and opposite signs. Here, the slope -1 is obtained by $-1 - 1 + 1$. Using the same reasoning as in the proof of Lemma 2.10, we have an exchange from the first decoration to the second without altering the isotopy classes of contact structures. ∎

2.4 $t_1 = 0$.

Lemma 2.14 *Suppose $t_1 = 0$. Then, there are at most*

$$\begin{cases} 8, & \text{if } t_2 \geq 3, \\ 6, & \text{if } t_2 = 2, \\ 4, & \text{if } t_2 = 1, \\ 2 - 2t_2, & \text{if } t_2 \leq 0 \end{cases}$$

appropriate tight contact structures on $\Sigma \times S^1$ with the given boundary slopes. All of them have 0-twisting vertical Legendrian circles.

Proof Since $s_1 = \infty$, the appropriate tight contact structures on $\Sigma \times S^1$ always contain 0-twisting vertical Legendrian circles.

The boundary slopes of $\Sigma \times S^1$ are t_0, ∞ and $-\frac{1}{t_2}$. We can factorize $\Sigma \times S^1 = L_0' \cup L_2' \cup \Sigma' \times S^1$, where the boundary slopes of $\Sigma' \times S^1$ are all ∞, the boundary slopes of L_0' are ∞ and t_0, and the boundary slopes of L_2' are ∞ and $-\frac{1}{t_2}$. There are exactly 2 minimally twisting tight contact structures on L_0'. If $t_2 \leq 0$, $= 1$, $= 2$, or ≥ 3, then there are $1 - t_2$, 2, 3, or 4 minimally twisting tight contact structures on L_2', respectively.

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Therefore, if $t_2 \leq 0$, $= 1$, $= 2$, or ≥ 3, then there are $2 - 2t_2$, 4, 6, or 8 appropriate tight contact structures on $\Sigma \times S^1$, respectively. ∎

If $t_2 \geq 3$, the 8 contact structures on $\Sigma \times S^1$ are denoted using the decorations $(\pm)((\pm)(\pm))$. For $t_2 = 2$, we use the decorations $(\pm)(\pm\pm)$ to represent the 6 contact structures on $\Sigma \times S^1$. When $t_2 = 1$, we use the decorations $(\pm)(\pm)$ to denote the 4 contact structures on $\Sigma \times S^1$. If $t_2 \leq 0$, we use the decorations $(\pm)(\underbrace{\pm\cdots\pm}_{-t_2})$ to denote the $2 - 2t_2$ contact structures on $\Sigma \times S^1$.

2.5 Some tight contact structures

We use the notation $(T^2 \times [0,1], s_0, s_1)$ to represent a basic slice with boundary slopes s_0 and s_1 on $T^2 \times \{i\}$, where $i = 0, 1$. There is a geodesic in the Farey graph connecting s_0 and s_1. Moreover, any boundary parallel convex torus of this slice has a dividing slope within the range of $[s_0, s_1]$ corresponding to the clockwise arc on the boundary of the Poincare disk shown in Figure 3.

Lemma 2.15 *There are 6 tight contact structures on $\Sigma \times S^1$ with boundary slopes t_0, $-\frac{1}{t_1}$, and $-\frac{1}{t_2}$, where $t_1, t_2 \neq 0$, and satisfying that*

- *$\Sigma \times S^1$ can be decomposed as $L_0' \cup L_1' \cup L_2' \cup \Sigma' \times S^1$, where $\Sigma' \times S^1$ have boundary slopes ∞,*
- *L_0' is a basic slice,*
- *L_i', $i = 1, 2$, is a thickened torus, all of whose basic slices have the same signs,*
- *the signs of L_0', L_1', and L_2' are $\pm\mp\mp$, $\pm\mp\pm$ or $\pm\pm\mp$.*

Proof Suppose they have 0-twisting vertical Legendrian circles. By Lemma 2.3, each of them can be decomposed as $L_0' \cup L_1' \cup L_2' \cup \Sigma' \times S^1$, where the boundary slopes of $\Sigma' \times S^1$ are all ∞, L_0' is a basic slice $(T^2 \times [0,1]; \infty, t_0)$, and the innermost basic slice B_i' of L_i' is $(T^2 \times [0,1]; \infty, \lceil -\frac{1}{t_i} \rceil)$ for $i = 1, 2$. Using Part 2 of [14, Lemma 5.1], we know that there are 6 universally tight contact structures on $L_0' \cup B_1' \cup B_2' \cup \Sigma' \times S^1$ which are determined by the signs of L_0', B_1', and B_2'. Note that the three signs are not the same. Each of them can be extended to a universally tight $\tilde{\Sigma} \times S^1$ whose boundary slopes are all ∞. The contact structure on $L_0' \cup L_1' \cup L_2' \cup \Sigma' \times S^1$ can be embedded into $\tilde{\Sigma} \times S^1$. Hence, the given contact $\Sigma \times S^1$ is tight. ∎

Lemma 2.16 *There are 4 tight contact structures on $\Sigma \times S^1$ with boundary slopes t_0, ∞, and $-\frac{1}{t_2}$, where $t_2 \neq 0$, and satisfying that*

- *$\Sigma \times S^1$ can be decomposed as $L_0' \cup L_2' \cup \Sigma' \times S^1$, where $\Sigma' \times S^1$ have boundary slopes ∞,*
- *L_0' is a basic slice,*
- *L_2' is a thickened torus, all of whose basic slices have the same signs,*
- *the signs of L_0' and L_2' are $\pm\pm$ or $\pm\mp$.*

Proof Using [14, Lemma 5.2], the proof is similar to that of Lemma 2.15. ∎

3 Methods of construction of strongly exceptional Legendrian A_3 links

In practice, contact surgery diagrams are a common tool for representing strongly exceptional Legendrian links. Several works, such as [10], [11], [12], [16], and [9], employ this technique. In this paper, we utilize contact surgery diagrams to construct strongly exceptional Legendrian A_3 links. It is worth noting that if an exceptional Legendrian A_3 link can be constructed by this technique, then it must be strongly exceptional. This is because conducting contact surgery along such a Legendrian A_3 link results in a tight contact 3-manifold, whereas a Giroux torsion domain in $\Sigma \times S^1$ gives rise to an overtwisted disk after the surgery. Given a contact surgery diagram for an exceptional Legendrian A_3 link, the Thurston-Bennequin invariants and rotation numbers can be calculated using [15, Lemma 6.6]. Furthermore, the d_3-invariant of the ambient contact 3-sphere can be obtained according to [4].

Additionally, we introduce three other methods. The first method involves performing Legendrian connected sums of two Legendrian knots. The concept of Legendrian connected sums of Legendrian knots was defined in [8, Section 3].

Lemma 3.1 *Let $K_0' \cup K_1$ be a strongly exceptional Legendrian Hopf link in a contact $(S^3, \xi_{\frac{1}{2}})$ with $(t_0', r_0') = (t_1, r_1) = (1, 0)$ or $(t_0', r_0') = (0, \pm 1), t_1 \geq 2, r_1 = \pm(t_1 - 1)$. Let $K_0'' \cup K_2$ be a strongly exceptional Legendrian Hopf link in a contact S^3. Then, the Legendrian connected sum $(K_0' \# K_0'') \cup K_1 \cup K_2$ is a strongly exceptional Legendrian A_3 link in a contact S^3.*

Proof Suppose $t_0' = 0, t_1 \geq 1$. Let t_0'' be the Thurston-Bennequin invariant of K_0''. If the pair (t_0'', t_2) is not $(2, 1)$ or $(1, 2)$, then any strongly exceptional Legendrian Hopf link $K_0'' \cup K_2$ has a contact surgery diagram [11]. As a result, $(K_0' \# K_0'') \cup K_1 \cup K_2$ has a contact surgery diagram as shown in the middle and right of Figure 6. We then perform contact (-1)-surgery along K_1 and cancel the contact $(+1)$-surgery along the Legendrian unknots. By ignoring the Legendrian unknots with contact (-1)-surgeries, we obtain a contact surgery diagram for the Legendrian link $K_0'' \cup K_2$. As per [11], some contact surgeries along $K_0'' \cup K_2$ will result in closed tight contact 3-manifolds. Since contact (-1)-surgery on closed contact 3-manifold preserves tightness [17], some contact surgery along $(K_0' \# K_0'') \cup K_1 \cup K_2$ will yield a tight contact 3-manifold. Therefore, $(K_0' \# K_0'') \cup K_1 \cup K_2$ is strongly exceptional.

In the case where (t_0'', t_2) is either $(2, 1)$ or $(1, 2)$, [11] tells us that its exterior is a universally tight thickened torus and can therefore be contact embedded into a tight contact T^3. The contact (-1)-surgery along links in a tight contact T^3 results in a tight 3-manifold. As such, the contact (-1)-surgery along links in the exterior of $K_0'' \cup K_2$ will also yield a tight 3-manifold. Therefore, the contact (-1)-surgery along K_1 will result in a tight contact 3-manifold. This means that $(K_0' \# K_0'') \cup K_1 \cup K_2$ is strongly exceptional.

Assuming $t_0' = t_1 = 1$. If the pair (t_0'', t_2) is not $(2, 1)$ or $(1, 2)$, then $(K_0' \# K_0'') \cup K_1 \cup K_2$ will have a contact surgery diagram as shown in the left of Figure 6. We then perform contact $(-\frac{1}{2})$-surgery along K_1 and cancel the contact $(+1)$-surgery along the two Legendrian unknots. By doing so, we obtain a contact surgery diagram for

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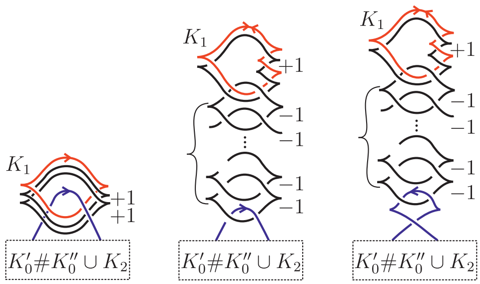

Figure 6: In the middle and right picture, for t_1 even, K_0' and K_1 bear the same orientation, and for t_1 odd, the opposite one.

the strongly exceptional Legendrian link $K_0''\cup K_2$. This means that the exterior of $(K_0'\#K_0'')\cup K_1\cup K_2$ is appropriate tight.

If the pair (t_0'', t_2) is either $(2,1)$ or $(1,2)$, we can apply the same argument as in the previous case. ∎

We recall that the d_3-invariant of the contact connected sum of two contact 3-spheres (S^3,ξ) and (S^3,ξ') is given by $d_3(\xi)+d_3(\xi')+\frac{1}{2}$. Suppose K_0'' has Thurston-Bennequin invariant t_0'' and rotation number r_0''. Then, $K_0'\#K_0''$ has Thurston-Bennequin invariant $t_0'+t_0''+1$ and rotation number $r_0'+r_0''$.

The second method involves adding local Legendrian meridians. In a contact 3-sphere, consider a Legendrian knot intersecting a Darboux ball in a simple arc. A Legendrian unknot within the Darboux ball, which serves as a meridian of the given Legendrian knot, is called a local Legendrian meridian. The following lemma is straightforward.

Lemma 3.2 *Suppose $K_0\cup K_2$ is a strongly exceptional Legendrian Hopf link. Let K_1 be a local Legendrian meridian of K_0. Then, $K_0\cup K_1\cup K_2$ is a strongly exceptional Legendrian A_3 link with $t_1<0$ and $r_1\in\{t_1+1, t_1+3,\cdots,-t_1-1\}$.*

The third method involves extending an (appropriate) tight contact $\Sigma\times S^1$ admitting a 0-twisting vertical Legendrian circle to an overtwisted contact S^3.

Suppose an (appropriate) tight contact structure ξ on $\Sigma\times S^1$ has a 0-twisting vertical Legendrian circle γ. We attach three contact solid tori $D_i^2\times S^1$, $i=0,1,2$, to $(\Sigma\times S^1,\xi)$ such that ∂D_0^2 is identified to h, ∂D_1^2 is identified to c_1, and ∂D_2^2 is identified to c_2. Then, the resulting manifold $\Sigma\times S^1\cup D_0^2\times S^1\cup D_1^2\times S^1\cup D_2^2\times S^1$ is diffeomorphic to S^3.

If the contact structure on $D_i^2\times S^1$ has a minimal convex boundary with slope given by a longitude (i.e., the dividing set of the convex boundary intersects the meridional circle in exactly two points), then it admits a unique tight contact structure. Additionally, the core of such a contact solid torus is Legendrian.

Since the dividing set of T_i intersects the meridional disk of $D_i^2\times S^1$ in exactly two points, the contact structure ξ on $\Sigma\times S^1$ uniquely extends to a contact structure on S^3. However, since ∂D_0^2 is identified to h, the Legendrian vertical circle γ bounds an overtwisted disk in S^3. Therefore, the resulting contact structure on S^3 is overtwisted.

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Lemma 3.3 *Let ξ be an (appropriate) tight contact structure on $\Sigma \times S^1$ that admits a 0-twisting vertical Legendrian circle. Extending ξ to a contact 3-sphere as above by adding three tight contact solid tori. Let K_i, $i = 0,1,2$, be the core of three attached contact solid tori. Then, $K_0 \cup K_1 \cup K_2$ is a (strongly) exceptional Legendrian A_3 link in an overtwisted contact 3-sphere.*

Moreover, we have the following observations.

Lemma 3.4 *Let ξ_1 and ξ_2 be two tight contact structures on $\Sigma \times S^1$ with 0-twisting vertical Legendrian circles. Suppose they both have minimal convex boundaries with slopes t_0, $-\frac{1}{t_1}$, and $-\frac{1}{t_2}$. Suppose their factorizations $L_0' \cup L_1' \cup L_2' \cup \Sigma' \times S^1$ (or $L_0' \cup L_2' \cup \Sigma' \times S^1$ when $t_1 = 0$) differ only in the signs of basic slices in $L_0' \cup L_1' \cup L_2'$ (or $L_0' \cup L_2'$ when $t_1 = 0$). If ξ_1 is appropriate tight, then so is ξ_2.*

Proof This is because the computation of Giroux torsion of an embedded torus T in a contact 3-manifold only depends on the slopes of the convex tori parallel to T. ∎

Lemma 3.5 *Suppose $\mathcal{L}$ is an exceptional Legendrian A_3 link whose exterior contains a 0-twisting Legendrian vertical circle. Then, the components K_0 and K_i with $t_i \neq 0$, where $i = 1,2$, of $\mathcal{L}$ can always be destabilized.*

Proof There is a basic slice L_0' in the exterior of $\mathcal{L}$ which is $(T^2 \times [0,1], \infty, t_0)$. We can find a basic slice $(T^2 \times [0,1], t_0 + 1, t_0)$ in L_0'. So the component K_0 can be destabilized. For $i = 1,2$, since there is a basic slice $(T^2 \times [0,1], -\frac{1}{t_i+1}, -\frac{1}{t_i})$ in the thickened torus L_i', the component K_i can be destabilized. ∎

4 Realizations of strongly exceptional Legendrian A_3 links

In this section, we construct strongly exceptional Legendrian A_3 links.

Throughout this paper, in the contact surgery diagrams representing a Legendrian A_3 link, if a component is a Legendrian push-off of some K_i, $i = 0,1,2$, then its contact surgery coefficient is +1; otherwise, its contact surgery coefficient is −1.

4.1 $t_1 < 0$ and $t_2 < 0$.

The boundary slopes of $\Sigma \times S^1$ are $s_0 = t_0$, $s_1 = -\frac{1}{t_1} \in (0,1]$, and $s_2 = -\frac{1}{t_2} \in (0,1]$.

Lemma 4.1 *For any $t_0 \in \mathbb{Z}$, there are 6 exceptional Legendrian A_3 links whose exteriors have 0-twisting vertical Legendrian circles and have decorations $\pm(+)(\underbrace{-\cdots-}_{-t_1})(\underbrace{-\cdots-}_{-t_2})$, $\pm(+)(\underbrace{-\cdots-}_{-t_1})(\underbrace{+\cdots+}_{-t_2})$ and $\pm(+)(\underbrace{+\cdots+}_{-t_1})(\underbrace{-\cdots-}_{-t_2})$. Their rotation numbers are*

$$r_0 = \pm(t_0 - 1), r_1 = \pm(1 - t_1), r_2 = \pm(1 - t_2);$$
$$r_0 = \pm(t_0 - 1), r_1 = \pm(1 - t_1), r_2 = \pm(t_2 + 1);$$
$$r_0 = \pm(t_0 - 1), r_1 = \pm(t_1 + 1), r_2 = \pm(1 - t_2).$$

The corresponding d_3-invariants are independent of t_0 if t_1 and t_2 are fixed.

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Proof The first statement follows from Lemma 2.15 and Lemma 3.3. The rotation number of a Legendrian knot in a contact 3-sphere is the evaluation of the relative Euler class on a Seifert surface of the knot. We compute the rotation numbers in a similar way as that in [9, Section 2.5]. The Seifert surface of K_0 can be obtained by capping the pair of pants Σ by two disks along the boundary components c_1 and c_2. The Seifert surface of K_i, $i = 1, 2$, is a union of a meridian disk of K_0 and an annulus. For instance, if the signs of L'_0, L'_1, and L'_2 are $+--$ (see Figure 4 for an example), then the rotation numbers can be computed using relative Euler class as follows. We denote $\frac{a}{b} \ominus \frac{c}{d}$ to be $\frac{a-c}{b-d}$, and $\frac{a}{b} \bullet \frac{c}{d}$ to be $ad - bc$ [9, Section 2.5]. The denominators are assumed to be nonnegative. The rotation number of K_0 is

$$\begin{aligned} r_0 = &-\left(\frac{-1}{-t_1} \ominus \frac{-1}{-t_1-1}\right) \bullet \frac{0}{1} - \left(\frac{-1}{-t_1-1} \ominus \frac{-1}{-t_1-2}\right) \bullet \frac{0}{1} - \cdots - \left(\frac{-1}{1} \ominus \frac{-1}{0}\right) \bullet \frac{0}{1} \\ &-\left(\frac{-1}{-t_2} \ominus \frac{-1}{-t_2-1}\right) \bullet \frac{0}{1} - \left(\frac{-1}{-t_2-1} \ominus \frac{-1}{-t_2-2}\right) \bullet \frac{0}{1} - \cdots - \left(\frac{-1}{1} \ominus \frac{-1}{0}\right) \bullet \frac{0}{1} \\ &+\left(\frac{1}{0} \ominus \frac{t_0}{1}\right) \bullet \frac{0}{1} = 1 - t_0. \end{aligned}$$

The rotation number of K_1 is

$$\begin{aligned} r_1 &= \left(\frac{-t_0}{1} \ominus \frac{-1}{0}\right) \bullet \frac{1}{0} - \left(\frac{1}{0} \ominus \frac{1}{1}\right) \bullet \frac{1}{0} - \left(\frac{1}{1} \ominus \frac{1}{2}\right) \bullet \frac{1}{0} - \cdots - \left(\frac{1}{-t_1-1} \ominus \frac{1}{-t_1}\right) \bullet \frac{1}{0} \\ &= t_1 - 1. \end{aligned}$$

The rotation number of K_2 is

$$\begin{aligned} r_2 &= \left(\frac{-t_0}{1} \ominus \frac{-1}{0}\right) \bullet \frac{1}{0} - \left(\frac{1}{0} \ominus \frac{1}{1}\right) \bullet \frac{1}{0} - \left(\frac{1}{1} \ominus \frac{1}{2}\right) \bullet \frac{1}{0} - \cdots - \left(\frac{1}{-t_2-1} \ominus \frac{1}{-t_2}\right) \bullet \frac{1}{0} \\ &= t_2 - 1. \end{aligned}$$

In the computation above, when calculating r_0, it is necessary to reverse the signs of the dividing slopes in the thickened tori L'_1 and L'_2. Similarly, when calculating r_1 and r_2, the signs of the dividing slopes in the thickened torus L'_0 should be reversed.

The last statement follows directly from Lemma 3.5. ■

In a similar way, we can use relative Euler classes and the given decorations to compute the rotation numbers of any other Legendrian A_3 links whose exteriors contain a 0-twisting vertical Legendrian circle.

Proof of Theorem 1.1 Recall that the numbers of strongly exceptional Legendrian A_3 links have upper bounds listed in Lemma 2.8. We will show that these upper bounds can be attained.

Lemma 4.2 *The oriented link $K_0 \cup K_1 \cup K_2$ in the surgery diagram in Figure 7 is a topological A_3 link in S^3.*

Proof The proof is similar to that of [11, Lemma 5.1, part (i)]. ■

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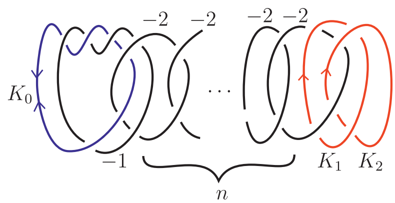

Figure 7: For n even, K_0 and K_i, $i = 1, 2$, bear the same orientation, and for n odd, the opposite one.

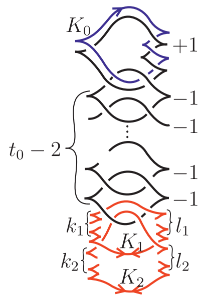

Figure 8: $t_0 \geq 2$, $t_1 \leq 0$, $t_2 \leq 0$. For $i = 1, 2$, $k_i + l_i = -t_i$. For t_0 even, K_0 and K_i, $i = 1, 2$, bear the same orientation, and for t_0 odd, the opposite one.

(1) Suppose $t_0 \geq 2$.

Lemma 4.3 *If $t_0 \geq 2, t_1 < 0, t_2 < 0$, there exist $2t_1t_2 - 2t_1 - 2t_2 + 2$ strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{1}{2}})$ whose rotation numbers are*

$$r_0 = \pm(t_0 - 1), r_i \in \pm\{t_i + 1, t_i + 3, \cdots, -t_i + 1\}, i = 1, 2.$$

Proof There are $2t_1t_2 - 2t_1 - 2t_2 + 2$ strongly exceptional Legendrian A_3 links as illustrated in Figure 8. According to Lemma 4.2, $K_0 \cup K_1 \cup K_2$ forms a topological A_3 link. By performing the same calculations as in the proof of Theorem 1.2 (b1) in [11], we can determine that their rotation numbers are as listed. The corresponding d_3-invariant is $\frac{1}{2}$. The strong exceptionality property arises from carrying out contact (-1)-surgery along K_0 which cancels the contact $(+1)$-surgery. ∎

(2) Suppose $t_0 = 1$.

Lemma 4.4 *If $t_0 = 1, t_1 < 0, t_2 < 0$, then there exist $t_1t_2 - 2t_1 - 2t_2 + 2$ strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{1}{2}})$ whose rotation numbers are*

$$r_0 = 0, r_i \in \{t_i + 1, t_i + 3, \cdots, -t_i + 1\}, i = 1, 2;$$
$$r_0 = 0, r_1 = t_1 - 1, r_2 \in \{t_2 - 1, t_2 + 1, \cdots, -t_2 - 1\};$$
$$r_0 = 0, r_1 \in \{t_1 + 1, t_1 + 3, \cdots, -t_1 - 1\}, r_2 = t_2 - 1.$$

Proof There are $t_1t_2 - 2t_1 - 2t_2 + 2$ strongly exceptional Legendrian A_3 links as shown in Figure 9. The linking number of the components K_1 and K_2 in Figure 9

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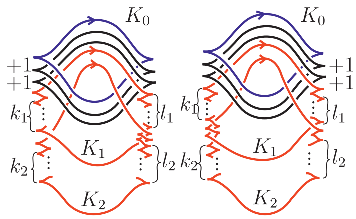

Figure 9: $t_0 = 1$, $t_1 < 0$, $t_2 < 0$. For $i = 1, 2$, $k_i + l_i = -t_i + 1$. In the left diagram, $l_1, l_2 \geq 1$, while in the right diagram, $k_1, k_2 \geq 1$.

is -2. Using similar Kirby diagrams as in [11, Lemma 5.1, part (iii), Figure 3], we can show that it is a topological A_3 link. By performing the same calculations as in the proof of Theorem 1.2 (b2) in [11], we can determine that their rotation numbers are as listed. In the left diagram of Figure 9,

$$r_0 = 0, r_i \in \{t_i + 1, t_i + 3, \cdots, -t_i + 1\} \text{ for } i = 1, 2,$$

while in the right diagram,

$$r_0 = 0, r_i \in \{t_i - 1, t_i + 1, \cdots, -t_i - 1\} \text{ for } i = 1, 2.$$

There are exactly $t_1 t_2$ Legendrian A_3 links represented by both the left and the right diagrams. Moreover, the corresponding d_3-invariant is $\frac{1}{2}$. ∎

(3) Suppose $t_0 = 0$.

Lemma 4.5 *If $t_0 = 0$, $t_1 < 0$, $t_2 < 0$, then there exist $-2t_1 - 2t_2 + 2$ strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{1}{2}})$ whose rotation numbers are*

$$r_0 = \pm 1, r_1 = \pm(t_1 - 1), r_2 \in \{t_2 + 1, t_2 + 3, \cdots, -t_2 - 1\};$$
$$r_0 = \pm 1, r_1 \in \{t_1 + 1, t_1 + 3, \cdots, -t_1 - 1\}, r_2 = \pm(t_2 - 1);$$
$$r_0 = \pm 1, r_1 = \pm(t_1 - 1), r_2 = \pm(t_2 - 1).$$

Proof By [11, Theorem 1.2], there are two strongly exceptional Legendrian Hopf links $K_0 \cup K_1$ in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0, r_0) = (0, \pm 1)$, $t_1 < 0$ and $r_1 = \pm(t_1 - 1)$. Let K_2 be a local Legendrian meridian of K_0. Then by Lemma 3.2, there are $-2t_2$ strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{1}{2}})$ whose rotation numbers are

$$r_0 = \pm 1, r_1 = \pm(t_1 - 1), r_2 \in \{t_2 + 1, t_2 + 3, \cdots, -t_2 - 1\}.$$

Similarly, there are $-2t_1$ strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{1}{2}})$ whose rotation numbers are

$$r_0 = \pm 1, r_1 \in \{t_1 + 1, t_1 + 3, \cdots, -t_1 - 1\}, r_2 = \pm(t_2 - 1).$$

Moreover, by Lemma 4.1 and Lemma 3.4, there are 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{1}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm 1, \pm(t_1 - 1), \pm(t_2 - 1))$. ∎

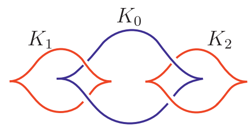

Figure 10: A Legendrian A_3 link in (S^3, ξ_{st}).

(4) Suppose $t_0 < 0$.

Lemma 4.6 *If $t_i < 0$ for $i = 0, 1, 2$, then there exist $-t_0 t_1 t_2$ strongly exceptional Legendrian A_3 links in (S^3, ξ_{st}) whose rotation numbers are*

$$r_i \in \{t_i + 1, t_i + 3, \cdots, -t_i - 1\}, \textit{ for } i = 0, 1, 2.$$

Proof By stabilizations of the Legendrian A_3 link shown in Figure 10, we obtain $-t_0 t_1 t_2$ strongly exceptional Legendrian A_3 links in (S^3, ξ_{st}). Their rotation numbers are as listed. ■

So there are exactly $-t_0 t_1 t_2$ Legendrian A_3 links in contact 3-spheres whose complements are appropriate tight if $t_i < 0$ for $i = 0, 1, 2$.

The proof of Theorem 1.1 is completed.

4.2 $t_1 > 0$ and $t_2 > 0$.

The boundary slopes of $\Sigma \times S^1$ are $s_0 = t_0$, $s_1 = -\frac{1}{t_1} \in [-1, 0)$, and $s_2 = -\frac{1}{t_2} \in [-1, 0)$.

Lemma 4.7 *For any $t_0 \in \mathbb{Z}$, there are 6 exceptional Legendrian A_3 links whose exteriors have 0-twisting vertical Legendrian circles, and the signs of basic slices in L_0', L_1', L_2' are $\pm(+--)$, $\pm(++-)$, and $\pm(+-+)$, respectively. Their rotation numbers are*

$$r_0 = \pm(t_0 + 3), r_1 = \pm(t_1 + 1), r_2 = \pm(t_2 + 1);$$
$$r_0 = \pm(t_0 - 1), r_1 = \pm(1 - t_1), r_2 = \pm(t_2 + 1);$$
$$r_0 = \pm(t_0 - 1), r_1 = \pm(t_1 + 1), r_2 = \pm(1 - t_2).$$

The corresponding d_3-invariants are independent of t_0 if t_1 and t_2 are fixed.

Proof The first statement can be inferred from Lemma 2.15 and Lemma 3.3. For example, when the signs of L_0', L_1', and L_2' are $+--$, the rotation numbers can be computed using the relative Euler class as follows. See Figure 5 for the decoration. The rotation number of K_0 is

$$r_0 = -\left(\frac{1}{t_1} \ominus \frac{0}{1}\right) \bullet \frac{0}{1} - \left(\frac{0}{1} \ominus \frac{-1}{0}\right) \bullet \frac{0}{1} - \left(\frac{1}{t_2} \ominus \frac{0}{1}\right) \bullet \frac{0}{1} - \left(\frac{0}{1} \ominus \frac{-1}{0}\right) \bullet \frac{0}{1}$$
$$+ \left(\frac{1}{0} \ominus \frac{t_0}{1}\right) \bullet \frac{0}{1} = -t_0 - 3.$$

The rotation number of K_1 is

$$r_1 = \left(\frac{-t_0}{1} \ominus \frac{-1}{0}\right) \bullet \frac{1}{0} - \left(\frac{1}{0} \ominus \frac{0}{1}\right) \bullet \frac{1}{0} - \left(\frac{0}{1} \ominus \frac{-1}{t_1}\right) \bullet \frac{1}{0} = -t_1 - 1.$$

The rotation number of K_2 is

$$r_2 = \left(\frac{-t_0}{1} \ominus \frac{-1}{0}\right) \bullet \frac{1}{0} - \left(\frac{1}{0} \ominus \frac{0}{1}\right) \bullet \frac{1}{0} - \left(\frac{0}{1} \ominus \frac{-1}{t_2}\right) \bullet \frac{1}{0} = -t_2 - 1.$$

■

4.2.1 $t_1 = t_2 = 1$.

Proof of Theorem 1.2 The upper bound of strongly exceptional Legendrian A_3 links is given by Lemma 2.9. We will show that these upper bounds can be attained.

(1) Suppose $t_0 \geq 6$.

Lemma 4.8 *If $t_0 \geq 6$, $t_1 = t_2 = 1$, then there exist 8 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3 invariants $(r_0, r_1, r_2; d_3)$ are*

$$\left(\pm(t_0+3), \pm 2, \pm 2; -\frac{3}{2}\right), \left(\pm(t_0-1), \pm 2, 0; \frac{1}{2}\right), \left(\pm(t_0-1), 0, \pm 2; \frac{1}{2}\right), \left(\pm(t_0-5), 0, 0; \frac{5}{2}\right).$$

Proof There exist 8 strongly exceptional Legendrian A_3 links shown in Figure 11. Using the trick of Lemma 4.2, the upper branch in each of the surgery diagrams can be topologically reduced to a single unknot, and the lower two branches in each of the surgery diagrams can be split. Furthermore, using the trick in the proof of [11, Lemma 5.1, part (ii), Figure 5], we can show that $K_0 \cup K_1 \cup K_2$ is a topological A_3 link. Their rotation numbers are

$$r_0 = \pm(t_0+3), r_1 = \pm 2, r_2 = \pm 2; r_0 = \pm(t_0-1), r_1 = \pm 2, r_2 = 0;$$

$$r_0 = \pm(t_0-1), r_1 = 0, r_2 = \pm 2; r_0 = \pm(t_0-5), r_1 = r_2 = 0.$$

The corresponding d_3-invariants are $-\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{5}{2}$. These d_3-invariants are calculated using the algorithm described in [4]. ■

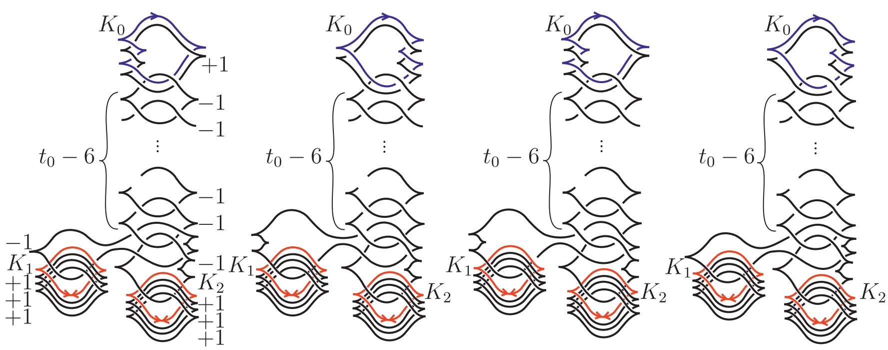

Figure 11: $t_0 \geq 6$, $t_1 = t_2 = 1$. For t_0 even, K_0 and K_i, $i = 1, 2$, bear the same orientation, and for t_0 odd, the opposite one.

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(2) Suppose $t_0 = 5$.

Lemma 4.9 *If $t_0 = 5, t_1 = t_2 = 1$, then there exist 7 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3 invariants $(r_0, r_1, r_2; d_3)$ are*

$$\left(\pm 4, 0, \pm 2; \frac{1}{2}\right), \left(0, 0, 0; \frac{5}{2}\right), \left(\pm 4, \pm 2, 0; \frac{1}{2}\right), \left(\pm 8, \pm 2, \pm 2; -\frac{3}{2}\right).$$

Proof By [11, Theorem 1.2, (c1), (c2)], there is a Legendrian Hopf link $K_0' \cup K_1$ in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0', r_0') = (t_1, r_1) = (1, 0)$, two Legendrian Hopf links $K_0'' \cup K_2$ in $(S^3, \xi_{-\frac{1}{2}})$ with $(t_0'', r_0'') = (3, \pm 4), (t_2, r_2) = (1, \pm 2)$, and a Legendrian Hopf links $K_0'' \cup K_2$ in $(S^3, \xi_{\frac{3}{2}})$ with $(t_0'', r_0'') = (3, 0), (t_2, r_2) = (1, 0)$. Connected summing K_0' and K_0'', by Lemma 3.1, we obtain 3 strongly exceptional Legendrian A_3 links with $t_0 = 5, t_1 = t_2 = 1$. Their rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are $(\pm 4, 0, \pm 2; \frac{1}{2})$ and $(0, 0, 0; \frac{5}{2})$.

By exchanging the roles of K_1 and K_2, we obtain 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{1}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm 4, \pm 2, 0)$.

By Lemma 4.7 and Lemma 3.4, there are 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{-\frac{3}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm 8, \pm 2, \pm 2)$. Their exteriors have decorations $\pm(+)(-)(-)$. ∎

(3) Suppose $t_0 = 4$.

Lemma 4.10 *If $t_0 = 4, t_1 = t_2 = 1$, then there exist 6 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3 invariants $(r_0, r_1, r_2; d_3)$ are*

$$\left(\pm 3, 0, \pm 2; \frac{1}{2}\right), \left(\pm 3, \pm 2, 0; \frac{1}{2}\right), \left(\pm 7, \pm 2, \pm 2; -\frac{3}{2}\right).$$

Proof Suppose $t_0 = 4$. By [11, Theorem 1.2], there is a Legendrian Hopf link $K_0' \cup K_1$ in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0', r_0') = (t_1, r_1) = (1, 0)$, and two Legendrian Hopf links $K_0'' \cup K_2$ in $(S^3, \xi_{-\frac{1}{2}})$ with $(t_0'', r_0'') = (2, \pm 3), (t_2, r_2) = (1, \pm 2)$. Connected summing K_0' and K_0'', by Lemma 3.1, we obtain 2 strongly exceptional Legendrian A_3 link with $t_0 = 4, t_1 = t_2 = 1$ in $(S^3, \xi_{\frac{1}{2}})$. Their rotation numbers (r_0, r_1, r_2) are $(\pm 3, 0, \pm 2)$.

By exchanging the roles of K_1 and K_2, we obtain 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{1}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm 3, \pm 2, 0)$.

By Lemma 4.7 and Lemma 3.4, there are 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{-\frac{3}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm 7, \pm 2, \pm 2)$. Their exteriors have decorations $\pm(+)(-)(-)$. ∎

(4) Suppose $t_0 \leq 3$.

Lemma 4.11 *If $t_0 \leq 3, t_1 = t_2 = 1$, then there exist $4 - t_0$ strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{3}{2}})$ whose rotation numbers are*

$$r_0 \in \{t_0 - 3, t_0 - 1, \cdots, 3 - t_0\}, r_1 = r_2 = 0.$$

Proof Suppose $t_0 \leq 3$. By [11, Theorem 1.2, (c1), (b2)], there is a Legendrian Hopf link $K_0' \cup K_1$ in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0', r_0') = (t_1, r_1) = (1, 0)$, and a Legendrian Hopf link $K_0'' \cup K_2$ in $(S^3, \xi_{\frac{1}{2}})$ with $t_0'' \leq 1, r_0'' \in \{t_0'' - 1, t_0'' + 1, \cdots, -t_0'' + 1\}, (t_2, r_2) = (1, 0)$.

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By Lemma 3.1, we can construct $4-t_0$ strongly exceptional Legendrian A_3 links in $(S^3,\xi_{\frac{3}{2}})$ with $t_0 \le 3$, $t_1=t_2=1$. Their rotation numbers are as listed. ∎

These $4-t_0$ strongly exceptional Legendrian A_3 links are obtained by stabilizations along K_0 of the Legendrian A_3 link with $t_0=3$, $t_1=t_2=1$.

The proof of Theorem 1.2 is completed.

4.2.2 $t_1 \ge 2$ and $t_2 = 1$.

The boundary slopes of $\Sigma\times S^1$ are $s_0=t_0$, $s_1=-\frac{1}{t_1}$, and $s_2=-1$.

Proof of Theorem 1.3 The upper bound of strongly exceptional Legendrian A_3 links is given by Lemma 2.10. We will show that these upper bounds can be attained.

(1) Suppose $t_0\ge 5$ and $t_1=2$.

Lemma 4.12 *If $t_0\ge 5$, $t_1=2$, and $t_2=1$, then there exist 12 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3 invariants $(r_0,r_1,r_2;d_3)$ are*

$$\left(\pm(t_0-5),\mp 1,0;\frac{5}{2}\right),\left(\pm(t_0-3),\pm 1,0;\frac{5}{2}\right),\left(\pm(t_0-1),\pm 3,0;\frac{1}{2}\right),$$
$$\left(\pm(t_0-1),\mp 1,\pm 2;\frac{1}{2}\right),\left(\pm(t_0+1),\pm 1,\pm 2;\frac{1}{2}\right),\left(\pm(t_0+3),\pm 3,\pm 2;-\frac{3}{2}\right).$$

Proof There exist 12 strongly exceptional Legendrian A_3 links shown in Figure 12. Using the trick of Lemma 4.2 and the proof of [11, Theorem 1.2, (c3)], we can show that $K_0\cup K_1\cup K_2$ is a topological A_3 link. Their rotation numbers and corresponding d_3-invariants are as listed. ∎

(2) Suppose $t_0=4$ and $t_1=2$.

Lemma 4.13 *If $t_0=4$, $t_1=2$, and $t_2=1$, then there exist 10 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0,r_1,r_2;d_3)$ are*

$$\left(\pm 5,\pm 1,\pm 2;\frac{1}{2}\right),\left(\pm 3,\mp 1,\pm 2;\frac{1}{2}\right),\left(\pm 1,\pm 1,0;\frac{5}{2}\right),\left(\pm 3,\pm 3,0;\frac{1}{2}\right),\left(\pm 7,\pm 3,\pm 2;-\frac{3}{2}\right).$$

Proof By [11, Theorem 1.2, (c2), (d)], there are two Legendrian Hopf links $K_0'\cup K_1$ in $(S^3,\xi_{\frac{1}{2}})$ with $(t_0',r_0')=(0,\pm 1),(t_1,r_1)=(2,\pm 1)$, two Legendrian Hopf links $K_0''\cup K_2$ in $(S^3,\xi_{-\frac{1}{2}})$ with $(t_0'',r_0'')=(3,\pm 4),(t_2,r_2)=(1,\pm 2)$, and a Legendrian Hopf link $K_0''\cup K_2$ in $(S^3,\xi_{\frac{3}{2}})$ with $(t_0'',r_0'')=(3,0),(t_2,r_2)=(1,0)$. By Lemma 3.1, we can obtain 6 strongly exceptional Legendrian A_3 links whose rotation numbers and d_3-invariants $(r_0,r_1,r_2;d_3)$ are $(\pm 5,\pm 1,\pm 2;\frac{1}{2})$, $(\pm 3,\mp 1,\pm 2;\frac{1}{2})$, and $(\pm 1,\pm 1,0;\frac{5}{2})$.

By [11, Theorem 1.2, (c1), (c2)], there is a Legendrian Hopf link $K_0'\cup K_2$ in $(S^3,\xi_{\frac{1}{2}})$ with $(t_0',r_0')=(t_2,r_2)=(1,0)$, and four Legendrian Hopf links $K_0''\cup K_1$ with $(t_0'',r_0'')=(t_1,r_1)=(2,\pm 3)$ in $(S^3,\xi_{-\frac{1}{2}})$ or $(2,\pm 1)$ in $(S^3,\xi_{\frac{3}{2}})$. By Lemma 3.1, we can obtain 2 more strongly exceptional Legendrian A_3 links whose rotation numbers and d_3-invariants $(r_0,r_1,r_2;d_3)$ are $(\pm 3,\pm 3,0;\frac{1}{2})$.

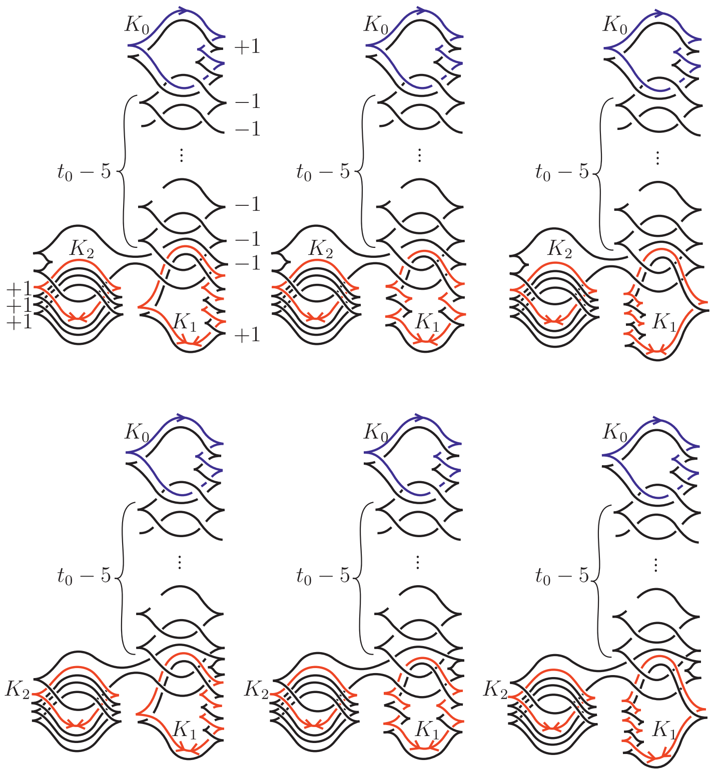

Figure 12: $t_0 \geq 5$, $t_1 = 2$, $t_2 = 1$. For t_0 odd, K_0 and K_2 bear the same orientation, while K_0 and K_1 bear the opposite orientation. For t_0 even, K_0 and K_1 bear the same orientation, while K_0 and K_2 bear the opposite orientation.

By Lemma 4.7 and Lemma 3.4, there exist 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{-\frac{3}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm 7, \pm 3, \pm 2)$. Their exteriors have decorations $\pm(+)(--)(-)$. ■

So there exist 10 strongly exceptional Legendrian A_3 links with $t_0 = 4, t_1 = 2, t_2 = 1$. As a corollary, the 10 contact structures on $\Sigma \times S^1$ with boundary slopes $s_0 = 4, s_1 = -\frac{1}{2}, s_2 = -1$ listed in Lemma 2.10 are all appropriate tight.

(3) Suppose $t_0 = 3$ and $t_1 = 2$.

Lemma 4.14 *If $t_0 = 3$, $t_1 = 2$, and $t_2 = 1$, then there exist 8 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3 invariants $(r_0, r_1, r_2; d_3)$ are*

$$\left(\pm 2, \pm 3, 0; \frac{1}{2}\right), \left(\pm 2, \mp 1, \pm 2; \frac{1}{2}\right), \left(\pm 4, \pm 1, \pm 2; \frac{1}{2}\right), \left(\pm 6, \pm 3, \pm 2; -\frac{3}{2}\right).$$

Proof By [11, Theorem 1.2, (c2), (c1)], there are two Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{-\frac{1}{2}})$ with $(t_0', r_0') = (1, \pm 2), (t_1, r_1) = (2, \pm 3)$, and one Legendrian Hopf link

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$K_0'' \cup K_2$ in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0'', r_0'') = (t_2, r_2) = (1, 0)$. By Lemma 3.1, we can obtain 2 strongly exceptional Legendrian A_3 links whose rotation numbers and d_3-invariants $(r_0, r_1, r_2; d_3)$ are $(\pm 2, \pm 3, 0; \frac{1}{2})$.

By [11, Theorem 1.2, (d), (c2)], there are two Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0', r_0') = (0, \pm 1), (t_1, r_1) = (2, \pm 1)$, and two Legendrian Hopf links $K_0'' \cup K_2$ in $(S^3, \xi_{-\frac{1}{2}})$ with $(t_0'', r_0'') = (2, \pm 3), (t_2, r_2) = (1, \pm 2)$. By Lemma 3.1, we can obtain 4 strongly exceptional Legendrian A_3 links whose rotation numbers and d_3-invariants $(r_0, r_1, r_2; d_3)$ are $(\pm 2, \mp 1, \pm 2; \frac{1}{2})$ and $(\pm 4, \pm 1, \pm 2; \frac{1}{2})$.

By Lemma 4.7 and Lemma 3.4, there are 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{-\frac{3}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm 6, \pm 3, \pm 2)$. Their exteriors have decorations $\pm(+)(--)(-)$. ■

So there are 8 strongly exceptional Legendrian A_3 links with $t_0 = 3, t_1 = 2, t_2 = 1$. As a corollary, the 8 contact structures on $\Sigma \times S^1$ with boundary slopes $s_0 = 3, s_1 = -\frac{1}{2}, s_2 = -1$ listed in Lemma 2.10 are all appropriate tight.

(4) Suppose $t_0 \geq 5$ and $t_1 \geq 3$.

Lemma 4.15 *If $t_0 \geq 5$, $t_1 \geq 3$, and $t_2 = 1$, then there exist 16 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3 invariants $(r_0, r_1, r_2; d_3)$ are*

$$\left(\pm(t_0+1), \pm(t_1-1), \pm 2; \frac{1}{2}\right), \left(\pm(t_0+3), \pm(t_1+1), \pm 2; -\frac{3}{2}\right),$$
$$\left(\pm(t_0-1), \pm(1-t_1), \pm 2; \frac{1}{2}\right), \left(\pm(t_0+1), \pm(3-t_1), \pm 2; \frac{1}{2}\right),$$
$$\left(\pm(t_0-3), \pm(t_1-1), 0; \frac{5}{2}\right), \left(\pm(t_0-1), \pm(t_1+1), 0; \frac{1}{2}\right),$$
$$\left(\pm(t_0-5), \pm(1-t_1), 0; \frac{5}{2}\right), \left(\pm(t_0-3), \pm(3-t_1), 0; \frac{5}{2}\right).$$

Proof There exist 16 strongly exceptional Legendrian A_3 links shown in Figure 13. Using the trick of Lemma 4.2 and the proof of [11, Theorem 1.2, (c3), (c4)], we can show that $K_0 \cup K_1 \cup K_2$ is a topological A_3 link. Their rotation numbers and corresponding d_3-invariants are as listed. ■

(5) Suppose $t_0 = 4$ and $t_1 \geq 3$.

Lemma 4.16 *If $t_0 = 4$, $t_1 \geq 3$, and $t_2 = 1$, then there exist 14 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3 invariants $(r_0, r_1, r_2; d_3)$ are*

$$\left(\pm 3, \pm(t_1+1), 0; \frac{1}{2}\right), \left(\mp 1, \pm(1-t_1), 0; \frac{5}{2}\right), \left(\pm 1, \pm(3-t_1), 0; \frac{5}{2}\right),$$
$$\left(\pm 5, \pm(t_1-1), \pm 2; \frac{1}{2}\right), \left(\pm 3, \pm(1-t_1), \pm 2; \frac{1}{2}\right),$$
$$\left(\pm 7, \pm(t_1+1), \pm 2; -\frac{3}{2}\right), \left(\pm 5, \pm(3-t_1), \pm 2; \frac{1}{2}\right).$$

Proof By [11, Theorem 1.2, (c3), (c1)], there are two Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{-\frac{1}{2}})$ with $(t_0', r_0') = (2, \pm 3), t_1 \geq 3, r_1 = \pm(t_1+1)$, two Legendrian Hopf

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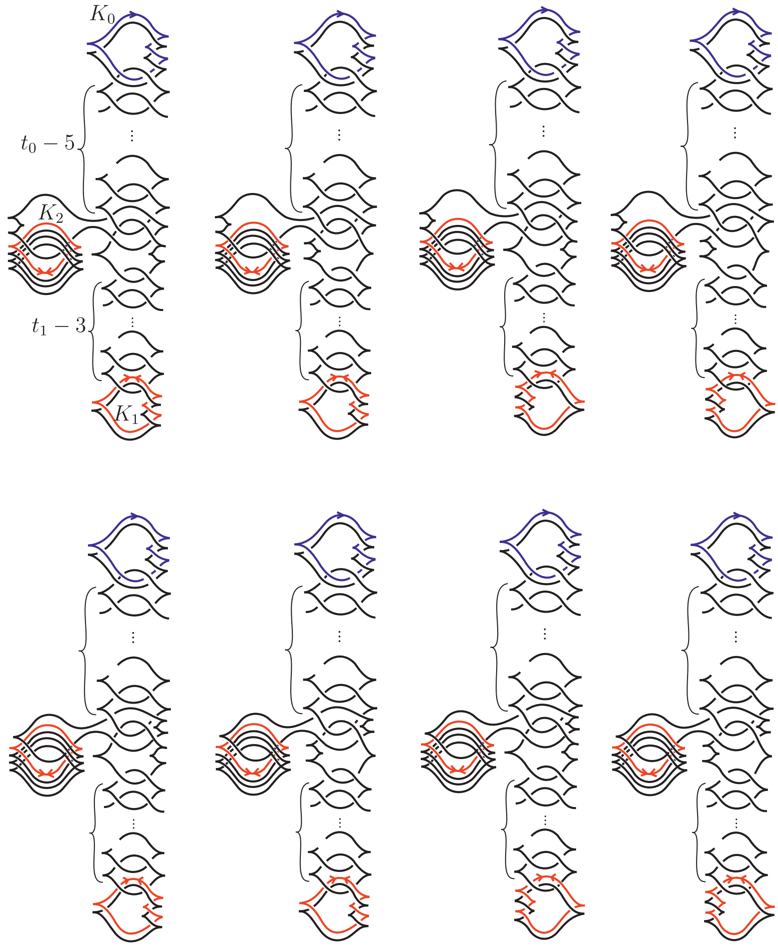

Figure 13: $t_0 \geq 5$, $t_1 \geq 3$, $t_2 = 1$. For $t_0 + t_1$ even, K_0 and K_1 bear the same orientation, and for $t_0 + t_1$ odd, the opposite one. For t_0 odd, K_0 and K_2 bear the same orientation, and for t_0 even, the opposite one.

links $K_0' \cup K_1$ in $(S^3, \xi_{\frac{3}{2}})$ with $(t_0', r_0') = (2, \pm 1)$, $t_1 \geq 3$, $r_1 = \pm(t_1 - 1)$, two Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{\frac{3}{2}})$ with $(t_0', r_0') = (2, \mp 1)$, $t_1 \geq 3$, $r_1 = \pm(t_1 - 3)$, and one Legendrian Hopf link $K_0'' \cup K_2$ in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0'', r_0'') = (t_2, r_2) = (1, 0)$. By Lemma 3.1, we can obtain 6 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are $(\pm 3, \pm(t_1 + 1), 0; \frac{1}{2})$, $(\mp 1, \pm(1 - t_1), 0; \frac{5}{2})$, and $(\pm 1, \pm(3 - t_1), 0; \frac{5}{2})$.

By [11, Theorem 1.2, (d), (c2)], there are two Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0', r_0') = (0, \pm 1)$, $t_1 \geq 3$, $r_1 = \pm(t_1 - 1)$, and two Legendrian Hopf links $K_0'' \cup K_2$ in $(S^3, \xi_{-\frac{1}{2}})$ with $(t_0'', r_0'') = (3, \pm 4)$, $(t_2, r_2) = (1, \pm 2)$. By Lemma 3.1, we can obtain 4 more strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are $(\pm 5, \pm(t_1 - 1), \pm 2; \frac{1}{2})$ and $(\pm 3, \pm(1 - t_1), \pm 2; \frac{1}{2})$.

By Lemma 4.7 and Lemma 3.4, there are 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{-\frac{3}{2}})$ whose rotation numbers are $(\pm 7, \pm(t_1 + 1), \pm 2)$. The decorations of their exteriors are $\pm(+)((-)(-))(-)$.

There are 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{1}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm 5, \pm(3 - t_1), \pm 2)$. The decorations of their exteriors are $\pm(+)((-)(+))(-)$. These exteriors can be embedded into an appropriate tight contact $\Sigma \times S^1$ with boundary slopes $4, -\frac{1}{2}, -1$ and decorations $\pm(+)(-+)(-)$. This can be achieved by adding basic slices $(T^2 \times [0,1], -\frac{1}{t_1}, -\frac{1}{t_1-1}), \cdots, (T^2 \times [0,1], -\frac{1}{3}, -\frac{1}{2})$ to the boundary T_1, as per the Gluing Theorem [14, Theorem 1.3]. So these exteriors are appropriate tight. ∎

(6) Suppose $t_0 = 3$ and $t_1 \geq 3$.

Lemma 4.17 *If $t_0 = 3$, $t_1 \geq 3$, and $t_2 = 1$, then there exist 12 (11 if $t_1 = 3$) strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are*

$$\left(\pm 2, \pm(t_1 + 1), 0; \frac{1}{2}\right), \left(0, \pm(3 - t_1), 0; \frac{5}{2}\right), \left(\pm 4, \pm(t_1 - 1), \pm 2; \frac{1}{2}\right),$$
$$\left(\pm 2, \pm(1 - t_1), \pm 2; \frac{1}{2}\right), \left(\pm 6, \pm(t_1 + 1), \pm 2; -\frac{3}{2}\right), \left(\pm 4, \pm(3 - t_1), \pm 2; \frac{1}{2}\right).$$

Proof By [11, Theorem 1.2, (c3), (c2), (c1)], there are two Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{-\frac{1}{2}})$ with $(t_0', r_0') = (1, \pm 2), t_1 \geq 3, r_1 = \pm(t_1 + 1)$, two (one if $t_1 = 3$) Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{\frac{3}{2}})$ with $(t_0', r_0') = (1, 0), t_1 \geq 3, r_1 = \pm(t_1 - 3)$, and one Legendrian Hopf link $K_0'' \cup K_2$ in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0'', r_0'') = (t_2, r_2) = (1, 0)$. By Lemma 3.1, we can obtain 4 (3 if $t_1 = 3$) strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are $(\pm 2, \pm(t_1 + 1), 0; \frac{1}{2})$ and $(0, \pm(3 - t_1), 0; \frac{5}{2})$.

By [11, Theorem 1.2, (d), (c2)], there are two Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0', r_0') = (0, \pm 1), t_1 \geq 3, r_1 = \pm(t_1 - 1)$, and two Legendrian Hopf links $K_0'' \cup K_2$ in $(S^3, \xi_{-\frac{1}{2}})$ with $(t_0'', r_0'') = (2, \pm 3), (t_2, r_2) = (1, \pm 2)$. By Lemma 3.1, we can obtain strongly exceptional Legendrian A_3 links with $t_0 = 3, t_1 \geq 3, t_2 = 1$. So there are 4 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are $(\pm 4, \pm(t_1 - 1), \pm 2; \frac{1}{2})$ and $(\pm 2, \pm(1 - t_1), \pm 2; \frac{1}{2})$.

By Lemma 4.7 and Lemma 3.4, there are 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{-\frac{3}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm 6, \pm(t_1 + 1), \pm 2)$. The decorations of their exteriors are $\pm(+)((-)(-))(-)$.

There are 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{1}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm 4, \pm(3 - t_1), \pm 2)$. The decorations of their exteriors are $\pm(+)((-)(+))(-)$. These exteriors are appropriate tight since they can be embedded into an appropriate tight contact $\Sigma \times S^1$ with boundary slopes $3, -\frac{1}{2}, -1$ and decorations $\pm(+)(-+)(-)$ by adding basic slices $(T^2 \times [0,1], -\frac{1}{t_1}, -\frac{1}{t_1-1}), \cdots, (T^2 \times [0,1], -\frac{1}{3}, -\frac{1}{2})$ to the boundary T_1. ∎

So there are exactly 12 (resp. 11) strongly exceptional Legendrian A_3 links with $t_0 = 3$, $t_1 \geq 4$ (resp. $t_1 = 3$), $t_2 = 1$. If $t_0 = t_1 = 3$ and $t_2 = 1$, then the decorations $(+)((-)(+))(+)$ and $(-)((+)(-))(-)$ correspond to the same Legendrian A_3 links with rotation numbers $r_0 = r_1 = r_2 = 0$.

(7) Suppose $t_0 \leq 2$.

Lemma 4.18 *If $t_0 \leq 2$, $t_1 > 1$, and $t_2 = 1$, then there exist $6 - 2t_0$ strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{3}{2}})$ whose rotation numbers are*

$$r_0 \in \pm\{t_0 - 1, t_0 + 1, \cdots, -t_0 + 1, -t_0 + 3\}, r_1 = \pm(t_1 - 1), r_2 = 0.$$

Proof By [11, Theorem 1.2, (b1), (c1)], there is a Legendrian Hopf link $K_0' \cup K_1$ in $(S^3, \xi_{\frac{1}{2}})$ with $t_0' \leq 1, r_0' \in \pm\{t_0' + 1, t_0' + 3, \cdots, -t_0' - 1, -t_0' + 1\}, t_1 \geq 2, r_1 = \pm(t_1 - 1)$, and a Legendrian Hopf link $K_0'' \cup K_2$ in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0'', r_0'') = (t_2, r_2) = (1, 0)$. By Lemma 3.1, we can construct $6 - 2t_0$ strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{3}{2}})$ with $t_0 \leq 2, t_1 > 1, t_2 = 1$. Their rotation numbers are as listed. ∎

These $6 - 2t_0$ strongly exceptional Legendrian A_3 links are stabilizations of the Legendrian A_3 links with $t_0 = 2, t_1 > 1, t_2 = 1$.

The proof of Theorem 1.3 is completed.

4.2.3 $t_1 \geq 2$ and $t_2 \geq 2$.

Proof of Theorem 1.4 The upper bound of strongly exceptional Legendrian A_3 links is given by Lemma 2.11. We will show that these upper bounds can be attained.

(1) Suppose $t_0 \geq 4$ and $t_1 = t_2 = 2$.

Lemma 4.19 *If $t_0 \geq 4$ and $t_1 = t_2 = 2$, then there exist 18 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are*

$$\left(\pm(t_0 - 1), \pm 3, \mp 1; \frac{1}{2}\right), \left(\pm(t_0 + 1), \pm 3, \pm 1; \frac{1}{2}\right), \left(\pm(t_0 + 3), \pm 3, \pm 3; -\frac{3}{2}\right),$$
$$\left(\pm(t_0 - 3), \pm 1, \mp 1; \frac{5}{2}\right), \left(\pm(t_0 - 1), \pm 1, \pm 1; \frac{5}{2}\right), \left(\pm(t_0 + 1), \pm 1, \pm 3; \frac{1}{2}\right),$$
$$\left(\pm(t_0 - 5), \mp 1, \mp 1; \frac{5}{2}\right), \left(\pm(t_0 - 3), \mp 1, \pm 1; \frac{5}{2}\right), \left(\pm(t_0 - 1), \mp 1, \pm 3; \frac{1}{2}\right).$$

Proof If $t_0 \geq 4$ and $t_1 = t_2 = 2$, then there exist 18 strongly exceptional Legendrian A_3 links shown in Figure 14. Using the trick of Lemma 4.2 and the proof of [11, Theorem 1.2, (c3)], we can show that $K_0 \cup K_1 \cup K_2$ is a topological A_3 link. Their rotation numbers and corresponding d_3-invariants are as listed. ∎

(2) Suppose $t_0 = 3$ and $t_1 = t_2 = 2$.

Lemma 4.20 *If $t_0 = 3$ and $t_1 = t_2 = 2$, then there exist 14 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are*

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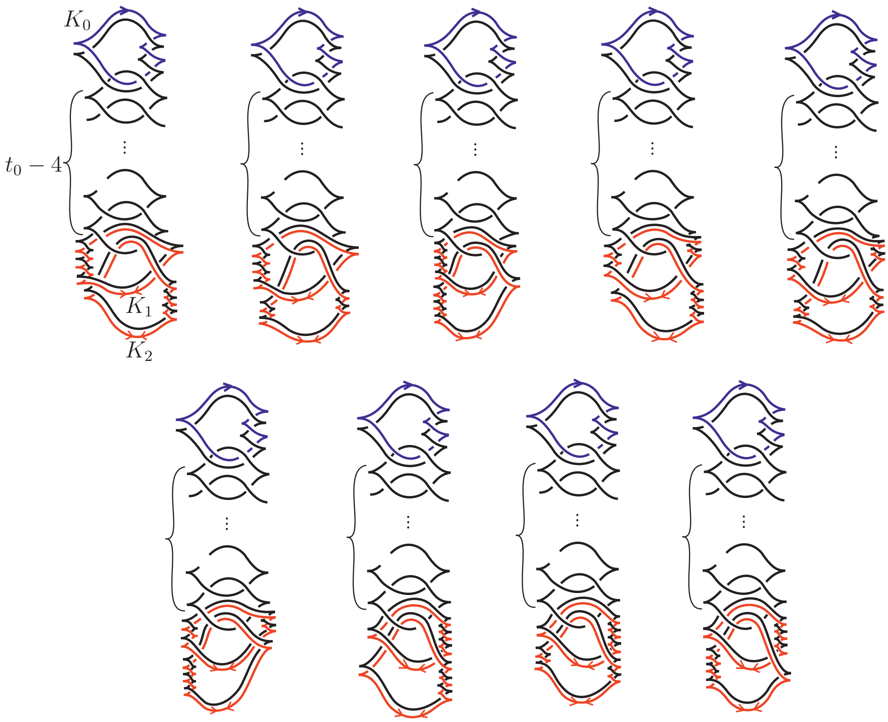

Figure 14: $t_0 \geq 4$, $t_1 = t_2 = 2$. For t_0 odd, K_0 and K_i are given the same orientation, and for t_0 even, the opposite one, where $i = 1, 2$.

$$\left(\pm 4, \pm 3, \pm 1; \frac{1}{2}\right), \left(\pm 4, \pm 1, \pm 3; \frac{1}{2}\right), \left(\pm 2, \pm 3, \mp 1; \frac{1}{2}\right), \left(\pm 2, \mp 1, \pm 3; \frac{1}{2}\right),$$
$$\left(\mp 2, \mp 1, \mp 1; \frac{5}{2}\right), \left(0, \mp 1, \pm 1; \frac{5}{2}\right), \left(\pm 6, \pm 3, \pm 3; -\frac{3}{2}\right).$$

Proof By [11, Theorem 1.2, (c2), (d)], there are two Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{-\frac{1}{2}})$ with $(t_0', r_0') = (t_1, r_1) = (2, \pm 3)$, two Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{\frac{3}{2}})$ with $(t_0', r_0') = (t_1, r_1) = (2, \pm 1)$, and two Legendrian Hopf links $K_0'' \cup K_2$ in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0'', r_0'') = (0, \pm 1), (t_2, r_2) = (2, \pm 1)$. By Lemma 3.1, we can obtain strongly exceptional Legendrian A_3 links with $t_0 = 3, t_1 = t_2 = 2$. So by exchanging the roles of K_1 and K_2, there are 12 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are $(\pm 4, \pm 3, \pm 1; \frac{1}{2})$, $(\pm 4, \pm 1, \pm 3; \frac{1}{2})$, $(\pm 2, \pm 3, \mp 1; \frac{1}{2})$, $(\pm 2, \mp 1, \pm 3; \frac{1}{2})$, $(\mp 2, \mp 1, \mp 1; \frac{5}{2})$, and $(0, \mp 1, \pm 1; \frac{5}{2})$.

By Lemma 4.7 and Lemma 3.4, there are 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{-\frac{3}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm 6, \pm 3, \pm 3)$. The decorations of their exteriors are $\pm(+)(--)(--)$. ∎

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So there are exactly 14 strongly exceptional Legendrian A_3 links with $t_0 = 3, t_1 = 2, t_2 = 2$. As a corollary, the 14 contact structures on $\Sigma \times S^1$ with boundary slopes $s_0 = 3, s_1 = -\frac{1}{2}, s_2 = -\frac{1}{2}$ listed in Lemma 2.11 are all appropriate tight.

(3) Suppose $t_0 = t_1 = t_2 = 2$.

Lemma 4.21 *If $t_0 = t_1 = t_2 = 2$, then there exist 10 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are*

$$\left(\pm 3, \pm 3, \pm 1; \frac{1}{2}\right), \left(\pm 3, \pm 1, \pm 3; \frac{1}{2}\right), \left(\pm 1, \pm 3, \mp 1; \frac{1}{2}\right), \left(\pm 1, \mp 1, \pm 3; \frac{1}{2}\right), \left(\pm 5, \pm 3, \pm 3; -\frac{3}{2}\right).$$

Proof By [11, Theorem 1.2, (c2), (d)], there are two Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{-\frac{1}{2}})$ with $(t_0', r_0') = (1, \pm 2), (t_1, r_1) = (2, \pm 3)$, and two Legendrian Hopf links $K_0'' \cup K_2$ in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0'', r_0'') = (0, \pm 1), (t_2, r_2) = (2, \pm 1)$. By Lemma 3.1, we can obtain strongly exceptional Legendrian A_3 links with $t_0 = t_1 = t_2 = 2$. So by exchanging the roles of K_1 and K_2, there are 8 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are are $(\pm 3, \pm 3, \pm 1; \frac{1}{2})$, $(\pm 3, \pm 1, \pm 3; \frac{1}{2})$, $(\pm 1, \pm 3, \mp 1; \frac{1}{2})$, and $(\pm 1, \mp 1, \pm 3; \frac{1}{2})$.

By Lemma 4.7 and Lemma 3.4, there are 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{-\frac{3}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm 5, \pm 3, \pm 3)$. The decorations of their exteriors are $\pm(+)(--)(--)$. ■

So there are exactly 10 strongly exceptional Legendrian A_3 links with $t_0 = 2, t_1 = 2, t_2 = 2$. As a corollary, the 10 contact structures on $\Sigma \times S^1$ with boundary slopes $s_0 = 2, s_1 = -\frac{1}{2}, s_2 = -\frac{1}{2}$ listed in Lemma 2.11 are all appropriate tight.

(4) Suppose $t_0 \geq 4$, $t_1 \geq 3$, and $t_2 = 2$.

Lemma 4.22 *If $t_0 \geq 4$, $t_1 \geq 3$, and $t_2 = 2$, then there exist 24 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3 invariants $(r_0, r_1, r_2; d_3)$ are*

$$\begin{gathered}
\left(\pm(t_0 - 3), \pm(3 - t_1), \mp 1; \frac{5}{2}\right), \left(\pm(t_0 - 1), \pm(3 - t_1), \pm 1; \frac{5}{2}\right), \\
\left(\pm(t_0 + 1), \pm(3 - t_1), \pm 3; \frac{1}{2}\right), \left(\pm(t_0 - 5), \pm(1 - t_1), \mp 1; \frac{5}{2}\right), \\
\left(\pm(t_0 - 3), \pm(1 - t_1), \pm 1; \frac{5}{2}\right), \left(\pm(t_0 - 1), \pm(1 - t_1), \pm 3; \frac{1}{2}\right), \\
\left(\pm(t_0 + 1), \pm(t_1 - 1), \pm 3; \frac{1}{2}\right), \left(\pm(t_0 - 1), \pm(t_1 - 1), \pm 1; \frac{5}{2}\right), \\
\left(\pm(t_0 - 3), \pm(t_1 - 1), \mp 1; \frac{5}{2}\right), \left(\pm(t_0 + 3), \pm(t_1 + 1), \pm 3; -\frac{3}{2}\right), \\
\left(\pm(t_0 + 1), \pm(t_1 + 1), \pm 1; \frac{1}{2}\right), \left(\pm(t_0 - 1), \pm(t_1 + 1), \mp 1; \frac{1}{2}\right).
\end{gathered}$$

Proof If $t_0 \geq 4$, $t_1 \geq 3$, and $t_2 = 2$, then there are exactly 24 strongly exceptional Legendrian A_3 links shown in Figure 15. Using the trick of Lemma 4.2 and the proof of [11, Theorem 1.2, (c3), (c4)], we can show that $K_0 \cup K_1 \cup K_2$ is a topological A_3 link. Their rotation numbers and corresponding d_3-invariants are as listed. ■

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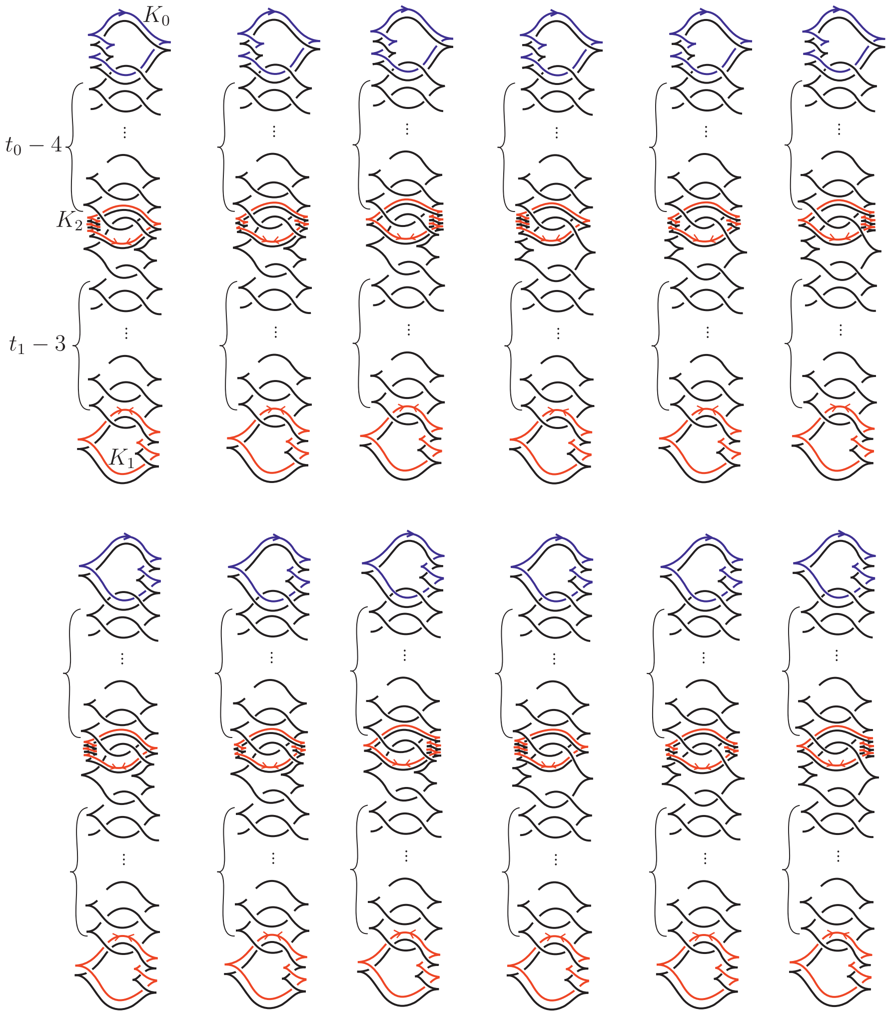

Figure 15: $t_0 \geq 4$, $t_1 \geq 3$, $t_2 = 2$. If $t_0 + t_1$ is odd, then K_0 and K_1 bear the same orientation. If $t_0 + t_1$ is even, then the opposite one. If t_0 is odd, then K_0 and K_2 bear the same orientation. If t_0 is even, then the opposite one.

(5) Suppose $t_0 = 3$, $t_1 \geq 3$, and $t_2 = 2$.

Lemma 4.23 *If $t_0 = 3, t_1 \geq 3, t_2 = 2$, then there exist 20 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are*

$$\left(\pm 4, \pm(t_1 - 1), \pm 3; \frac{1}{2}\right), \left(\pm 2, \pm(1 - t_1), \pm 3; \frac{1}{2}\right), \left(\mp 2, \pm(1 - t_1), \mp 1; \frac{5}{2}\right),$$
$$\left(0, \pm(1 - t_1), \pm 1; \frac{5}{2}\right), \left(\pm 4, \pm(t_1 + 1), \pm 1; \frac{1}{2}\right), \left(\pm 2, \pm(t_1 + 1), \mp 1; \frac{1}{2}\right),$$

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$$\left(0, \pm(3-t_1), \mp 1; \frac{5}{2}\right), \left(\pm 2, \pm(3-t_1), \pm 1; \frac{5}{2}\right),$$
$$\left(\pm 6, \pm(t_1+1), \pm 3; -\frac{3}{2}\right), \left(\pm 4, \pm(3-t_1), \pm 3; \frac{1}{2}\right).$$

Proof By [11, Theorem 1.2, (d), (c2)], there are two Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0', r_0') = (0, \pm 1)$, $t_1 \geq 3$, $r_1 = \pm(t_1 - 1)$, two Legendrian Hopf links $K_0'' \cup K_2$ in $(S^3, \xi_{-\frac{1}{2}})$ with $(t_0'', r_0'') = (t_2, r_2) = (2, \pm 3)$, and two Legendrian Hopf links $K_0'' \cup K_2$ in $(S^3, \xi_{\frac{3}{2}})$ with $(t_0'', r_0'') = (t_1, r_2) = (2, \pm 1)$. By Lemma 3.1, we can obtain 8 strongly exceptional Legendrian A_3 links whose rotation numbers and d_3-invariants $(r_0, r_1, r_2; d_3)$ are $(\pm 4, \pm(t_1-1), \pm 3; \frac{1}{2})$, $(\pm 2, \pm(1-t_1), \pm 3; \frac{1}{2})$, $(\mp 2, \pm(1-t_1), \mp 1; \frac{5}{2})$, and $(0, \pm(1-t_1), \pm 1; \frac{5}{2})$.

By [11, Theorem 1.2, (c3), (d)], there are two Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{-\frac{1}{2}})$ with $(t_0', r_0') = (2, \pm 3)$, $t_1 \geq 3$, $r_1 = \pm(t_1 + 1)$, two Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{\frac{3}{2}})$ with $(t_0', r_0') = (2, \mp 1)$, $t_1 \geq 3$, $r_1 = \pm(t_1 - 3)$, and two Legendrian Hopf links $K_0'' \cup K_2$ in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0'', r_0'') = (0, \pm 1)$, $(t_2, r_2) = (2, \pm 1)$. By Lemma 3.1, we can obtain strongly exceptional Legendrian A_3 links with $t_0 = 3$, $t_1 \geq 3$, $t_2 = 2$. Then, there are 8 strongly exceptional Legendrian A_3 links whose rotation numbers and d_3-invariants are $(\pm 4, \pm(t_1+1), \pm 1; \frac{1}{2})$, $(\pm 2, \pm(t_1+1), \mp 1; \frac{1}{2})$, $(0, \pm(3-t_1), \mp 1; \frac{5}{2})$, and $(\pm 2, \pm(3-t_1), \pm 1; \frac{5}{2})$.

By Lemma 4.7 and Lemma 3.4, there are 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{-\frac{3}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm 6, \pm(t_1+1), \pm 3)$. The decorations of their exteriors are $\pm(+)((-)(-))(--)$.

There are 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{1}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm 4, \pm(3-t_1), \pm 3)$. The decorations of their exteriors are $\pm(+)((-)(+))(--)$. These exteriors are appropriate tight since they can be embedded into an appropriate tight contact $\Sigma \times S^1$ with boundary slopes $3, -\frac{1}{2}, -\frac{1}{2}$ and decorations $\pm(+)(-+)(--)$ by adding basic slices $(T^2 \times [0,1], -\frac{1}{t_1}, -\frac{1}{t_1-1})$, $\cdots$, $(T^2 \times [0,1], -\frac{1}{3}, -\frac{1}{2})$ to the boundary T_1. ■

So there are exactly 20 strongly exceptional Legendrian A_3 links with $t_0 = 3$, $t_1 \geq 3$, $t_2 = 2$. As a corollary, the 20 contact structures on $\Sigma \times S^1$ with boundary slopes $s_0 = 3$, $s_1 = -\frac{1}{t_1}$, $s_2 = -\frac{1}{2}$ listed in Lemma 2.11 are all appropriate tight.

(6) Suppose $t_0 = 2$, $t_1 \geq 3$, and $t_2 = 2$.

Lemma 4.24 *If $t_0 = 2$, $t_1 \geq 3$, and $t_2 = 2$, then there exist* 16 *strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are*

$$\left(\pm 3, \pm(t_1+1), \pm 1; \frac{1}{2}\right), \left(\pm 1, \pm(t_1+1), \mp 1; \frac{1}{2}\right), \left(\mp 1, \pm(3-t_1), \mp 1; \frac{5}{2}\right),$$
$$\left(\pm 1, \pm(3-t_1), \pm 1; \frac{5}{2}\right), \left(\pm 1, \pm(1-t_1), \pm 3; \frac{1}{2}\right),$$
$$\left(\pm 3, \pm(t_1-1), \pm 3; \frac{1}{2}\right), \left(\pm 5, \pm(t_1+1), \pm 3; -\frac{3}{2}\right), \left(\pm 3, \pm(3-t_1), \pm 3; \frac{1}{2}\right).$$

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Proof By [11, Theorem 1.2, (c2), (c3), (d)], there are two Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{-\frac{1}{2}})$ with $(t_0', r_0') = (1, \pm 2), t_1 \geq 3, r_1 = \pm(t_1 + 1)$, two (one if $t_1 = 3$) Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{\frac{3}{2}})$ with $(t_0', r_0') = (1, 0), t_1 \geq 3, r_1 = \pm(t_1 - 3)$, and two Legendrian Hopf links $K_0'' \cup K_2$ in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0'', r_0'') = (0, \pm 1), (t_2, r_2) = (2, \pm 1)$. By Lemma 3.1, we can obtain strongly exceptional Legendrian A_3 links with $t_0 = 2, t_1 \geq 3, t_2 = 2$. Then, there are 8 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are $(\pm 3, \pm(t_1 + 1), \pm 1; \frac{1}{2}), (\pm 1, \pm(t_1 + 1), \mp 1; \frac{1}{2}), (\mp 1, \pm(3 - t_1), \mp 1; \frac{5}{2})$, and $(\pm 1, \pm(3 - t_1), \pm 1; \frac{5}{2})$.

By [11, Theorem 1.2, (d), (c2)], there are two Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0', r_0') = (0, \pm 1), t_1 \geq 3, r_1 = \pm(t_1 - 1)$, and two Legendrian Hopf links $K_0'' \cup K_2$ in $(S^3, \xi_{-\frac{1}{2}})$ with $(t_0'', r_0'') = (1, \pm 2), (t_2, r_2) = (2, \pm 3)$. By Lemma 3.1, we can obtain strongly exceptional Legendrian A_3 links with $t_0 = 2, t_1 \geq 3, t_2 = 2$. Then, there are 4 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are $(\pm 1, \pm(1 - t_1), \pm 3; \frac{1}{2})$ and $(\pm 3, \pm(t_1 - 1), \pm 3; \frac{1}{2})$.

By Lemma 4.7 and Lemma 3.4, there are 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{-\frac{3}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm 5, \pm(t_1 + 1), \pm 3)$. The decorations of their exteriors are $\pm(+)((-)(-))(--)$.

There are 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{1}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm 3, \pm(3 - t_1), \pm 3)$. The decorations of their exteriors are $\pm(+)((-)(+))(--)$. These exteriors are appropriate tight since they can be embedded into an appropriate tight contact $\Sigma \times S^1$ with boundary slopes $2, -\frac{1}{2}, -\frac{1}{2}$ and decorations $\pm(+)(-+)(--)$ by adding basic slices $(T^2 \times [0,1], -\frac{1}{t_1}, -\frac{1}{t_1 - 1}), \cdots, (T^2 \times [0,1], -\frac{1}{3}, -\frac{1}{2})$ to the boundary T_1. ■

So there are exactly 16 strongly exceptional Legendrian A_3 links with $t_0 = 2, t_1 \geq 3, t_2 = 2$. As a corollary, the 16 contact structures on $\Sigma \times S^1$ with boundary slopes $s_0 = 3, s_1 = -\frac{1}{t_1}, s_2 = -\frac{1}{2}$ listed in Lemma 2.11 are all appropriate tight.

(7) Suppose $t_0 \geq 4$, $t_1 \geq 3$, and $t_2 \geq 3$.

Lemma 4.25 *If $t_0 \geq 4$, $t_1 \geq 3$, and $t_2 \geq 3$, then there exist 32 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are*

$$\left(\pm(t_0 + 1), \pm(t_1 - 1), \pm(t_2 + 1); \frac{1}{2}\right), \left(\pm(t_0 - 1), \pm(1 - t_1), \pm(t_2 + 1); \frac{1}{2}\right),$$

$$\left(\pm(t_0 + 3), \pm(t_1 + 1), \pm(t_2 + 1); -\frac{3}{2}\right), \left(\pm(t_0 + 1), \pm(3 - t_1), \pm(t_2 + 1); \frac{1}{2}\right),$$

$$\left(\pm(t_0 - 1), \pm(t_1 - 1), \pm(3 - t_2); \frac{5}{2}\right), \left(\pm(t_0 - 3), \pm(1 - t_1), \pm(3 - t_2); \frac{5}{2}\right),$$

$$\left(\pm(t_0 + 1), \pm(t_1 + 1), \pm(3 - t_2); \frac{1}{2}\right), \left(\pm(t_0 - 1), \pm(3 - t_1), \pm(3 - t_2); \frac{5}{2}\right),$$

$$\left(\pm(t_0 - 1), \pm(t_1 - 1), \pm(t_2 - 1); \frac{5}{2}\right), \left(\pm(t_0 - 3), \pm(1 - t_1), \pm(t_2 - 1); \frac{5}{2}\right),$$

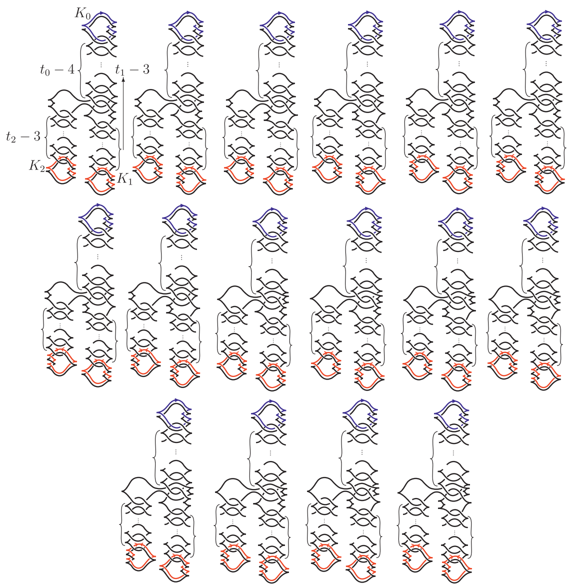

Figure 16: $t_0 \geq 4$, $t_1 \geq 3$, $t_2 \geq 3$. If $t_0 + t_i$ is odd, then K_0 and K_i bear the same orientation, $i = 1, 2$. If $t_0 + t_i$ is even, then the opposite one.

$$\left(\pm(t_0+1), \pm(t_1+1), \pm(t_2-1); \frac{1}{2}\right), \left(\pm(t_0-1), \pm(3-t_1), \pm(t_2-1); \frac{5}{2}\right),$$

$$\left(\pm(t_0-3), \pm(t_1-1), \pm(1-t_2); \frac{5}{2}\right), \left(\pm(t_0-5), \pm(1-t_1), \pm(1-t_2); \frac{5}{2}\right),$$

$$\left(\pm(t_0-1), \pm(t_1+1), \pm(1-t_2); \frac{1}{2}\right), \left(\pm(t_0-3), \pm(3-t_1), \pm(1-t_2); \frac{5}{2}\right).$$

Proof If $t_0 \geq 4$, $t_1 \geq 3$, and $t_2 \geq 3$, then there are exactly 32 strongly exceptional Legendrian A_3 links shown in Figure 16. Using the trick of Lemma 4.2 and the proof of [11, Theorem 1.2, (c4)], we can show that $K_0 \cup K_1 \cup K_2$ is a topological A_3 link. Their rotation numbers and corresponding d_3-invariants are as listed. ■

(8) Suppose $t_0 = 3$, $t_1 \geq 3$, and $t_2 \geq 3$.

Lemma 4.26 *If $t_0 = 3$, $t_1 \geq 3$, and $t_2 \geq 3$, then there exist 28 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are*

$$\left(\pm 4, \pm(t_1+1), \pm(t_2-1); \frac{1}{2}\right), \left(\pm 4, \pm(t_1-1), \pm(t_2+1); \frac{1}{2}\right),$$

$$\left(\pm 2, \pm(t_1+1), \pm(1-t_2); \frac{1}{2}\right), \left(\pm 2, \pm(1-t_1), \pm(t_2+1); \frac{1}{2}\right),$$

$$\left(\mp 2, \pm(1-t_1), \pm(1-t_2); \frac{5}{2}\right), \left(0, \pm(t_1-1), \pm(1-t_2); \frac{5}{2}\right),$$

$$\left(0, \pm(3-t_1), \pm(1-t_2); \frac{5}{2}\right), \left(0, \pm(1-t_1), \pm(3-t_2); \frac{5}{2}\right),$$

$$\left(\pm 2, \pm(3-t_1), \pm(t_2-1); \frac{5}{2}\right), \left(\pm 2, \pm(t_1-1), \pm(3-t_2); \frac{5}{2}\right),$$

$$\left(\pm 6, \pm(t_1+1), \pm(t_2+1); -\frac{3}{2}\right), \left(\pm 2, \pm(3-t_1), \pm(3-t_2); \frac{5}{2}\right),$$

$$\left(\pm 4, \pm(t_1+1), \pm(3-t_2); \frac{1}{2}\right), \left(\pm 4, \pm(3-t_1), \pm(t_2+1); \frac{1}{2}\right).$$

Proof By [11, Theorem 1.2, (c3), (d)], there are two Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{-\frac{1}{2}})$ with $(t_0', r_0') = (2, \pm 3)$, $t_1 \geq 3$, $r_1 = \pm(t_1+1)$, two Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{\frac{3}{2}})$ with $(t_0', r_0') = (2, \pm 1)$, $t_1 \geq 3$, $r_1 = \pm(t_1-1)$, two Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{\frac{3}{2}})$ with $(t_0', r_0') = (2, \mp 1)$, $t_1 \geq 3$, $r_1 = \pm(t_1-3)$, and two Legendrian Hopf links $K_0'' \cup K_2$ in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0'', r_0'') = (0, \pm 1)$, $t_2 \geq 3$, $r_2 = \pm(t_2-1)$. By Lemma 3.1, we can obtain strongly exceptional Legendrian A_3 links with $t_0 = 3$, $t_1 \geq 3$, $t_2 \geq 3$. Then, after exchanging the roles of K_1 and K_2, there are 20 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are

$$\left(\pm 4, \pm(t_1+1), \pm(t_2-1); \frac{1}{2}\right), \left(\pm 4, \pm(t_1-1), \pm(t_2+1); \frac{1}{2}\right),$$

$$\left(\pm 2, \pm(t_1+1), \pm(1-t_2); \frac{1}{2}\right), \left(\pm 2, \pm(1-t_1), \pm(t_2+1); \frac{1}{2}\right),$$

$$\left(\mp 2, \pm(1-t_1), \pm(1-t_2); \frac{5}{2}\right), \left(0, \pm(t_1-1), \pm(1-t_2); \frac{5}{2}\right),$$

$$\left(0, \pm(3-t_1), \pm(1-t_2); \frac{5}{2}\right), \left(0, \pm(1-t_1), \pm(3-t_2); \frac{5}{2}\right),$$

$$\left(\pm 2, \pm(3-t_1), \pm(t_2-1); \frac{5}{2}\right), \left(\pm 2, \pm(t_1-1), \pm(3-t_2); \frac{5}{2}\right).$$

By Lemma 4.7 and Lemma 3.4, there are 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{-\frac{3}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm 6, \pm(t_1+1), \pm(t_2+1))$. The decorations of their exteriors are $\pm(+)((-)(-))((-)(-))$.

There are other 6 more strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are $(\pm 2, \pm(3-t_1), \pm(3-t_2); \frac{5}{2})$, $(\pm 4, \pm(t_1+1), \pm(3-t_2); \frac{1}{2})$, and $(\pm 4, \pm(3-t_1), \pm(t_2+1); \frac{1}{2})$. The decorations of their exteriors are

$$\pm(+)((-)(+))((-)(+)), \pm(+)((-)(-))((-)(+)), \text{ and } \pm(+)((-)(+))((-)(-)),$$

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respectively. These exteriors are tight since they can be embedded into a tight contact $\Sigma \times S^1$ with boundary slopes $3, -\frac{1}{t_1}, -\frac{1}{2}$ and decorations

$$\pm(+)((-)(+))(-+), \pm(+)((-)(-))(-+) \text{ and } \pm(+)((-)(+))(--)$$

by adding basic slices $(T^2 \times [0,1], -\frac{1}{t_2}, -\frac{1}{t_2-1}), \cdots, (T^2 \times [0,1], -\frac{1}{3}, -\frac{1}{2})$ to the boundary T_2, respectively. ■

(9) Suppose $t_0 = 2$, $t_1 \geq 3$, and $t_2 \geq 3$.

Lemma 4.27 *If $t_0 = 2$, $t_1 \geq 3$, and $t_2 \geq 3$, then there exist 24 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are*

$$\left(\pm 3, \pm(t_1+1), \pm(t_2-1); \frac{1}{2}\right), \left(\pm 3, \pm(t_1-1), \pm(t_2+1); \frac{1}{2}\right),$$

$$\left(\pm 1, \pm(t_1+1), \pm(1-t_2); \frac{1}{2}\right), \left(\pm 1, \pm(1-t_1), \pm(t_2+1); \frac{1}{2}\right),$$

$$\left(\mp 1, \pm(3-t_1), \pm(1-t_2); \frac{5}{2}\right), \left(\mp 1, \pm(1-t_1), \pm(3-t_2); \frac{5}{2}\right),$$

$$\left(\pm 1, \pm(3-t_1), \pm(t_2-1); \frac{5}{2}\right), \left(\pm 1, \pm(t_1-1), \pm(3-t_2); \frac{5}{2}\right),$$

$$\left(\pm 5, \pm(t_1+1), \pm(t_2+1); -\frac{3}{2}\right), \left(\pm 1, \pm(3-t_1), \pm(3-t_2); \frac{5}{2}\right),$$

$$\left(\pm 3, \pm(t_1+1), \pm(3-t_2); \frac{1}{2}\right), \left(\pm 3, \pm(3-t_1), \pm(t_2+1); \frac{1}{2}\right).$$

Proof By [11, Theorem 1.2, (c2), (c3), (d)], there are two Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{-\frac{1}{2}})$ with $(t_0', r_0') = (1, \pm 2)$, $t_1 \geq 3$, $r_1 = \pm(t_1+1)$, two (one if $t_1 = 3$) Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{\frac{3}{2}})$ with $(t_0', r_0') = (1, 0)$, $t_1 \geq 3$, $r_1 = \pm(t_1-3)$, and two Legendrian Hopf links $K_0'' \cup K_2$ in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0'', r_0'') = (0, \pm 1)$, $t_2 \geq 3$, $r_2 = \pm(t_2-1)$. By Lemma 3.1, we can obtain strongly exceptional Legendrian A_3 links with $t_0 = 2$, $t_1 \geq 3$, $t_2 \geq 3$. So, after exchanging the roles of K_1 and K_2, there are 16 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are

$$\left(\pm 3, \pm(t_1+1), \pm(t_2-1); \frac{1}{2}\right), \left(\pm 3, \pm(t_1-1), \pm(t_2+1); \frac{1}{2}\right),$$

$$\left(\pm 1, \pm(t_1+1), \pm(1-t_2); \frac{1}{2}\right), \left(\pm 1, \pm(1-t_1), \pm(t_2+1); \frac{1}{2}\right),$$

$$\left(\mp 1, \pm(3-t_1), \pm(1-t_2); \frac{5}{2}\right), \left(\mp 1, \pm(1-t_1), \pm(3-t_2); \frac{5}{2}\right),$$

$$\left(\pm 1, \pm(3-t_1), \pm(t_2-1); \frac{5}{2}\right), \left(\pm 1, \pm(t_1-1), \pm(3-t_2); \frac{5}{2}\right).$$

By Lemma 4.7 and Lemma 3.4, there are 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{-\frac{3}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm 5, \pm(t_1+1), \pm(t_2+1))$. The decorations of their exteriors are $\pm(+)((-)(-))(--)$.

There are 6 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are $(\pm 1, \pm(3-t_1), \pm(3-t_2); \frac{5}{2})$, $(\pm 3, \pm(t_1+1), \pm(3-t_2); \frac{1}{2})$, and $(\pm 3, \pm(3-t_1), \pm(t_2+1); \frac{1}{2})$. The decorations of their exteriors are

$$\pm(+)((-)(+))((-)(+)), \pm(+)((-)(-))((-)(+)), \text{ and } \pm(+)((-)(+))((-)(-)),$$

respectively. These exteriors are appropriate tight since they can be embedded into an appropriate tight contact $\Sigma \times S^1$ with boundary slopes $2, -\frac{1}{t_1}, -\frac{1}{2}$ and decorations

$$\pm(+)((-)(+))(-+), \pm(+)((-)(-))(-+), \text{ and } \pm(+)((-)(+))(--)$$

by adding basic slices $(T^2 \times [0,1], -\frac{1}{t_2}, -\frac{1}{t_2-1})$, $\cdots$, $(T^2 \times [0,1], -\frac{1}{3}, -\frac{1}{2})$ to the boundary T_2, respectively. ■

(10) Suppose $t_0 \le 1$.

Lemma 4.28 *If $t_0 \le 1$, $t_1 \ge 2$, and $t_2 \ge 2$, then there exist $8-4t_0$ strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{3}{2}})$ whose rotation numbers are*

$$r_0 \in \pm\{t_0+1, t_0+3, \cdots, -t_0+1, -t_0+3\}, r_1 = \pm(t_1-1), r_2 = \pm(t_2-1);$$

$$r_0 \in \pm\{t_0-1, t_0+1, \cdots, -t_0-1, -t_0+1\}, r_1 = \pm(1-t_1), r_2 = \pm(t_2-1).$$

Proof By [11, Theorem 1.2, (d)], there are two Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0', r_0') = (0, \pm 1)$, $t_1 \ge 3$, $r_1 = \pm(t_1-1)$. By [11, Theorem 1.2. (b1)], there are $2(1-t_0'')$ Legendrian Hopf links $K_0 \cup K_2$ in $(S^3, \xi_{\frac{1}{2}})$ with $t_0'' \le 0$, $r_0'' \in \pm\{t_0''+1, t_0''+3, \cdots, -t_0''-1, -t_0''+1\}$, $t_2 \ge 2$, $r_2 = \pm(t_2-1)$. Using Lemma 3.1, we construct $8-4t_0$ Legendrian A_3 links in $(S^3, \xi_{3/2})$ with $t_0 \le 1, t_1 \ge 2, t_2 \ge 2$. Their rotation numbers are as listed. ■

These $8-4t_0$ strongly exceptional Legendrian A_3 links are stabilizations of the Legendrian A_3 links with $t_0 = 1, t_1 \ge 2, t_2 \ge 2$.

The proof of Theorem 1.4 is completed.

4.3 $t_1 < 0$ and $t_2 > 0$.

Lemma 4.29 *For any $t_0 \in \mathbb{Z}$, there are 6 exceptional Legendrian A_3 links whose exteriors have 0-twisting vertical Legendrian circles, and the signs of basic slices in*

L_0', L_1', L_2' are $\pm(+--)$, $\pm(++-)$, and $\pm(+-+)$, respectively. Their rotation numbers are

$$r_0 = \pm(t_0+1), r_1 = \pm(1-t_1), r_2 = \pm(t_2+1);$$
$$r_0 = \pm(t_0+1), r_1 = \pm(t_1+1), r_2 = \pm(t_2+1);$$
$$r_0 = \pm(t_0-3), r_1 = \pm(1-t_1), r_2 = \pm(1-t_2).$$

The corresponding d_3-invariants are independent of t_0 if t_1 and t_2 are fixed.

Proof The first statement follows from Lemma 2.15 and Lemma 3.3. The calculation of rotation numbers is analogous to that in the proof of Lemma 4.7. ∎

4.3.1 $t_1 < 0$ and $t_2 = 1$.

The boundary slopes of $\Sigma \times S^1$ are $s_0 = t_0$, $s_1 = -\frac{1}{t_1}$, and $s_2 = -1$.

Proof of Theorem 1.5 The upper bound of strongly exceptional Legendrian A_3 links is given by Lemma 2.12. We will show that these upper bounds can be attained.

(1) Suppose $t_0 \geq 4$.

Lemma 4.30 *If $t_0 \geq 4$, $t_1 < 0$, and $t_2 = 1$, then there exist $2-2t_1$ strongly exceptional Legendrian A_3 links in $(S^3, \xi_{-\frac{1}{2}})$ with rotation numbers*

$$r_0 = \pm(t_0+1), r_1 \in \mp\{t_1-1, t_1+1, \cdots, -t_1-1\}, r_2 = \pm 2$$

and $2-2t_1$ strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{3}{2}})$ with rotation numbers

$$r_0 = \pm(t_0-3), r_1 \in \mp\{t_1-1, t_1+1, \cdots, -t_1-1\}, r_2 = 0.$$

Proof There exist $4-4t_1$ strongly exceptional Legendrian representatives shown in Figure 17. Using the trick of Lemma 4.2 and the proof of [11, Theorem 1.2, (b1), (c3)], we can show that $K_0 \cup K_1 \cup K_2$ is a topological A_3 link. Their rotation numbers and corresponding d_3-invariants are

$$r_0 = \pm(t_0+1), r_1 \in \mp\{t_1-1, t_1+1, \cdots, -t_1-1\}, r_2 = \pm 2; d_3 = -\frac{1}{2},$$
$$r_0 = \pm(t_0-3), r_1 \in \mp\{t_1-1, t_1+1, \cdots, -t_1-1\}, r_2 = 0; d_3 = \frac{3}{2}.$$

∎

(2) Suppose $t_0 = 3$.

Lemma 4.31 *If $t_0 = 3$, $t_1 < 0$, and $t_2 = 1$, then there exist $2-2t_1$ strongly exceptional Legendrian A_3 links in $(S^3, \xi_{-\frac{1}{2}})$ with rotation numbers*

$$r_0 = \pm 4, r_1 \in \mp\{t_1-1, t_1+1, \cdots, -t_1-1\}, r_2 = \pm 2$$

and $2-t_1$ strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{3}{2}})$ with rotation numbers

$$r_0 = 0, r_1 \in \mp\{t_1-1, t_1+1, \cdots, -t_1-1\}, r_2 = 0.$$

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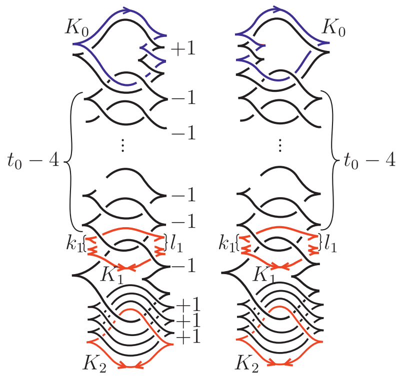

Figure 17: $t_0 \geq 4$, $t_1 \leq 0$, $t_2 = 1$. $k_1 + l_1 = -t_1$. If t_0 is even, then K_0 and K_i, $i = 1, 2$, bear the same orientations. If t_0 is odd, then the opposite orientation.

Proof By [11, Theorem 1.2], there are two Legendrian Hopf links $K_0 \cup K_2$ with $(t_0, r_0) = (3, \pm 4)$ and $(t_2, r_2) = (1, \pm 2)$ in $(S^3, \xi_{-\frac{1}{2}})$, and a Legendrian Hopf link with $(t_0, r_0) = (3, 0)$ and $(t_2, r_2) = (1, 0)$ in $(S^3, \xi_{\frac{3}{2}})$. Let K_1 be a local Legendrian meridian of K_0; then, there are $-3t_1$ strongly exceptional Legendrian A_3 links. Their rotation numbers and corresponding d_3-invariants are

$$r_0 = \pm 4, r_1 \in \{t_1 + 1, t_1 + 3, \cdots, -t_1 - 1\}, r_2 = \pm 2; d_3 = -\frac{1}{2},$$

$$r_0 = 0, r_1 \in \{t_1 + 1, t_1 + 3, \cdots, -t_1 - 1\}, r_2 = 0; d_3 = \frac{3}{2}.$$

By Lemma 4.29 and Lemma 3.4, there are 4 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are $(\pm 4, \pm(1 - t_1), \pm 2; -\frac{1}{2})$ and $(0, \pm(t_1 - 1), 0; \frac{3}{2})$. ■

(3) Suppose $t_0 = 2$.

Lemma 4.32 *If $t_0 = 2$, $t_1 < 0$, and $t_2 = 1$, then there exist $2 - 2t_1$ strongly exceptional Legendrian A_3 links in $(S^3, \xi_{-\frac{1}{2}})$ with rotation numbers*

$$r_0 = \pm 3, r_1 \in \mp\{t_1 - 1, t_1 + 1, \cdots, -t_1 - 1\}, r_2 = \pm 2$$

and 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{3}{2}})$ with rotation numbers

$$r_0 = \pm 1, r_1 = \pm(t_1 - 1), r_2 = 0.$$

Proof If $t_0 = 2$, then by [11, Theorem 1.2. (c2)], there exist two Legendrian Hopf links $K_0 \cup K_2$ in $(S^3, \xi_{-\frac{1}{2}})$ with $(t_0, r_0) = (2, \pm 3)$ and $(t_2, r_2) = (1, \pm 2)$. Let K_1 be a local Legendrian meridian of K_0; then, by Lemma 3.2, we can realize $-2t_1$ strongly exceptional Legendrian A_3 links in $(S^3, \xi_{-\frac{1}{2}})$ whose rotation numbers are

$$r_0 = \pm 3, r_1 \in \{t_1 + 1, t_1 + 3, \cdots, -t_1 - 1\}, r_2 = \pm 2.$$

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By Lemma 4.29 and Lemma 3.4, there are 4 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are $(\pm 3, \pm(1-t_1), \pm 2; -\frac{3}{2})$ and $(\pm 1, \pm(t_1-1), 0; \frac{1}{2})$. ∎

(4) Suppose $t_0 \leq 1$.

Lemma 4.33 *If $t_0 \leq 1$, $t_1 < 0$, and $t_2 = 1$, then there exist $t_0 t_1 - 2t_1$ strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{1}{2}})$ with rotation numbers*

$$r_0 \in \{t_0 - 1, t_0 + 1, \cdots, 1 - t_0\}, r_1 \in \{t_1 + 1, t_1 + 3, \cdots, -t_1 - 1\}, r_2 = 0.$$

Proof By [11, Theorem 1.2. (b2), (e)], there are $2 - t_0$ strongly exceptional Legendrian Hopf links $K_0 \cup K_2$ in $(S^3, \xi_{1/2})$ with

$$r_0 \in \{t_0 - 1, t_0 + 1, \cdots, 1 - t_0\}, t_2 = 1, r_2 = 0.$$

Let K_1 be a local Legendrian meridian of K_0. Then, by Lemma 3.2, there are $(2 - t_0)(-t_1) = t_0 t_1 - 2t_1$ strongly exceptional Legendrian A_3 links in $(S^3, \xi_{1/2})$ with rotation numbers are as listed. ∎

These $t_0 t_1 - 2t_1$ strongly exceptional Legendrian A_3 links are stabilizations of the Legendrian A_3 links with $t_0 = 1, t_1 = -1, t_2 = 1$.

The proof of Theorem 1.5 is completed.

4.3.2 $t_1 < 0$ and $t_2 \geq 2$.

The boundary slopes of $\Sigma \times S^1$ are $s_0 = t_0$, $s_1 = -\frac{1}{t_1}$, and $s_2 = -\frac{1}{t_2}$.

Proof of Theorem 1.6 The upper bound of strongly exceptional Legendrian A_3 links is given by Lemma 2.13. We will show that the upper bounds can be attained except in the cases that $t_0 = 1$, $t_1 < 0$, and $t_2 = 3$.

(1) Suppose $t_0 \geq 3$ and $t_2 = 2$.

Lemma 4.34 *If $t_0 \geq 3$, $t_1 < 0$, and $t_2 = 2$, then there exist $6 - 6t_1$ strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants are*

$$r_0 = \pm(t_0 + 1), r_1 \in \pm\{t_1 + 1, \cdots, -t_1 - 1, -t_1 + 1\}, r_2 = \pm 3; d_3 = -\frac{1}{2},$$

$$r_0 = \pm(t_0 - 1), r_1 \in \pm\{t_1 + 1, \cdots, -t_1 - 1, -t_1 + 1\}, r_2 = \pm 1; d_3 = \frac{3}{2},$$

$$r_0 = \pm(t_0 - 3), r_1 \in \pm\{t_1 + 1, \cdots, -t_1 - 1, -t_1 + 1\}, r_2 = \mp 1; d_3 = \frac{3}{2}.$$

Proof There are $6 - 6t_1$ strongly exceptional Legendrian A_3 links shown in Figure 18. Using the trick of Lemma 4.2 and the proof of [11, Theorem 1.2, (b1), (c3)], we can show that $K_0 \cup K_1 \cup K_2$ is a topological A_3 link. Their rotation numbers and corresponding d_3-invariants are as listed. ∎

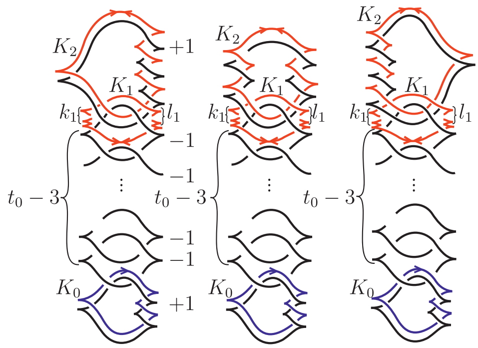

Figure 18: $t_0 \geq 3$, $t_1 \leq 0$, $t_2 = 2$. $k_1 + l_1 = -t_1$. If t_0 is even, then K_0 and K_2 bear the same orientation. If t_0 is odd, then the opposite one. If t_0 is odd, then K_0 and K_1 bear the same orientation. If t_0 is even, then the opposite one.

(2) Suppose $t_0 = t_2 = 2$.

Lemma 4.35 *If $t_0 = t_2 = 2$ and $t_1 < 0$, then there exist $6 - 4t_1$ strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants are*

$$r_0 = \pm 3, r_1 \in \pm\{t_1 + 1, \cdots, -t_1 - 1, -t_1 + 1\}, r_2 = \pm 3; d_3 = -\frac{1}{2},$$

$$r_0 = \pm 1, r_1 = \pm(1 - t_1), r_2 = \pm 1; d_3 = \frac{3}{2},$$

$$r_0 = \mp 1, r_1 \in \pm\{t_1 + 1, \cdots, -t_1 - 1, -t_1 + 1\}, r_2 = \mp 1; d_3 = \frac{3}{2}.$$

Proof If $t_0 = t_2 = 2$, then by [11, Theorem 1.2, (c2)], there are two strongly exceptional Legendrian Hopf links $K_0 \cup K_2$ in $(S^3, \xi_{-\frac{1}{2}})$ with $(t_0, r_0) = (2, \pm 3)$ and $(t_2, r_2) = (2, \pm 3)$, and two strongly exceptional Legendrian Hopf links $K_0 \cup K_2$ in $(S^3, \xi_{\frac{3}{2}})$ with $(t_0, r_0) = (2, \pm 1)$ and $(t_2, r_2) = (2, \pm 1)$. Let K_1 be a local Legendrian meridian of K_0. Then by Lemma 3.2, there are $-4t_1$ strongly exceptional Legendrian A_3 links. Their rotation numbers and corresponding d_3-invariants are

$$r_0 = \pm 3, r_1 \in \{t_1 + 1, t_1 + 3, \cdots, -t_1 - 1\}, r_2 = \pm 3; d_3 = -\frac{1}{2},$$

$$r_0 = \pm 1, r_1 \in \{t_1 + 1, t_1 + 3, \cdots, -t_1 - 1\}, r_2 = \pm 1; d_3 = \frac{3}{2}.$$

By Lemma 4.29 and Lemma 3.4, there are 4 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are $(\pm 3, \pm(1 - t_1), \pm 3; -\frac{1}{2})$ and $(\mp 1, \pm(1 - t_1), \mp 1; \frac{3}{2})$. The decorations of their exteriors are

$$\pm(+)(\underbrace{-\cdots-}_{-t_1})(--) \text{ and } \pm(+)(\underbrace{-\cdots-}_{-t_1})(++),$$

respectively.

By [11, Theorem 1.2, (b2), (d)], there are two Legendrian Hopf links in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0', r_0') = (1, 0), t_1 < 0, r_1 = \mp(t_1 - 1)$, and two Legendrian Hopf links in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0', r_0') = (0, \pm 1), (t_2, r_2) = (2, \pm 1)$. By Lemma 3.1, we can construct 2 strongly exceptional Legendrian A_3 in $(S^3, \xi_{\frac{3}{2}})$ links with $t_0 = t_2 = 2, t_1 < 0$. Their rotation numbers (r_0, r_1, r_2) are $(\pm 1, \pm(1 - t_1), \pm 1)$. The decorations of their exteriors are

$$\pm(+)(\underbrace{-\cdots-}_{-t_1})(+-).$$

■

So there are $6 - 4t_1$ strongly exceptional Legendrian A_3 links with $t_0 = 2$, $t_1 < 0, t_2 = 2$. As a corollary, the $6 - 4t_1$ contact structures on $\Sigma \times S^1$ with boundary slopes $s_0 = 2, s_1 = -\frac{1}{t_1}, s_2 = -\frac{1}{2}$ listed in Lemma 2.13 are all appropriate tight.

(3) Suppose $t_0 = 1$ and $t_2 = 2$.

Lemma 4.36 *If $t_0 = 1$, $t_1 < 0$, and $t_2 = 2$, there exist $6 - 2t_1$ strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants are*

$$r_0 = \pm 2, r_1 \in \pm\{t_1 + 1, \cdots, -t_1 - 1, -t_1 + 1\}, r_2 = \pm 3; d_3 = -\frac{1}{2},$$

$$r_0 = \mp 2, r_1 = \pm(1 - t_1), r_2 = \mp 1; d_3 = \frac{3}{2},$$

$$r_0 = 0, r_1 = \pm(1 - t_1), r_2 = \pm 1; d_3 = \frac{3}{2}.$$

Proof If $t_0 = 1$ and $t_2 = 2$, then by [11, Theorem 1.2], there are two strongly exceptional Legendrian Hopf links $K_0 \cup K_2$ with $(t_0, r_0) = (1, \pm 2)$ and $(t_2, r_2) = (2, \pm 3)$ in $(S^3, \xi_{-\frac{1}{2}})$. Let K_1 be a local Legendrian meridian of K_0. Then by Lemma 3.2, we can realize $-2t_1$ strongly exceptional Legendrian A_3 links in $(S^3, \xi_{-\frac{1}{2}})$ whose rotation numbers are

$$r_0 = \pm 2, r_1 \in \{t_1 + 1, t_1 + 3, \cdots, -t_1 - 1\}, r_2 = \pm 3.$$

By Lemma 4.29 and Lemma 3.4, there are 4 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are $(\pm 2, \pm(1 - t_1), \pm 3; -\frac{1}{2})$ and $(\mp 2, \pm(1 - t_1), \mp 1; \frac{3}{2})$. The decorations of their exteriors are

$$\pm(+)(\underbrace{-\cdots-}_{-t_1})(--) \text{ and } \pm(+)(\underbrace{-\cdots-}_{-t_1})(++),$$

respectively.

By [11, Theorem 1.2, (d)], there are two Legendrian Hopf links in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0', r_0') = (0, \mp 1), t_1 < 0, r_1 = \mp(t_1 - 1)$, and two Legendrian Hopf links in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0', r_0') = (0, \pm 1), (t_2, r_2) = (2, \pm 1)$. By Lemma 3.1, we can construct 2 strongly

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exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{3}{2}})$ with $t_0 = 1, t_1 < 0, t_2 = 2$. Their rotation numbers (r_0, r_1, r_2) are $(0, \pm(1 - t_1), \pm 1)$. The decorations of their exteriors are

$$\pm(+)(\underbrace{-\cdots-}_{-t_1})(+-).$$

■

So there are $6 - 2t_1$ strongly exceptional Legendrian A_3 links with $t_0 = 1, t_1 < 0$, $t_2 = 2$. As a corollary, the $6 - 2t_1$ contact structures on $\Sigma \times S^1$ with boundary slopes $s_0 = 1, s_1 = -\frac{1}{t_1}, s_2 = -\frac{1}{2}$ listed in Lemma 2.13 are all appropriate tight.

(4) Suppose $t_0 \geq 3$ and $t_2 \geq 3$.

Lemma 4.37 *If $t_0 \geq 3$, $t_1 < 0$, and $t_2 \geq 3$, then there are $8 - 8t_1$ strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants are*

$$r_0 = \pm(t_0 + 1), r_1 \in \pm\{t_1 + 1, t_1 + 3, \cdots, -t_1 + 1\}, r_2 = \pm(t_2 + 1); d_3 = -\frac{1}{2},$$
$$r_0 = \pm(t_0 - 1), r_1 \in \pm\{t_1 + 1, t_1 + 3, \cdots, -t_1 + 1\}, r_2 = \pm(t_2 - 1); d_3 = \frac{3}{2},$$
$$r_0 = \pm(t_0 - 3), r_1 \in \pm\{t_1 + 1, t_1 + 3, \cdots, -t_1 + 1\}, r_2 = \pm(1 - t_2); d_3 = \frac{3}{2},$$
$$r_0 = \pm(t_0 - 1), r_1 \in \pm\{t_1 + 1, t_1 + 3, \cdots, -t_1 + 1\}, r_2 = \pm(3 - t_2); d_3 = \frac{3}{2}.$$

Proof If $t_0 \geq 3$ and $t_2 \geq 3$, then there are $8 - 8t_1$ strongly exceptional Legendrian A_3 links shown in Figure 19. Using the trick of Lemma 4.2 and the proof of [11, Theorem 1.2, (b1), (c4)], we can show that $K_0 \cup K_1 \cup K_2$ is a topological A_3 link. Their rotation numbers are as listed. ■

(5) Suppose $t_0 = 2$ and $t_2 \geq 3$.

Lemma 4.38 *If $t_0 = 2$, $t_1 < 0$, and $t_2 \geq 3$, then there exist $8 - 6t_1$ strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants are*

$$r_0 = \pm 3, r_1 \in \pm\{t_1 + 1, t_1 + 3, \cdots, -t_1 + 1\}, r_2 = \pm(t_2 + 1); d_3 = -\frac{1}{2},$$
$$r_0 = \pm 1, r_1 \in \pm\{t_1 + 1, t_1 + 3, \cdots, -t_1 + 1\}, r_2 = \pm(t_2 - 1); d_3 = \frac{3}{2},$$
$$r_0 = \mp 1, r_1 = \pm(1 - t_1), r_2 = \pm(1 - t_2); d_3 = \frac{3}{2},$$
$$r_0 = \pm 1, r_1 \in \pm\{t_1 + 1, t_1 + 3, \cdots, -t_1 + 1\}, r_2 = \pm(3 - t_2); d_3 = \frac{3}{2}.$$

Proof If $t_0 = 2$ and $t_2 \geq 3$, then by [11, Theorem 1.2, (c3)], there are two Legendrian Hopf links $K_0' \cup K_2$ in $(S^3, \xi_{-\frac{1}{2}})$ with $(t_0', r_0') = (2, \pm 3)$, $t_2 \geq 3$, $r_2 = \pm(t_2 + 1)$, two Legendrian Hopf links $K_0' \cup K_2$ in $(S^3, \xi_{\frac{3}{2}})$ with $(t_0', r_0') = (2, \pm 1)$, $t_2 \geq 3$, $r_2 = \pm(t_2 - 1)$, and two Legendrian Hopf links $K_0' \cup K_2$ in $(S^3, \xi_{\frac{3}{2}})$ with $(t_0', r_0') = (2, \mp 1)$, $t_2 \geq 3$, $r_2 = \pm(t_2 - 3)$. Let K_1 be a local Legendrian meridian of K_0; then by Lemma 3.2, we can

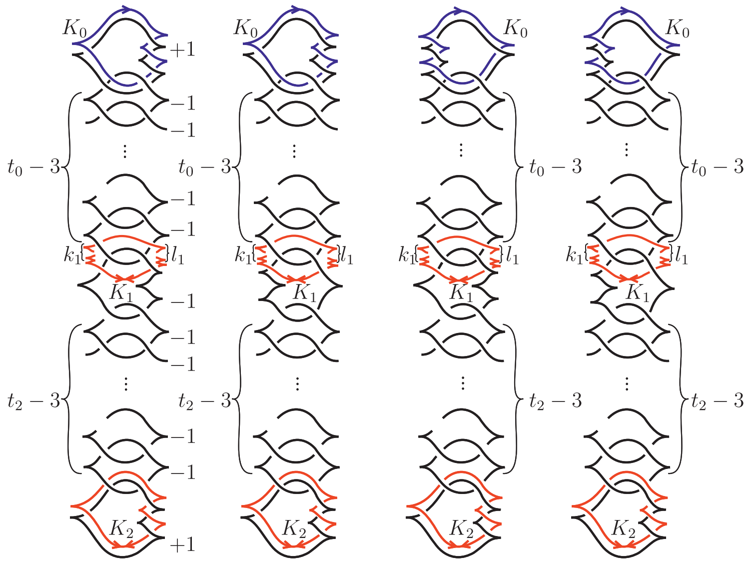

Figure 19: $t_0 \geq 3$, $t_1 \leq 0$, $t_2 \geq 3$. $k_1 + l_1 = -t_1$. For $t_0 + t_2$ even, K_0 and K_2 bear the same orientation, and for $t_0 + t_2$ odd, the opposite one. For t_0 odd, K_0 and K_1 bear the same orientation, and for t_0 even, the opposite one.

realize $-6t_1$ strongly exceptional Legendrian representatives. There are $-2t_1$ of them belonging to $(S^3, \xi_{-\frac{1}{2}})$ with rotation numbers

$$r_0 = \pm 3, r_1 \in \{t_1 + 1, t_1 + 3, \cdots, -t_1 - 1\}, r_2 = \pm(t_2 + 1).$$

There are $-4t_1$ of them belonging to $(S^3, \xi_{\frac{3}{2}})$ with rotation numbers

$$r_0 = \pm 1, r_1 \in \{t_1 + 1, t_1 + 3, \cdots, -t_1 - 1\}, r_2 = \pm(t_2 - 1);$$

$$r_0 = \mp 1, r_1 \in \{t_1 + 1, t_1 + 3, \cdots, -t_1 - 1\}, r_2 = \pm(t_2 - 3).$$

By Lemma 4.29 and Lemma 3.4, there are 4 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are $(\pm 3, \pm(1 - t_1), \pm(t_2 + 1); -\frac{1}{2})$ and $(\mp 1, \pm(1 - t_1), \pm(1 - t_2); \frac{3}{2})$. The decorations of their exteriors are

$$\pm(+)(\underbrace{-\cdots-}_{-t_1})((-)(-)) \text{ and } \pm(+)(\underbrace{-\cdots-}_{-t_1})((+)(+)),$$

respectively.

There are 4 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are $(\pm 1, \pm(1 - t_1), \pm(t_2 - 1); \frac{3}{2})$ and $(\pm 1, \pm(1 - t_1), \pm(3 - t_2); \frac{3}{2})$. The decorations of their exteriors are

$$\pm(+)(\underbrace{-\cdots-}_{-t_1})((-)(+)) \text{ and } \pm(+)(\underbrace{-\cdots-}_{-t_1})((+)(-)),$$

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respectively. These exteriors are appropriate tight since they can be embedded into an appropriate tight contact $\Sigma \times S^1$ with boundary slopes $2, -\frac{1}{t_1}, -\frac{1}{2}$ and decorations $\pm(+)(\underbrace{-\cdots-}_{-t_1})(-+)$ by adding basic slices $(T^2 \times [0,1], -\frac{1}{t_2}, -\frac{1}{t_2-1})$, $\cdots$, $(T^2 \times [0,1], -\frac{1}{3}, -\frac{1}{2})$ to the boundary T_2, respectively. ■

(6) Suppose $t_0 = 1$ and $t_2 \geq 3$.

Lemma 4.39 *If $t_0 = 1$, $t_1 < 0$, and $t_2 \geq 4$ (resp. $t_2 = 3$), then there exist $8 - 4t_1$ (resp. $8 - 3t_1$) strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants are*

$$r_0 = \pm 2, r_1 \in \{t_1+1, t_1+3, \cdots, -t_1-1\} \cup \{\pm(1-t_1)\}, r_2 = \pm(t_2+1); d_3 = -\frac{1}{2},$$

$$r_0 = 0, r_1 \in \{t_1+1, t_1+3, \cdots, -t_1-1\} \cup \{\pm(t_1-1)\}, r_2 = \pm(t_2-3); d_3 = \frac{3}{2},$$

$$r_0 = \mp 2, r_1 = \pm(1-t_1), r_2 = \pm(1-t_2); d_3 = \frac{3}{2},$$

$$r_0 = 0, r_1 = \pm(1-t_1), r_2 = \pm(t_2-1); d_3 = \frac{3}{2}.$$

Proof If $t_0 = 1$ and $t_2 = 3$, then by [11, Theorem 1.2, (c2)], there are two strongly exceptional Legendrian Hopf links $K_0 \cup K_2$ in $(S^3, \xi_{-\frac{1}{2}})$ with $(t_0, r_0) = (1, \pm 2)$ and $(t_2, r_2) = (3, \pm 4)$, and one strongly exceptional Legendrian Hopf link $K_0 \cup K_2$ in $(S^3, \xi_{\frac{3}{2}})$ with $(t_0, r_0) = (1, 0)$ and $(t_2, r_2) = (3, 0)$. Let K_1 be a local Legendrian meridian of K_0. Then by Lemma 3.2, we can realize $-3t_1$ strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants are

$$r_0 = \pm 2, r_1 \in \{t_1+1, t_1+3, \cdots, -t_1-1\}, r_2 = \pm 4; d_3 = -\frac{1}{2},$$

$$r_0 = 0, r_1 \in \{t_1+1, t_1+3, \cdots, -t_1-1\}, r_2 = 0; d_3 = \frac{3}{2}.$$

If $t_0 = 1$ and $t_2 \geq 4$, then by [11, Theorem 1.2, (c3)], there are two strongly exceptional Legendrian Hopf links $K_0 \cup K_2$ in $(S^3, \xi_{-\frac{1}{2}})$ with $(t_0, r_0) = (1, \pm 2)$, $t_2 \geq 4$, and $r_2 = \pm(t_2+1)$, and two strongly exceptional Legendrian Hopf links $K_0 \cup K_2$ in $(S^3, \xi_{\frac{3}{2}})$ with $(t_0, r_0) = (1, 0)$, $t_2 \geq 4$ and $r_2 = \pm(t_2-3)$. Let K_1 be a local Legendrian meridian of K_0. Then by Lemma 3.2, we can realize $-4t_1$ strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants are

$$r_0 = \pm 2, r_1 \in \{t_1+1, t_1+3, \cdots, -t_1-1\}, r_2 = \pm(t_2+1); d_3 = -\frac{1}{2},$$

$$r_0 = 0, r_1 \in \{t_1+1, t_1+3, \cdots, -t_1-1\}, r_2 = \pm(t_2-3); d_3 = \frac{3}{2}.$$

For any $t_2 \geq 3$, by Lemma 4.29 and Lemma 3.4, there are 4 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are $(\pm 2, \pm(1-t_1), \pm(t_2+1); -\frac{1}{2})$ and $(\mp 2, \pm(1-t_1), \pm(1-t_2); \frac{3}{2})$. The decorations of their exteriors are

$$\pm(+)(\underbrace{-\cdots-}_{-t_1})((-)(-)) \text{ and } \pm(+)(\underbrace{-\cdots-}_{-t_1})((+)(+)),$$

respectively.

For any $t_2 \geq 3$, there are 4 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are $(0, \pm(1-t_1), \pm(t_2-1); \frac{3}{2})$ and $(0, \pm(t_1-1), \pm(t_2-3); \frac{3}{2})$. The decorations of their exteriors are

$$\pm(+)(\underbrace{-\cdots-}_{-t_1})((+)(-)) \text{ and } \pm(+)(\underbrace{-\cdots-}_{-t_1})((-)(+)),$$

respectively. These exteriors are appropriate tight since they can be embedded into an appropriate tight contact $\Sigma \times S^1$ with boundary slopes $1, -\frac{1}{t_1}, -\frac{1}{2}$ and decorations $\pm(+)(\underbrace{-\cdots-}_{-t_1})(+-)$ by adding basic slices $(T^2 \times [0,1], -\frac{1}{t_2}, -\frac{1}{t_2-1}), \cdots, (T^2 \times [0,1], -\frac{1}{3}, -\frac{1}{2})$ to the boundary T_2, respectively. ■

So, there are exactly $8 - 4t_1$ (resp. exactly $8 - 3t_1$) strongly exceptional Legendrian A_3 links with $t_0 = 1, t_1 < 0, t_2 \geq 4$ (resp. $t_2 = 3$). If $t_0 = 1$, $t_1 < 0$, and $t_2 = 3$, then the decorations

$$(+)(\underbrace{+\cdots+}_{l}\underbrace{-\cdots-}_{k})((-)(+)) \text{ and } (-)(\underbrace{-\cdots-}_{k+1}\underbrace{+\cdots+}_{l-1})((+)(-))$$

correspond to the same Legendrian A_3 links with rotation numbers $r_0 = r_2 = 0, r_1 = l - k - 1$, where $k \geq 0, l \geq 1, k + l = -t_1$.

(7) Suppose $t_0 \leq 0$.

Lemma 4.40 *If $t_0 \leq 0$, $t_1 < 0$, and $t_2 > 1$, then there exist $2t_0t_1 - 2t_1$ strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{1}{2}})$ whose rotation numbers are*

$$r_0 \in \pm\{t_0 + 1, t_0 + 3, \cdots, -t_0 - 1, -t_0 + 1\},$$
$$r_1 \in \{t_1 + 1, t_1 + 3, \cdots, -t_1 - 1\}, r_2 = \pm(t_2 - 1).$$

Proof By [11, Theorem 1.2. (b1)], there are $2(1 - t_0)$ Legendrian Hopf links $K_0 \cup K_2$ in $(S^3, \xi_{1/2})$ whose rotation numbers are

$$r_0 \in \pm\{t_0 + 1, t_0 + 3, \cdots, -t_0 - 1, -t_0 + 1\}, t_2 \geq 2, r_2 = \pm(t_2 - 1).$$

Let K_1 be a local Legendrian meridian of K_0. Then by Lemma 3.2, there are $2(1 - t_0)(-t_1) = 2t_0t_1 - 2t_1$ isotopy classes. Their rotation numbers are as listed. ■

These $2t_0t_1 - 2t_1$ strongly exceptional Legendrian A_3 links are stabilizations of the Legendrian A_3 links with $t_0 = 0, t_1 = -1, t_2 > 1$.

The proof of Theorem 1.6 is completed.

4.4 $t_1 = 0$.

The boundary slopes of $\Sigma \times S^1$ are $s_0 = t_0$, $s_1 = \infty$, and $s_2 = -\frac{1}{t_2}$. The appropriate tight contact structures on $\Sigma \times S^1$ can be decomposed as $L_0' \cup L_2' \cup \Sigma' \times S^1$.

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Lemma 4.41 *For any $t_0 \in \mathbb{Z}$, there are 4 exceptional Legendrian A_3 links whose signs of basic slices in L'_0, L'_2 are $\pm(+-)$ and $\pm(++)$, respectively. Their rotation numbers are*

$$r_0 = \pm(t_0+1), r_1 = \pm 1, r_2 = \pm(t_2+1); r_0 = \pm(t_0-3), r_1 = \pm 1, r_2 = \pm(1-t_2).$$

The corresponding d_3-invariants are independent of t_0 if t_2 is fixed.

Proof The first statement follows from Lemma 2.16 and Lemma 3.3. Suppose the signs of the basic slices in L'_0 and L'_2 are + and −, respectively. Then,

$$\begin{aligned} r_0 &= -\left(\frac{1}{t_2} \ominus \frac{0}{1}\right) \bullet \frac{0}{1} - \left(\frac{0}{1} \ominus \frac{-1}{0}\right) \bullet \frac{0}{1} + \left(\frac{1}{0} \ominus \frac{t_0}{1}\right) \bullet \frac{0}{1} = -(t_0+1), \\ r_1 &= \left(\frac{-t_0}{1} \ominus \frac{-1}{0}\right) \bullet \frac{1}{0} = -1, \\ r_2 &= \left(\frac{-t_0}{1} \ominus \frac{-1}{0}\right) \bullet \frac{1}{0} - \left(\frac{1}{0} \ominus \frac{0}{1}\right) \bullet \frac{0}{1} - \left(\frac{0}{1} \ominus \frac{-1}{t_2}\right) \bullet \frac{1}{0} = -(t_2+1). \end{aligned}$$

The computation of other cases are similar. ∎

Lemma 4.42 *Suppose $t_0 \le 2$, $t_1 = 0$, and $t_2 \ge 2$. Then, there are 4 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{3}{2}})$ whose rotation numbers are*

$$r_0 = \pm(t_0-1), r_1 = \pm 1, r_2 = \pm(t_2-1); r_0 = \pm(t_0-3), r_1 = \pm 1, r_2 = \pm(1-t_2).$$

Proof By [11, Theorem 1.2, (d)], there are two Legendrian Hopf links $K'_0 \cup K_1$ in $(S^3, \xi_{\frac{1}{2}})$ with $t'_0 \le 1, r'_0 = \pm(t'_0-1), (t_1, r_1) = (0, \pm 1)$, and two Legendrian Hopf links $K''_0 \cup K_2$ in $(S^3, \xi_{\frac{1}{2}})$ with $(t''_0, r''_0) = (0, \pm 1), t_2 \ge 2, r_2 = \pm(t_2-1)$. By Lemma 3.1, we can obtain strongly exceptional Legendrian A_3 links with $t_0 \le 2, t_1 = 0, t_2 \ge 2$. So there are 4 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{3}{2}})$ whose rotation numbers are as listed. ∎

Proof of Theorem 1.7 The upper bound of strongly exceptional Legendrian A_3 links is given by Lemma 2.14. We will show that these upper bounds can be attained.

(1) Suppose $t_2 \le 0$.

Lemma 4.43 *If $t_1 = 0$ and $t_2 \le 0$, then there exist $2 - 2t_2$ strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{1}{2}})$ whose rotation numbers are*

$$r_0 = \pm(t_0-1), r_1 = \pm 1, r_2 \in \pm\{t_2+1, t_2+3, \cdots, -t_2+1\}.$$

Proof If $t_2 \le 0$ and $t_0 \le 0$, there exist $2(1-t_2)$ strongly exceptional Legendrian A_3 links shown in Figure 20. Similar to the proof of [11, Lemma 5.1, part (iii), Figure 6], we can show that the link $K_0 \cup K_1 \cup K_2$ in Figure 20 is indeed a topological A_3 link. By performing the same calculations as in the proof of Theorem 1.2 (d) in [11], we can determine that their rotation numbers are as listed. Moreover, the corresponding d_3-invariant is $\frac{1}{2}$.

If $t_2 \le 0$ and $t_0 = 1$ (resp. $t_0 \ge 2$), then there exist $2(1-t_2)$ strongly exceptional Legendrian A_3 links shown in Figure 9 (resp. Figure 8) with $k_1 = l_1 = 0$. Their rotation numbers and the corresponding d_3-invariants are as listed. ∎

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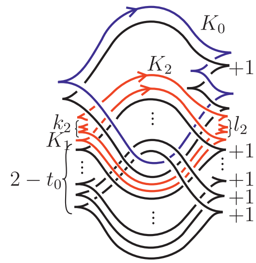

Figure 20: $t_0 \leq 0$, $t_1 = 0$, $t_2 \leq 0$, $k_2 + l_2 = -t_2$.

(2) Suppose $t_2 = 1$.

Lemma 4.44 *If $t_1 = 0$ and $t_2 = 1$, then there exist 4 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are*

$$(\pm(t_0 - 3), \pm 1, 0; \frac{3}{2}), (\pm(t_0 + 1), \pm 1, \pm 2; -\frac{1}{2}).$$

Proof If $t_2 = 1$ and $t_0 \geq 4$, then there exist 4 strongly exceptional Legendrian A_3 links shown in Figure 17 with $k_1 = l_1 = 0$. Their rotation numbers and corresponding d_3-invariants are as listed.

Suppose $t_2 = 1$ and $t_0 \leq 3$. By [11, Theorem 1.2, (d), (c1)], there are two Legendrian Hopf links $K_0' \cup K_1$ in $(S^3, \xi_{\frac{1}{2}})$ with $t_0' \leq 1, r_0' = \pm(t_0' - 1), (t_1, r_1) = (0, \pm 1)$, and one Legendrian Hopf link $K_0'' \cup K_2$ in $(S^3, \xi_{\frac{1}{2}})$ with $(t_0'', r_0'') = (t_2, r_2) = (1, 0)$. By Lemma 3.1, we can obtain strongly exceptional Legendrian A_3 links with $t_0 \leq 3, t_1 = 0, t_2 = 1$. So there are 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{3}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm(t_0 - 3), \pm 1, 0)$.

Moreover, by Lemma 4.41 and Lemma 3.4, there are other 2 Legendrian A_3 links in $(S^3, \xi_{-\frac{1}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm(t_0 + 1), \pm 1, \pm 2)$. ■

(3) Suppose $t_2 = 2$.

Lemma 4.45 *If $t_1 = 0$ and $t_2 = 2$, then there exist 6 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are*

$$\left(\pm(t_0 + 1), \pm 1, \pm 3; -\frac{1}{2}\right), \left(\pm(t_0 - 1), \pm 1, \pm 1; \frac{3}{2}\right), \left(\pm(t_0 - 3), \pm 1, \mp 1; \frac{3}{2}\right).$$

Proof If $t_2 = 2$ and $t_0 \geq 3$, then there exist 6 strongly exceptional Legendrian A_3 links shown in Figure 18 with $k_1 = l_1 = 0$. Their rotation numbers and corresponding d_3-invariants are as listed.

If $t_2 = 2$ and $t_0 \leq 2$, then by Lemma 4.41 and Lemma 3.4, there exist 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{-\frac{1}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm(t_0 + 1), \pm 1, \pm 3)$.

Moreover, by Lemma 4.42, there exist 4 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{3}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm(t_0 - 1), \pm 1, \pm 1)$ and $(\pm(t_0 - 3), \pm 1, \mp 1)$. ■

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(4) Suppose $t_2 \geq 3$.

Lemma 4.46 *If $t_1 = 0$ and $t_2 \geq 3$, then there exist 8 strongly exceptional Legendrian A_3 links whose rotation numbers and corresponding d_3-invariants $(r_0, r_1, r_2; d_3)$ are*

$$\left(\pm(t_0+1), \pm 1, \pm(t_2+1); -\frac{1}{2}\right), \left(\pm(t_0-1), \pm 1, \pm(t_2-1); \frac{3}{2}\right),$$
$$\left(\pm(t_0-1), \pm 1, \pm(3-t_2); \frac{3}{2}\right), \left(\pm(t_0-3), \pm 1, \pm(1-t_2); \frac{3}{2}\right).$$

Proof If $t_2 \geq 3$ and $t_0 \geq 3$, then there are exactly 8 strongly exceptional Legendrian A_3 links shown in Figure 19 with $k_1 = l_1 = 0$. Their rotation numbers and corresponding d_3-invariants are as listed.

Suppose $t_2 \geq 3$ and $t_0 \leq 2$. By Lemma 4.41 and Lemma 3.4, there exist 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{-\frac{1}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm(t_0+1), \pm 1, \pm(t_2+1))$.

By Lemma 4.42, there exist 4 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{3}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm(t_0-1), \pm 1, \pm(t_2-1))$ and $(\pm(t_0-3), \pm 1, \pm(1-t_2))$.

Moreover, there are 2 strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{3}{2}})$ whose rotation numbers (r_0, r_1, r_2) are $(\pm(t_0-1), \pm 1, \pm(3-t_2))$. The decorations of their exteriors are $\pm(+)((-)(+))$. These exteriors are appropriate tight since they can be embedded into an appropriate tight contact $\Sigma \times S^1$ with boundary slopes $t_0, \infty, -\frac{1}{2}$ and decoration $\pm(+)(-+)$ by adding basic slices $(T^2 \times [0,1], -\frac{1}{t_2}, -\frac{1}{t_2-1}), \cdots, (T^2 \times [0,1], -\frac{1}{3}, -\frac{1}{2})$ to the boundary T_2. ∎

The proof of Theorem 1.7 is completed.

Proof of Theorem 1.8 It follows from the proof of Theorems 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7. ∎

5 Stabilizations

The aim of this section is to elucidate Remark 1.10.

5.1 Stabilizations of the component K_0.

For the strongly exceptional Legendrian A_3 links with $t_1, t_2 \neq 0$ and $t_0 + \lceil -\frac{1}{t_1} \rceil + \lceil -\frac{1}{t_2} \rceil \geq 2$, their exteriors have 0-twisting vertical Legendrian circles. So by Lemma 3.5, the component K_0 can always be destabilized. For the strongly exceptional Legendrian A_3 links with $t_1 = 0$, their exteriors obviously have 0-twisting vertical Legendrian circles. By the same reason, the component K_0 can be destabilized.

As examples, we list the mountain ranges of the component K_0 in some Legendrian A_3 links with fixed t_1, t_2.

(1) Strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{5}{2}})$ with $r_0 = \pm(t_0-5), r_1 = r_2 = 0$, where $t_0 \geq 5, t_1 = t_2 = 1$. See Lemmas 4.8 and 4.9. Their exteriors have decorations $\pm(+)(+)(+)$. The mountain range is depicted in the upper left of Figure 21. It is infinite on the upper side.

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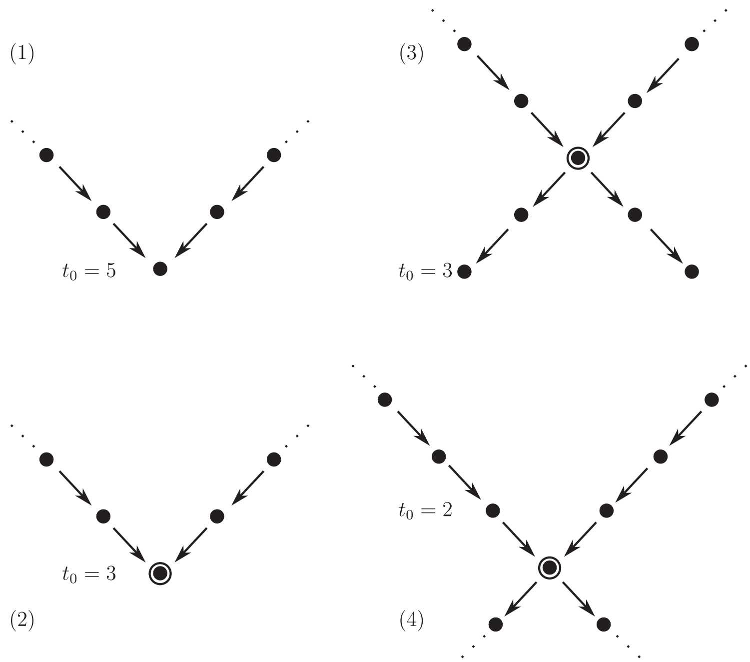

Figure 21: The mountain ranges of some strongly exceptional Legendrian A_3 links with fixed t_1 and t_2. Each dot represents a Legendrian A_3 link. A dot with a circle represents two Legendrian A_3 links. Each arrow represents a stabilization.

(2) Strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{5}{2}})$ with $r_0 = \pm(t_0 - 3), r_1 = \pm(t_1 - 1), r_2 = \pm(1 - t_2)$, where $t_0, t_1, t_2 \geq 3$. See Lemmas 4.25 and 4.26. Their exteriors have decorations $\pm(+)((+)(-))((+)(+))$. The mountain range is depicted in the lower left of Figure 21. It is infinite on the upper side.

(3) Strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{5}{2}})$ with $r_0 = \pm(t_0 - 5), r_1 = \pm(1 - t_1), r_2 = \pm(1 - t_2)$, where $t_0, t_1, t_2 \geq 3$. See Lemmas 4.25 and 4.26. Their exteriors have decorations $\pm(+)((+)(+))((+)(+))$. The mountain range is depicted in the upper right of Figure 21. It is infinite on the upper side.

(4) Exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{1}{2}})$ with $r_0 = \pm(t_0 - 1), r_1 = \pm(1 - t_1), r_2 = \pm(t_2 + 1)$, where $t_1, t_2 \geq 3$. Their exteriors have decorations $\pm(+)((+)(+))((-)(-))$. See Lemmas 4.7, 4.25, 4.26, and 4.27. The mountain range of such links is depicted in the lower right of Figure 21. It is infinite on both the upper and lower sides. The exteriors of such A_3 links have decorations $\pm(+)((+)(+))((-)(-))$. If $t_0 \geq 2$, then they are strongly exceptional. If $t_0 < 2$, then, based on Lemma 4.7 and Lemma 2.6, they are exceptional but not strongly exceptional.

In a more general setting, with a fixed decoration and nonzero integers t_1 and t_2, if L'_0 and the innermost basic slices of L'_1 and L'_2 have the same signs (possibly after shuffling), then the components K_0 of the strongly exceptional Legendrian A_3 links exhibit mountain ranges with shapes resembling a "V" or an "X" truncated from the lower side, as shown in the first three subfigures in Figure 21.

For the strongly exceptional Legendrian A_3 links with fixed $t_1, t_2 \neq 0$ and $t_0 + \lceil -\frac{1}{t_1} \rceil + \lceil -\frac{1}{t_2} \rceil \leq 1$, the mountain ranges of the component K_0 can be observed through Lemma 4.6, Lemma 4.11, Lemma 4.18, Lemma 4.28, Lemma 4.33, and Lemma 4.40.

5.2 Stabilizations of the component K_2 when $t_1 = 0$.

The strongly exceptional Legendrian A_3 links with $t_1 = 0$ are classified in Theorem 1.7. The exteriors of such Legendrian A_3 links contain 0-twisting vertical Legendrian circles. By Lemma 3.5, the component K_2 is always destabilizable unless $t_2 = 0$.

We list the mountain range of the component K_2 of the strongly exceptional Legendrian A_3 links with fixed t_0 and $t_1 = 0$. The exteriors of such Legendrian A_3 links can be decomposed into $L_0' \cup L_2' \cup \Sigma' \times S^1$. Recall that if $t_2 \geq 3$, then L_2' consists of 2 basic slices and is not a continued fraction block. If $t_2 = 2$, then L_2' is a continued fraction block consisting of 2 basic slices. If $t_2 = 1$, then L_2' is a basic slice. If $t_2 = 0$, then L_2' is an empty set. If $t_2 \leq -1$, then L_2' is a continued fraction block consisting of $-t_2$ basic slices.

(1) Strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{3}{2}})$ with $r_0 = \pm(t_0 - 3), r_1 = \pm 1, r_2 = \pm(1 - t_2)$, where $t_1 = 0, t_2 \geq 1$. See Lemmas 4.44, 4.45, and 4.46. The signs of the basic slices in L_0' and L_2' are all the same. The mountain range is depicted in the upper left of Figure 22.

(2) Strongly exceptional Legendrian A_3 links in $(S^3, \xi_{-\frac{1}{2}})$ with $r_0 = \pm(t_0 + 1), r_1 = \pm 1, r_2 = \pm(t_2 + 1)$, where $t_1 = 0, t_2 \geq 1$. See Lemmas 4.44, 4.45, and 4.46. The sign of L_0' and the sign of each of the basic slices in L_2' are opposite. The mountain range can be depicted in the upper left of Figure 22.

(3) Strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{3}{2}})$ with $r_0 = \pm(t_0 - 1), r_1 = \pm 1, r_2 = \pm(t_2 - 1)$ (or $r_2 = \pm(3 - t_2)$), where $t_1 = 0, t_2 \geq 2$. See Lemmas 4.45 and 4.46. Their exteriors have decorations $\pm(+)((+)(-))$ (or $\pm(+)((-)(+))$) if $t_2 \geq 3$, and $\pm(+)(+-)$ if $t_2 = 2$. Note that when $t_2 = 2$, L_2' is a continued fraction block, and hence,

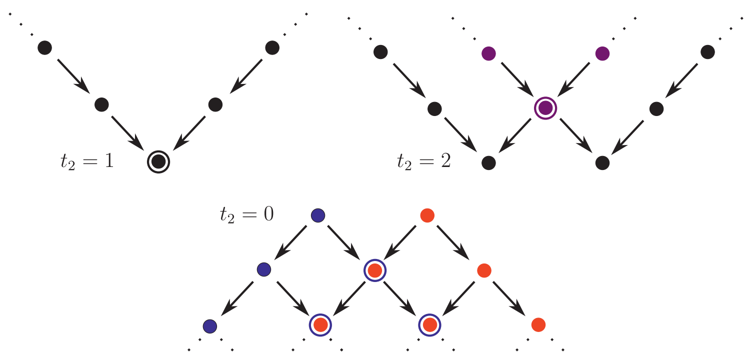

Figure 22: The mountain ranges of the strongly exceptional Legendrian A_3 links with fixed t_0 and $t_1 = 0$.

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the two decorations $(+)(+-)$ and $(+)(-+)$ (or $(-)(-+)$ and $(-)(+-)$) stand for the same Legendrian A_3 link. So the mountain range can be depicted in the upper right of Figure 22.

(4) Strongly exceptional Legendrian A_3 links in $(S^3, \xi_{\frac{1}{2}})$ with $r_0 = \pm(t_0 - 1), r_1 = \pm 1, r_2 \in \pm\{t_2 + 1, t_2 + 3, \cdots, -t_2 + 1\}$, where $t_1 = 0, t_2 \leq 0$. See Lemma 4.43. It is easy to know that the mountain range can be depicted in the lower part of Figure 22.

In conclusion, the whole mountain range of the strongly exceptional Legendrian A_3 links with fixed t_0 and $t_1 = 0$ consists of two copies of the upper left subfigure, the upper right subfigure, and the lower subfigure of Figure 22.

6 Some Computations

Here, we summarize how to compute the classical invariants of Legendrian realizations $A_3 = K_0 \cup K_1 \cup K_2$ of the connected sum of two Hopf links, and the d_3-invariant of the contact 3-sphere S^3 containing the realizations. We compute the invariants of the first surgery diagram on the top left of Figure 14. Similar arguments apply to all remaining examples. For the example in Figure 14, the linking matrix M is the $(t_0 - 1) \times (t_0 - 1)$-matrix, which we form by ordering the surgery curves from bottom to top where all are oriented clockwise:

$$M = \begin{bmatrix} -2 & -1 & -1 & & & & & & & \\ -1 & -2 & -1 & & & & & & & \\ -1 & -1 & -2 & -1 & & & & & & \\ & & -1 & -2 & -1 & & & & & \\ & & & -1 & -2 & -1 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & & -1 & \\ & & & & & & & -1 & -2 & -1 \\ & & & & & & & & -1 & -1 \end{bmatrix}.$$

The determinant of M is $\det M = (-1)^{t_0 - 1}$.

6.1 The d_3-invariant.

Let $(Y, \xi) = \partial X$ be a contact 3-manifold given by contact (± 1)-surgeries on a Legendrian link $\mathbb{L} \in (S^3, \xi_{st})$, all of which have the nonvanishing Thurston-Bennequin invariant. We compute the d_3-invariant of (Y, ξ) with $c_1(\xi)$ torsion by following the formula from [4, Corollary 3.6]:

$$d_3(\xi) = \frac{1}{4}(c^2 - 3\sigma(X) - 2\chi(X)) + q,$$

where q is the number of $(+1)$-surgery components in $\mathbb{L}$ and $c \in H^2(X)$ is the cohomology class determined by $c(\Sigma_i)$ for each $L_i \in \mathbb{L}$, where Σ_i is the Seifert surface of L_i glued with the core disk of the corresponding handle. We read σ and χ from the surgery diagram in Figure 14. The signature σ is the signature of the linking matrix M. The surgery diagram is topologically equivalent to $(t_0 - 1)$ unlinked -1-framed unknots, so the signature is $\sigma(X) = -(t_0 - 1)$. The Euler characteristic is

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$\chi(X) = t_0 - 1 + 1 = t_0$ since each surgery knot corresponds to attaching a 2-handle. We compute c^2 by following the algorithm in [4], $c^2 = \mathtt{x}^t M\mathtt{x} =< \mathtt{x}, \underline{\mathtt{rot}} >$, where $\underline{\mathtt{rot}} = (rot(L_1), \dots, rot(L_n))$ is the vector rotation number of the Legendrian surgery knots $L_i \subset \mathbb{L}$ and $\mathtt{x}$ is the solution vector of $M\mathtt{x} = \underline{\mathtt{rot}}$. For the surgery diagram on top left of Figure 14, the vector rotation number is

$$\underline{\mathtt{rot}} = (2, -2, 0, \dots, 0, 1)^t.$$

The solution vector $\mathtt{x}$ is

$$\mathtt{x} = (-1, 3, *, \dots, *, -(t_0 - 1))^t \text{ for } t_0 \text{ even,}$$

and

$$\mathtt{x} = (-3, 1, *, \dots, *, -(t_0 - 1))^t \text{ for } t_0 \text{ odd.}$$

This gives $c^2 =< \mathtt{x}, \underline{\mathtt{rot}} >= -6 - 2 + 0 + \cdots + 0 - (t_0 - 1) = -7 - t_0$. Observing that $q = 3$ in this example, we compute

$$d_3 = \frac{1}{4}(-7 - t_0 - 3(-(t_0 - 1)) - 2t_0) + 3 = \frac{1}{2}.$$

6.2 The Thurston-Bennequin invariant and the rotation number.

We use the formulae in [15, Lemma 6.6] to compute the Thurston-Bennequin invariant and the rotation number of a Legendrian knot L in a contact (± 1)-surgery diagram of surgery link $\mathbb{L}$ with the linking matrix M. The Thurston-Bennequin invariant is

$$tb(L) = tb(L_0) + \frac{\mathtt{det}\, M_0}{\mathtt{det}\, M},$$

where $tb(L_0)$ is the Thurston-Bennequin invariant of L as a knot in (S^3, ξ_{st}) before the contact surgeries, and M_0 is the extended linking matrix which is the linking matrix of $L_0 \cup \mathbb{L}$ with the convention that $lk(L_0, L_0) = 0$. The rotation number of L after surgery is

$$rot(L) = rot(L_0) - < \underline{\mathtt{rot}}, M^{-1}\underline{\mathtt{lk}} >,$$

where $rot(L_0)$ is the rotation number of L before surgeries, $\underline{\mathtt{rot}}$ is the vector rotation number of the Legendrian surgery knots $L_i \subset \mathbb{L}$, and $\underline{\mathtt{lk}} = (lk(L, L_1), \dots, lk(L, L_n))$ is the vector of the linking numbers.

For the surgery diagram on the top left of Figure 14, we assume that K_0, K_1, and K_2 are oriented clockwise. So the extended linking matrices for K_0, K_1 and, K_2 are, respectively,

$$M_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 & -1 & -2 \\ 0 & & & & & \\ \vdots & & & M & & \\ 0 & & & & & \\ -1 & & & & & \\ -2 & & & & & \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 & -1 & -3 & -1 & 0 \cdots & 0 \\ -1 & & & & & \\ -3 & & & & & \\ -1 & & & M & & \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & & & & & \end{bmatrix},$$

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$$M_2 = \begin{bmatrix} 0 & -3 & -1 & -1 & 0 \cdots & 0 \\ -3 & & & & & \\ -1 & & & & & \\ -1 & & & M & & \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & & & & & \end{bmatrix}.$$

The determinants are $\texttt{det}\, M_0 = (-1)^{t_0-1}(t_0+2)$ and $\texttt{det}\, M_1 = \texttt{det}\, M_2 = 5(-1)^{t_0-1}$. We compute the Thurston-Bennequin invariants as follows:

$$tb(K_0) = -2 + \frac{(-1)^{t_0-1}(t_0+2)}{(-1)^{t_0-1}} = t_0, \text{ and } tb(K_1) = tb(K_2) = -3 + \frac{5(-1)^{t_0-1}}{(-1)^{t_0-1}} = 2.$$

Recall that for t_0 odd, K_0 and K_i are given the same orientation, and for t_0 even, the opposite one, where $i = 1, 2$. If t_0 is odd, then K_i is oriented clockwise. If t_0 is even, then K_i is oriented counterclockwise. We compute the rotation numbers as follows:

$$r_0 = 1 - \left\langle \begin{bmatrix} 2 \\ -2 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, M^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -1 \\ -2 \end{bmatrix} \right\rangle = 1 - \left\langle \begin{bmatrix} 2 \\ -2 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} (-1)^{t_0-1} \\ (-1)^{t_0-1} \\ * \\ * \\ * \\ t_0 \end{bmatrix} \right\rangle$$

$$= 1 - (2 - 2 + 0 + \cdots + 0 + t_0) = -(t_0 - 1),$$

$$r_1 = 2(-1)^{t_0} - \left\langle \begin{bmatrix} 2 \\ -2 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, M^{-1} \begin{bmatrix} (-1)^{t_0} \\ 3(-1)^{t_0} \\ (-1)^{t_0} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\rangle = 2(-1)^{t_0} - \left\langle \begin{bmatrix} 2 \\ -2 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2(-1)^{t_0-1} \\ * \\ * \\ * \\ 1 \end{bmatrix} \right\rangle$$

$$= 2(-1)^{t_0} - (0 + 4(-1)^{t_0} + 0 + \cdots + 0 + 1) = \begin{cases} 1 & \text{if } t_0 \text{ is odd,} \\ -3 & \text{if } t_0 \text{ is even,} \end{cases}$$

$$r_2 = 2(-1)^{t_0-1} - \left\langle \begin{bmatrix} 2 \\ -2 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, M^{-1} \begin{bmatrix} 3(-1)^{t_0} \\ (-1)^{t_0} \\ (-1)^{t_0} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\rangle = 2(-1)^{t_0-1} - \left\langle \begin{bmatrix} 2 \\ -2 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2(-1)^{t_0-1} \\ 0 \\ * \\ * \\ * \\ 1 \end{bmatrix} \right\rangle$$

$$= 2(-1)^{t_0-1} - (4(-1)^{t_0-1} + 0 + \cdots + 0 + 1) = \begin{cases} -3 & \text{if } t_0 \text{ is odd,} \\ 1 & \text{if } t_0 \text{ is even.} \end{cases}$$

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