A FAMILY OF REGULAR MAPS OF TYPE $\{6, 6\}$

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Introduction. In (4), pp. 25-27, Coxeter divided the regular maps on a surface of genus 1 into three infinite families. They are:

(i) Maps of type $\{4, 4\}$.

(ii) Maps of type $\{6, 3\}$.

(iii) Maps of type $\{3, 6\}$ (the duals of (ii)).

We consider the family (iii). By adjoining an element to the group of any map in (iii) we shall derive the group of a regular map of type $\{6, 6\}$. Thus we produce a 1 - 1 correspondence between the members of the family (iii) and of the new family. Corresponding members in the two families have certain properties in common, the most interesting of which is the property of reflexibility. Our results are summarized in Theorems 1 and 2.

1. Regular Maps and their Groups. A map is a partitioning of an unbounded surface into simply-connected, non-overlapping regions called faces by means of line segments called edges. The intersections of the edges are called vertices. An automorphism of a map is a permutation of its elements that preserves incidences. If among the automorphisms there exists an element R which cyclically permutes the edges surrounding some face α of the map, and another element S which cyclically permutes the edges meeting at a

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vertex of α, then the map is said to be regular. It follows that if a given map is regular, then the same number, say p, of edges surround each face, and also the same number, say q, of edges meet at each vertex. The map is therefore said to be of type $\{p, q\}$.

The group generated by R and S is called the group of the map. R and S satisfy the relations[1]

$$(1.1) \qquad R^p = S^q = (RS)^2 = E ,$$

where E denotes the identity element.

The element RS leaves invariant an edge a of the map while interchanging its two vertices and the two faces which it borders. If, in addition, there is an automorphism R_1 which interchanges the vertices of a but leaves the two bordering faces fixed, we say that the regular map is reflexible. If the surface of the map is non-orientable, R_1 is contained in the group of the map; if the surface is orientable, then the group of the map is a subgroup of index 2 in the extended group (2, p. 125) with generators R_1, $R_2 = R_1R$, and $R_3 = R_2S$, satisfying the relations

$$(1.2) \qquad R_1^{\,2} = R_2^{\,2} = R_3^{\,2} = (R_1R_2)^p = (R_2R_3)^q = (R_3R_1)^2 = E.$$

It should be noted that the group of the map in our terminology is $\{R, S\}$, the group generated by R and S. Frequently (6, p. 100; 5, p. 386) this name is given to the extended group. We prefer our notation, which is due to Brahana (1, p. 269), because our attention will be turned entirely to the group $\{R, S\}$.

The relations (1.1) define the group denoted by $[p, q]^+$, which is the group of a regular map of type $\{p, q\}$ covering the sphere, the Euclidean plane, or the hyperbolic plane, according as $(p - 2)(q - 2)$ is less than, equal to, or greater than 4. In the latter two cases the map has an infinite number of

[1] Throughout this paper we read products from left to right. Thus if A and B are elements of a group, then AB is the element obtained by performing A first and then performing B.

vertices, edges, and faces, and is called a <u>regular tessellation</u> (6, pp. 52-56). If a regular map has a finite number of vertices, edges, and faces, say N_0, N_1, and N_2 respectively, then it lies on a surface of <u>Euler characteristic</u> χ, where

$$\chi = N_0 - N_1 + N_2 . \tag{1.3}$$

If the surface is orientable, it is of genus $\underline{p}$, where $\chi = 2 - 2\underline{p}$. A regular map on a surface of genus $\underline{p}$ is called for brevity a regular map of genus $\underline{p}$. The regular maps of genera 0, 1, 2, and 3 have all been tabulated (3, pp. 11, 12; 4, pp. 25-27; 6, p. 141; 8, p. 475).

2. <u>The Regular Maps</u> $\{6, 6\}_{b,c}$. Let us consider the group $H_{b,c}$ defined by

$$\rho^3 = \sigma^6 = (\rho\sigma)^2 = (\rho^{-1}\sigma^2)^b(\rho\sigma^{-2})^c = E \qquad (b \geq c \geq 0). \tag{2.1}$$

If $b = c = 0$, this is simply the group $[3, 6]^+$ of the regular tessellation $\{3, 6\}$. In all other cases it is the group of the regular map $\{3, 6\}_{b,c}$ of genus 1 having k vertices, 3k edges, and 2k faces, where $k = b^2 + bc + c^2$ (4, p. 25). The order of $H_{b,c}$ is 6k, unless $b = c = 0$, in which case it is infinite.

Now there is an (inner) automorphism of $H_{b,c}$, defined by the correspondence

$$\rho \to (\sigma\rho)\, \rho\, (\sigma\rho) = \sigma\rho\sigma^{-1}$$

$$\sigma \to (\sigma\rho)\, \sigma\, (\sigma\rho) = \rho^{-1}\sigma\rho.$$

We shall therefore adjoin to $H_{b,c}$ an element T of period 2, which transforms ρ and σ according to this automorphism. We thus obtain a group $G_{b,c}$, in which $H_{b,c}$ is a subgroup of index 2 (6, p. 5). $G_{b,c}$ has the abstract definition

$$\rho^3 = \sigma^6 = (\rho\sigma)^2 = (\rho^{-1}\sigma^2)^b(\rho\sigma^{-2})^c = T^2 = E , \tag{2.2}$$

$$T\rho T = \sigma\rho\sigma^{-1}, \quad T\sigma T = \rho^{-1}\sigma\rho.$$

We note that, since $T\sigma\rho$ commutes with ρ and σ,

$$G_{b,c} \cong H_{b,c} \times \mathcal{L}_2$$

$\mathcal{L}_2$ is the cyclic group of order 2 (6, p. 1).

The last relation of (2.2), together with the third and first, yields

$$(T\sigma)^2 = \rho^{-1}\sigma\rho\sigma = \rho^{-2} = \rho .$$

Therefore we let $R = T\sigma$ and $S = \sigma^{-1}$. Then $\rho = R^2$, $\sigma = S^{-1}$, and $T = RS$. The relations (2.2) then reduce to

$$(2.3) \quad R^6 = S^6 = (RS)^2 = (R^2S^{-1})^2 = (R^{-2}S^{-2})^b(R^2S^2)^c = E .$$

Thus $G_{b,c}$ may be defined in terms of the two generators R and S. It is therefore the group of a regular map of type $\{6, 6\}$, which, in analogy with $\{3, 6\}_{b,c}$, we shall denote by $\{6, 6\}_{b,c}$. In all cases except $b = c = 0$, $\{6, 6\}_{b,c}$ has a finite number of vertices, edges and faces, namely $2k$, $6k$, and $2k$ $(k = b^2 + bc + c^2)$ respectively. Applying formula (1.3), we see that the map is of genus $k + 1$.

It is easily seen that $\{6, 6\}_{1,0}$ is the map $\{6, 6\}_2$ of genus 2 (6, p. 141). The other regular maps $\{6, 6\}_{b,c}$ appear to be new. Of special interest are those maps for which $b > c > 0$, for they are non-reflexible, as the following argument proves. A regular map is reflexible if and only if there is an automorphism R_1 such that $R_1R^pR_1 = R^{-p}$ $(0 \leqslant p < 6)$, and $R_1SR_1 = RS^{-1}R^{-1}$ (cf. § 1). This is easily seen to be equivalent to the condition that there is an automorphism $R' (= R^{-1}R_1R)$ such that $R'R^pR' = R^{-p}$ and $R'SR' = R^{-1}S^{-1}R$. Thus $\{6, 6\}_{b,c}$ is reflexible if and only if there is an R_1 which transforms $\rho = R^2$ into $R^{-2} = \rho^{-1}$ and $\sigma = S^{-1}$ into $RSR^{-1} = S^{-1}R^{-2} = S^{-1}R^4 = R^{-2}SR^2$ (since $(R^{-2}S)^2 = E) = \rho^{-1}\sigma^{-1}\rho$; i.e., $\{6, 6\}_{b,c}$ is reflexible if and only if $\{3, 6\}_{b,c}$ is reflexible. But it is known that $\{3, 6\}_{b,c}$ is reflexible if and only if $bc(b - c) = 0$ (6, p. 107).

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The simplest non-reflexible map $\{6, 6\}_{b,c}$ is therefore $\{6, 6\}_{2,1}$, which has 14 faces and is of genus 8 (cf. Fig. 1). This is much simpler than the non-reflexible regular map of type $\{12, 3\}$ and genus 55 discovered by Frucht (7, p. 247), but not quite as simple as the one of type $\{7, 7\}$ and genus 7 recently discovered by J. R. Edmonds (cf. 5, p. 388).

Summarizing the above results, we have

THEOREM 1. <u>The group</u> $G_{b,c}$ <u>defined by</u> (2.3) <u>is of order</u> $12k$, <u>where</u> $k = b^2 + bc + c^2$, <u>unless</u> $b = c = 0$, <u>in which case it is infinite</u>. <u>If</u> $b \neq 0$, $G_{b,c}$ <u>is the group of a regular map</u> $\{6, 6\}_{b,c}$ <u>of type</u> $\{6, 6\}$ <u>and genus</u> $k + 1$. <u>The map is reflexible if and only if</u> $bc(b - c) = 0$.

Whenever a map of type $\{p, p\}$ is given, it is possible by the process of 'truncation' to derive a map of type $\{p, 4\}$ on the same surface (3, p. 145; 2, p. 128). Such a map may be denoted by the symbol $\left\{ {p \atop p} \right\}$. Now the regularity of $\{p, p\}$ does not necessarily imply the regularity of $\left\{ {p \atop p} \right\}$. In fact, $\left\{ {p \atop p} \right\}$ is regular if and only if there is an (outer) automorphism of the group G of $\{p, p\}$ which interchanges the two generators R and S (cf. 1.1). Denoting by $\left\{ {6 \atop 6} \right\}_{b,c}$ the map obtained by truncating $\{6, 6\}_{b,c}$, we prove

THEOREM 2. <u>The only regular maps</u> $\left\{ {6 \atop 6} \right\}_{b,c}$ <u>are</u> $\left\{ {6 \atop 6} \right\}_{1,0}$ <u>and</u> $\left\{ {6 \atop 6} \right\}_{2,0}$.

Proof. Suppose that $\left\{ {6 \atop 6} \right\}_{b,c}$ is regular. Then there is an automorphism of $G_{b,c}$ which interchanges the generators R and S. This implies that to each relation in (2.3), a corresponding relation, obtained by interchanging R and S, is true. In particular,

$$(R^2S^{-1})^2 = (S^2R^{-1})^2 = E .$$

Now $(RS)^2 = (S^2R^{-1})^2 = E$ implies that $R = S^3RS^3$. Hence

$$
\begin{aligned}
(R^2S^2)^2 &= S(S^{-1}R^2S^2R^2S^3)S^{-1} \\
&= S(R^{-2}S^3R^2S^3)S^{-1} \qquad (\text{Since } (R^2S^{-1})^2 = E) \\
&= S(R^{-2}R^2)S^{-1} \\
&= E .
\end{aligned}
$$

Conversely, $(R^2S^{-1})^2 = (R^2S^2)^2 = E$ implies

$$R^{-1}(R^3S^2R^3)R^{-1}S^2 = E ,$$

i.e.

$$(R^{-1}S^2)^2 = E .$$

Therefore $c = 0$ and $b = 1$ or 2.

$\{{6 \atop 6}\}_{1,0}$ is $\{6, 4|2\}$ of genus 2 (6, p. 141), while, by Theorem 1, $\{{6 \atop 6}\}_{2,0}$ is of genus 5.

Finally, as a generalization, it may be remarked that, given any regular map of type $\{2r + 1, q\}$ $(r = 1, 2, \ldots)$ and genus $\underline{p}$, we can, by a process entirely analogous to the above, derive a regular map of type $\{2(2r + 1), q\}$. The new map has the same number of faces as the original, but twice the number of edges and vertices. Therefore, if $\underline{p} \neq 1$, it is of genus

$$1 + \frac{4(p - 1)\,[1 + r(2 - q)]}{4 + (2r - 1)(2 - q)}$$

(this formula is easily derived using (1.3) and the formula $qN_0 = 2N_1 = (2r + 1)N_2$ (6, p. 101; 8, p. 454)). In particular, from $\{3, 2\}$, $\{3, 3\}$, $\{3, 4\}$, $\{5, 3\}$, and $\{3, 5\}$ of

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genus 0 we derive $\{2.3, 2\} = \{6, 2\}$ of genus 0, $\{2.3, 3\} = \{6, 3\}_{2,0}$ of genus 1, $\{2.3, 4\}$ of genus 3, $\{2.5, 3\}$ of genus 5, and $\{2.3, 5\}$ of genus 9 respectively (cf. 8, p. 460).

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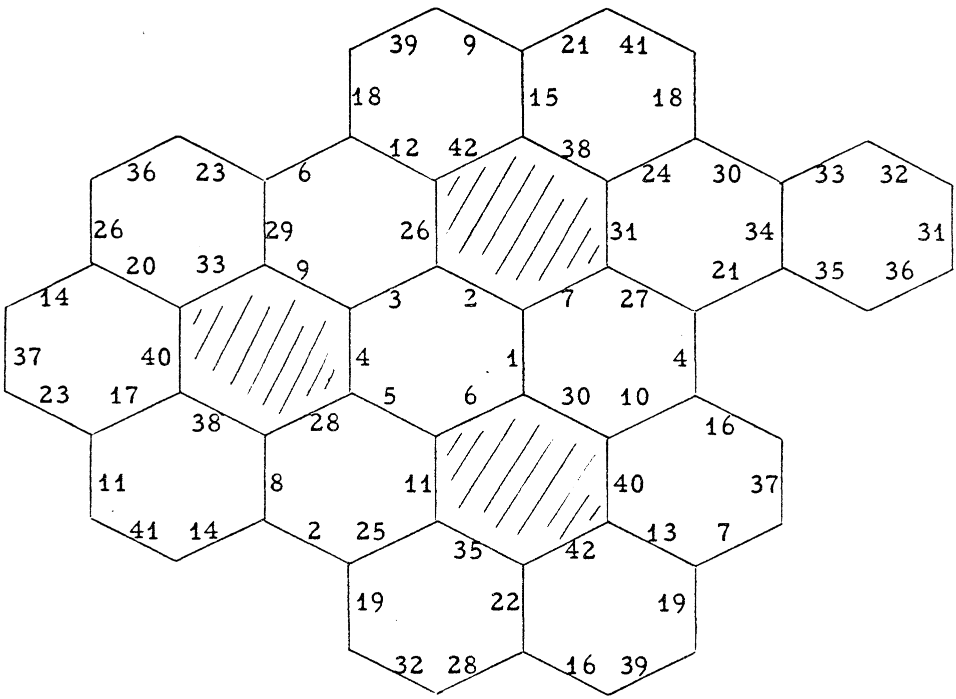

Fig. 1

$\{6,\ 6\}_{2,1}$

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