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THE STABILITY OF A FUNCTIONAL ANALOGUE OF THE WAVE EQUATION

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ABSTRACT. For $h \in \mathbb{R}$ and $\phi: \mathbb{R}^2 \longrightarrow \mathbb{R}$ define $L_h\phi: \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $(L_h\phi)(x,y) = \phi(x+h,y) + \phi(x-h,y) - \phi(x,y+h) - \phi(x,y-h)$ for all $(x,y) \in \mathbb{R}^2$. The aim of the paper is to establish the following "stability" theorem concerning the functional equation

$$(L_hf)(x,y) = 0 \text{ for all } x,y,h \in \mathbb{R}:$$

if $\delta > 0$, $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ and

$$|(L_hf)(x,y)| \leq \delta \quad \text{for all } x,y,h \in \mathbb{R}$$

then there exists $\varepsilon > 0$ and $\phi: \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that $(L_h\phi)(x,y) = 0$ for all x, $y,h \in \mathbb{R}$ and

$$|f(x,y) - \phi(x,y)| \leq \varepsilon \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

Suppose $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ is such that

$$f(x+h,y) + f(x-h,y) - f(x,y+h) - f(x,y-h) = 0 \tag{1}$$

for all $x,y,h \in \mathbb{R}$. If we define partial (central) difference operators, $\underset{1,h}{\Delta}$ and $\underset{2,h}{\Delta}$ for $h \in \mathbb{R}$, by

$$\underset{1,h}{\Delta}\varphi(x,y) = \varphi(x+\frac{h}{2},y) - \varphi(x-\frac{h}{2},y)$$

$$\underset{2,h}{\Delta}\varphi(x,y) = \varphi(x,y+\frac{h}{2}) - \varphi(x,y-\frac{h}{2})$$

for $x,y \in \mathbb{R}$ and $\varphi: \mathbb{R}^2 \longrightarrow \mathbb{R}$, then (1) can be written $\underset{1,h}{\Delta}^2 f(x,y) - \underset{2,h}{\Delta}^2 f(x,y) = 0$, a difference analogue of the wave equation. In [2] a "convolution method" for solving functional equations was introduced and used to show that if $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous then f satisfies (1) for all $x,y,h \in \mathbb{R}$ if and only if there exist continuous functions $\alpha, \beta: \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x,y) = \alpha(x+y) + \beta(x-y) \text{ for all } (x,y) \in \mathbb{R}.$$

A simpler proof of this fact was given by Haruki [5]. Fenyö [4] has found the locally integrable and even the distributional solutions of (1) on $\mathbb{R}^2$. It was also noted in [2]

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that if $\alpha, \beta : \mathbb{R} \longrightarrow \mathbb{R}$ are arbitrary functions and $A: \mathbb{R}^2 \longrightarrow \mathbb{R}$ is biadditive and skew-symmetric, ($A(x+y,z) = A(x,z) + A(y,z)$, and $A(y,x) = -A(x,y)$ for all $x, y, z \in \mathbb{R}$), and if $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ is defined by

$$f(x,y) = \alpha(x+y) + \beta(x-y) + A(x,y) \text{ for all } (x,y) \in \mathbb{R}$$

then (1) holds. McKiernan [7], showed that this is indeed the general solution of (1) by transforming it, as Haruki had done, into an equivalent equation as follows. For $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ define $g: \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $g(x,y) = f(x+y, x-y)$ for all $(x,y) \in \mathbb{R}$. Then f satisfies (1) if and only if g satisfies

$$g(x+h, y+h) - g(x+h, y) - g(x, y+h) + g(x,y) = 0 \text{ for all } x, y, h \in \mathbb{R}.$$

More generally, McKiernan proved the following

THEOREM. *Suppose $(G, +)$ is an abelian group, $(H,+)$ is a module over the diadic rationals and $f: G \longrightarrow H$. Then*

$$f(x+h, y+h) - f(x+h, y) - f(x, y+h) + f(x,y) = 0 \textit{ for all } x, y, h \in G \tag{2}$$

if and only if there exist functions $\varphi, \psi : G \longrightarrow H$ and $A: G \times G \longrightarrow H$ such that A is biadditive and skew-symmetric and

$$f(x,y) = \varphi(x) + \psi(y) + A(x,y) \textit{ for all } x, y \in G.$$

In recent years there has been considerable interest in studying the stability (in the sense of S. L. Ulam) of functional equations (see [6]). Our main aim in this paper is to prove stability theorems for the functional equations (1) and (2). Our proofs use ideas from McKiernan's paper [7] and a lemma from [1] concerning the stability of multiadditve functions.

THEOREM 1. *Suppose $(G,+)$ is an abelian group, X is a Banach space, $\delta > 0$ and $f: G \times G \longrightarrow X$ such that*

$$\|f(x+h, y+h) - f(x+h, y) - f(x, y+h) + f(x,y)\| \leq \delta \quad \textit{for all } x, y, h \in G. \tag{3}$$

Then there exist functions $\alpha, \beta : G \longrightarrow X$ and $A: G \times G \longrightarrow X$ such that A is biadditive and skew-symmetric and

$$\left\| f(x,y) - \left[\alpha(x) + \beta(y) + A(x,y) \right] \right\| \leq 20\delta \quad \textit{for all } x, y \in G.$$

PROOF. Let $g(x,y) = f(x,y) - f(x,0) - f(0,y) + f(0,0)$ for $x, y \in G$. Then

$$g(x,0) = g(0,y) = g(0,0) = 0 \quad \text{for all } x, y \in G \tag{4}$$

and

$$(5)\qquad \|g(x+h,y+h)-g(x+h,y)-g(x,y+h)+g(x,y)\| \le \delta \qquad \text{for all } x,y,h \in G.$$

Let

$$(6)\qquad \varepsilon(x,y,h) = g(x+h,y+h)-g(x+h,y)-g(x,y+h)+g(x,y) \qquad \text{for } x,y,h \in G$$

so that

$$(7)\qquad \|\varepsilon(x,y,h)\| \le \delta \qquad \text{for all } x,y,h \in G$$

Now, by (6), (4) and (7), for all $x, y, h \in G$,

$$(8)\qquad g(h,y+h) = g(h,y) + \varepsilon(0,y,h)$$

so that

$$(8')\qquad \|g(h,y+h)-g(h,y)\| \le \delta,$$

$$(9)\qquad g(x+h,h) = g(x,h) + \varepsilon(x,0,h)$$

so that

$$(9')\qquad \|g(x+h,h)-g(x,h)\| \le \delta,$$

and

$$(10)\qquad g(h,h) = \varepsilon(0,0,h)$$

so that

$$(10')\qquad \|g(h,h)\| \le \delta.$$

Moreover, by (6) and (4),

$$(11)\qquad \varepsilon(0,-h,h) = -g(h,-h), \qquad \text{for } h \in G,$$

so that, by (7),

$$(11')\qquad \|g(h,-h)\| \le \delta, \qquad \text{for } h \in G.$$

Now, for all $x, y \in G$, by (6),

$$\varepsilon(x,x,y-x) = g(y,y) - g(y,x) - g(x,y) + g(x,x)$$

and hence, by (10),

$$g(x,y)+g(y,x)=\varepsilon(0,0,x)+\varepsilon(0,0,y)-\varepsilon(x,x,y-x) \tag{12}$$

so that, in light of (7),

$$\|g(x,y)+g(y,x)\| \le 3\delta\,. \tag{12$'$}$$

Next observe that, by (8), for all $y,h\in G$,

$$\varepsilon(0,y-h,h)=g(h,y)-g(h,y-h) \tag{13}$$

so that, by (7),

$$\|g(h,y-h)-g(h,y)\| \le \delta\,. \tag{13$'$}$$

Similarly, for all $x,h\in G$,

$$\varepsilon(x-h,0,h)=g(x,h)-g(x-h,h) \tag{14}$$

so that, by (7),

$$\|g(x-h,h)-g(x,h)\| \le \delta\,. \tag{14$'$}$$

By (6) and (4),

$$\begin{aligned}\varepsilon(x,x+y,-x-y)&=g(-y,0)-g(-y,x+y)-g(x,0)+g(x,x+y)\\&=-g(-y,x+y)+g(x,x+y)\end{aligned}$$

so that

$$\|g(x,x+y)-g(-y,x+y)\| \le \delta \qquad \text{for all } x,y\in G. \tag{15}$$

But, by (8$'$) and (13$'$), for all $x,y\in G$,

$$\|g(x,x+y)-g(x,y)\| \le \delta$$

and

$$\|g(-y,x+y)-g(-y,x)\| \le \delta\,.$$

From the last three inequalities we find that

$$\|g(x,y)-g(-y,x)\| \le 3\delta \qquad \text{for all } x,y\in G. \tag{16}$$

In (5), put $x=u$, $y=v$ and $h=-u+w$ to get

$$\|g(w,v-u+w)-g(w,v)-g(u,v-u+w)+g(u,v)\| \le \delta\,.$$

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By (8′), $\|g(w, v-u+w) - g(w, v-u)\| \le \delta$ and thus

$$\|g(w, v-u) - g(w, v) - g(u, v-u+w) + g(u, v)\| \le 2\delta .$$

By (13′), $\|g(u, v-u+w) - g(u, v+w)\| \le \delta$ and thus

$$\|g(w, v-u) - g(w, v) - g(u, v+w) + g(u, v)\| \le 3\delta .$$

By (16), $\|g(v+w, -u) - g(u, v+w)\| \le 3\delta$ and $\|g(u, v) - g(v, -u)\| \le 3\delta$ so that

$$\|g(w, v-u) - g(w, v) - g(v+w, -u) + g(v, -u)\| \le 9\delta \qquad \text{for all } u, v, w \in G.$$

On replacing w by x, u by $-y$ and v by z we see that the last inequality may be rewritten

$$(17) \qquad \|g(x, y+z) - g(x, z) - g(x+z, y) + g(z, y)\| \le 9\delta \qquad \text{for all } x, y, z \in G.$$

If in (5) we set $x = u+v, y = v+w$ and $h = -u-v-w$ we find that

$$\|g(-w, -u) - g(-w, v+w) - g(u+v, -u) + g(u+v, v+w)\| \le \delta .$$

From (16) it follows that $\|g(-w, -u) - g(w, u)\| \le 6\delta$ and thus

$$\|g(w, u) - g(-w, v+w) - g(u+v, -u) + g(u+v, v+w)\| \le 7\delta .$$

But $\|g(-w, v+w) - g(-w, v)\| \le \delta$ by (13′) and $\|g(u+v, -u) - g(v, -u)\| \le \delta$ by (14′) so that

$$\|g(w, u) - g(-w, v) - g(v, -u) + g(u+v, v+w)\| \le 9\delta .$$

However, by (16), $\|g(-w, v) - g(v, w)\| \le 3\delta$ and $\|g(v, -u) - g(u, v)\| \le 3\delta$ so that

$$\|g(w, u) - g(v, w) - g(u, v) + g(u+v, v+w)\| \le 15\delta .$$

But, by (12′), $\|g(v, w) - \big(-g(w, v)\big)\| \le 3\delta$ and $\|g(u, v) - \big(-g(v, u)\big)\| \le 3\delta$ so that

$$\Big\|g(w, u) + g(w, v) + g(v, u) + g(u+v, v+w)\Big\| \le 21\delta .$$

However, by (5), $\|g(u+v, w+v) - \{g(u+v, w) + g(u, w+v) - g(u, w)\}\| \le \delta$ and thus

$$\|g(w, u) + g(w, v) + g(v, u) + g(u+v, w) + g(u, v+w) - g(u, w)\| \le 22\delta .$$

Thus, if we define $\eta \colon G^3 \longrightarrow X$ by

$$(18) \quad g(w, u) = -g(w, v) - g(v, u) - g(u+v, w) - g(u, v+w) + g(u, w) + \eta(u, v, w)$$

for all $u, v, w \in G$ then

$$\|\eta(u, v, w)\| \le 22\delta \qquad \text{for all } u, v, w \in G.$$

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Now put $x = u$, $y = w$ and $z = v$ in (17) to conclude that $\|\nu(u,v,w)\| \leq 9\delta$ if $\nu: G^3 \longrightarrow X$ is defined by

$$g(u, v+w) - g(u,v) - g(u+v,w) + g(v,w) = \nu(u,v,w). \tag{19}$$

Upon adding (18) to (19) we find that

$$\begin{aligned} 2g(u, v+w) = {} & \big[g(u,v) - g(v,u)\big] + \big[g(u,w) - g(w,u)\big] \\ & - \big[g(w,v) + g(v,w)\big] + \eta(u,v,w) + \nu(u,v,w). \end{aligned} \tag{20}$$

But

$$\|2g(u,v) - \big[g(u,v) - g(v,u)\big]\| = \|g(u,v) + g(v,u)\| \leq 3\delta,$$

by (12′); similarly,

$$\|2g(u,w) - \big[g(u,w) - g(w,u)\big]\| \leq 3\delta \quad \text{and} \quad \|g(w,v) + g(v,w)\| \leq 3\delta,$$

again by (12′). Thus, from (20), we conclude that

$$\begin{aligned} \|2g(u, v+w) - 2g(u,v) - 2g(u,w)\| & \leq 9\delta + \|\eta(u,v,w)\| + \|\nu(u,v,w)\| \\ & \leq 9\delta + 22\delta + 9\delta \end{aligned}$$

so that

$$\|g(u, v+w) - g(u,v) - g(u,w)\| \leq 20\delta \quad \text{for all } u, v, w \in G. \tag{21}$$

Let $k(x,y) = g(y,x)$ for $x, y \in G$. Then

$$\begin{aligned} & \|k(x+h, y+h) - k(x+h, y) - k(x, y+h) + k(x,y)\| \\ & \qquad = \|g(y+h, x+h) - g(y+h, x) - g(y, x+h) + g(y,x)\| \\ & \qquad = \|\varepsilon(y,x,h)\| \leq \delta \quad \text{for all } x, y, h \in G. \end{aligned}$$

Thus, (21) holds with g replaced by k. That is

$$\|g(v+w, u) - g(v,u) - g(w,u)\| \leq 20\delta \quad \text{for all } u, v, w \in G. \tag{22}$$

From (21), (22) and Theorem 2 of [1], there exists a biadditive function $A: G \times G \longrightarrow X$ such that

$$\|g(x,y) - A(x,y)\| \leq 20\delta \quad \text{for all } x, y \in G. \tag{23}$$

But, according to (12′),

$$\|g(x,y) + g(y,x)\| \leq 3\delta \quad \text{for all } x, y \in G,$$

and thus

$$\|A(x,y) + A(y,x)\| \leq 43\delta \quad \text{for all } x, y \in G. \tag{24}$$

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If we let $B(x,y) = A(x,y) + A(y,x)$ for $x, y \in G$ then B is biadditive and, according to (24), B is bounded. Thus, $B(x,y) = 0$ for all $x, y \in G$ so that A is skew-symmetric.

Finally, let $\alpha(x) = f(x,0)$ and $\beta(y) = f(0,y) - f(0,0)$ for all $x, y \in G$. From the first line of the proof we deduce that

$$\begin{aligned}\left\|f(x,y) - [\alpha(x) + \beta(y) + A(x,y)]\right\| &= \|g(x,y) - A(x,y)\| \\ &\leq 20\delta \qquad \text{for all } x, y \in G.\end{aligned}$$

COROLLARY. *Suppose $(G,+)$ is a module over the diadic rationals, X is a Banach space, $\delta > 0$ and $f\colon G \times G \longrightarrow X$ such that*

$$(*) \qquad \|f(x+h,y) + f(x-h,y) - f(x,y+h) - f(x,y-h)\| \leq \delta \qquad \textit{for all } x, y, h \in G.$$

Then there exist functions $a, b\colon G \longrightarrow X$ and $B\colon G \times G \longrightarrow X$ such that B is biadditive and skew-symmetric and

$$\left\|f(x,y) - \left[a(x+y) + b(x-y) + B(x,y)\right]\right\| \leq 20\delta$$

for all $x, y \in G$.

PROOF. Define $g\colon G \times G \longrightarrow X$ by $g(x,y) = f(x+y, x-y)$ for all $x, y \in G$. Let $x, y, h \in G$ and put $x' = x + y - h$, $y' = x - y - h$ and $h' = 2h$. It follows from the definition of g and $(*)$ that

$$\begin{aligned}&\|g(x+h,y) + g(x-h,y) - g(x,y+h) - g(x,y-h)\| \\ &\quad = \|f(x'+h',y'+h') + f(x',y') - f(x'+h',y') - f(x',y'+h')\| \leq \delta.\end{aligned}$$

Thus, according to Theorem 1, there exist functions $\alpha, \beta\colon G \longrightarrow X$ and $A\colon G \times G \longrightarrow X$ such that A is biadditive and skew-symmetric and

$$(**) \qquad \left\|g(x,y) - \left[\alpha(x) + \beta(y) + A(x,y)\right]\right\| \leq 20\delta \qquad \text{for all } x, y \in G.$$

However, $f(x,y) = g\big(\frac{1}{2}(x+y), \frac{1}{2}(x-y)\big)$ for all $x, y \in G$. Thus, by replacing x by $\frac{1}{2}(x+y)$ and y by $\frac{1}{2}(x-y)$ in $(**)$ we find that, for all $x, y \in G$,

$$(***) \qquad \left\|f(x,y) - \left[\alpha(\frac{1}{2}(x+y)) + \beta\big(\frac{1}{2}(x-y)\big) + A(\frac{1}{2}(x+y), \frac{1}{2}(x-y))\right]\right\| \leq 20\delta.$$

Since A is biadditive and skew-symmetric, it follows easily (as in [7], p. 263) that

$$A(\frac{1}{2}(x+y), \frac{1}{2}(x-y)) = -\frac{1}{2}A(x,y) \qquad \text{for all } x, y \in G.$$

The desired conclusion now follows by defining $a(x) = \alpha(\frac{1}{2}x)$, $b(x) = \beta(\frac{1}{2}x)$ and $B(x,y) = (-1/2)A(x,y)$ for all $x, y \in G$. ∎

As mentioned above, Fenyö [4] has found the solutions of (1) which are (Lebesgue) locally integrable on $\mathbb{R}^2$. From his result it easily follows that if $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ is locally integrable on $\mathbb{R}^2$ and (2) holds (with $G = \mathbb{R}$), then there exist functions $\varphi, \psi\colon \mathbb{R} \longrightarrow \mathbb{R}$ which are locally integrable on $\mathbb{R}$ such that

$$f(x,y) = \varphi(x) + \psi(y) \qquad \text{for all } x, y \in \mathbb{R}.$$

We will generalize this result as follows.

THEOREM 2. *Suppose* $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is Lebesgue measurable on* $\mathbb{R}^2$, $\delta \geq 0$ *and*

$$|f(x+h, y+h) - f(x+h, y) - f(x, y+h) + f(x, y)| \geq \delta \quad \text{for all } x, y, h \in \mathbb{R}.$$

Then there exist Lebesgue measurable functions $\varphi, \psi: \mathbb{R} \longrightarrow \mathbb{R}$ *such that* $\left|f(x,y) - \{\varphi(x) + \psi(y)\}\right| \leq 60\delta$ *for all* $x, y \in \mathbb{R}$.

PROOF. According to Theorem 1, there exist functions $\alpha, \beta: \mathbb{R} \longrightarrow \mathbb{R}$ and a skew symmetric biadditive $A: \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that

$$|f(x,y) - \{\alpha(x) + \beta(y) + A(x,y)\}| \leq 20\delta \quad \text{for all } x, y \in \mathbb{R}.$$

For a.e. $y \in \mathbb{R}$, $x \longrightarrow f(x,y)$ is measurable on $\mathbb{R}$. Denote by S the set of all such y so that $\mathbb{R} \setminus S$ has Lebesgue measure zero.

Suppose $y_1, y_2 \in S$. Then

$$|f(x,y_1) - \{\alpha(x) + \beta(y_1) + A(x,y_1)\}| \leq 20\delta \quad \text{and}$$
$$|f(x,y_2) - \{\alpha(x) + \beta(y_2) + A(x,y_2)\}| \leq 20\delta \quad \text{for all } x \in \mathbb{R}.$$

Thus

$$|f(x,y_1) - f(x,y_2) - \beta(y_1) + \beta(y_2) - A(x, y_1 - y_2)| \leq 40\delta$$

for all $x \in \mathbb{R}$ since A is additive in its second variable. But $x \longrightarrow f(x,y_1) - f(x,y_2)$ is measurable on $\mathbb{R}$, and hence it is bounded on some subset, say T, of $\mathbb{R}$ having positive Lebesgue measure. Thus $x \longrightarrow A(x, y_1 - y_2)$ is additive on $\mathbb{R}$ and bounded on T. It is well known (see, for example, Ostrowski [8]) that there therefore exists a real number $c(y_1 - y_2)$ such that $A(x, y_1 - y_2) = c(y_1 - y_2)x$ for all $x \in \mathbb{R}$.

Let $U = \{y_1 - y_2 | y_1, y_2 \in S\}$. We have shown that for each $z \in U$, there exists a $c(z) \in \mathbb{R}$ such that $A(x,z) = c(z)x$ for all $x \in \mathbb{R}$. But, by a well known theorem of Steinhaus [9], since S has positive measure, U contains a neighbourhood of 0, say V.

Let $y \in \mathbb{R}$. Choose $z \in V$ and a natural number n such that $y = nz$. Then $A(x,y) = nA(x,z) = nc(z)x$ for all $x \in \mathbb{R}$. Thus, for every $y \in \mathbb{R}$ there exists a $c(y) \in \mathbb{R}$ such that $A(x,y) = c(y)x$ for all $x \in \mathbb{R}$. But A is skew symmetric so that

$$c(y)x = A(x,y) = -A(y,x) = -c(x)y \quad \text{for all } x, y \in \mathbb{R}.$$

In particular, $c(x)x = -c(x)x$ for all $x \in \mathbb{R}$ so that $c(x) = 0$ for all nonzero x in $\mathbb{R}$. But, clearly, $c(0) = 0$ and it follows that $A(x,y) = 0$ for all $x, y \in \mathbb{R}$.

Thus we have

$$\left|f(x,y) - \{\alpha(x) + \beta(y)\}\right| \leq 20\delta \quad \text{for all } x, y \in \mathbb{R}.$$

Now choose $x_0, y_0 \in \mathbb{R}$ such that $x \longrightarrow f(x, y_0)$ and $y \longrightarrow f(x_0, y)$ are measurable on $\mathbb{R}$ and let $\varphi(x) = f(x, y_0) - \beta(y_0)$ and $\psi(y) = f(x_0, y) - \alpha(x_0)$ for $x, y \in \mathbb{R}$. Then ψ and φ are measurable on $\mathbb{R}$. Moreover,

$$|f(x,y_0) - \{\alpha(x) + \beta(y_0)\}| \leq 20\delta$$

and

$$|f(x_0,y) - \{\alpha(x_0) + \beta(y)\}| \leq 20\delta \quad \text{for all } x, y \in \mathbb{R}.$$

so that $|\varphi(x) - \alpha(x)| \leq 20\delta$ for all $x \in \mathbb{R}$ and $|\psi(y) - \beta(y)| \leq 20\delta$ for all $y \in \mathbb{R}$. Thus $|f(x,y) - \{\varphi(x) + \psi(y)\}| \leq 60\delta$ for all $x, y \in \mathbb{R}$.

COROLLARY. *Suppose* $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is Lebesgue measurable,* $\delta > 0$ *and*

$$|f(x+h,y+h)-f(x+h,y)-f(x,y+h)+f(x,y)| \leq \delta \text{ for all } x,y,h \in \mathbb{R}.$$

Suppose there exist $x_0, y_0 \in G$ *such that* $x \longrightarrow f(x,y_0)$ *and* $y \longrightarrow f(x_0,y)$ *are continuous on* $\mathbb{R}$. *Then there exist continuous functions* $a,b\colon \mathbb{R} \longrightarrow \mathbb{R}$ *such that*

$$|f(x,y)-\{a(x)+b(y)\}| \leq 180\delta \quad \text{for all } x,y \in \mathbb{R}.$$

PROOF. According to the last theorem, there exist measurable functions $\varphi, \psi : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$|f(x,y)-\{\varphi(x)+\psi(y)\}| \leq 60\delta \quad \text{for all } x,y \in \mathbb{R}.$$

Hence,

$$|f(x,y_0)-\{\varphi(x)+\psi(y_0)\}| \leq 60\delta$$

and

$$|f(x_0,y)-\{\varphi(x_0)+\psi(y)\}| \leq 60\delta \quad \text{for all } x,y \in \mathbb{R}.$$

Let $a(x) = f(x,y_0) - \psi(y_0)$ and $b(y) = f(x_0,y) - \varphi(x_0)$ for $x,y \in \mathbb{R}$. Then a and b are continuous on $\mathbb{R}$, $|a(x)-\varphi(x)| \leq 60\delta$ for all $x \in \mathbb{R}$ and $|b(y)-\psi(y)| \leq 60\delta$ for all $y \in \mathbb{R}$. Hence, $|f(x,y)-\{a(x)+b(y)\}| \leq 180\delta$ for all $x,y \in \mathbb{R}$.

Remarks

(1) We do not claim that the constants such as 20 in Theorem 1, 60 in Theorem 2, etc. are optimal.

(2) The technique used to deduce the corollary to Theorem 1 (what McKiernan [7] calls a change of variable) can be used to deduce stability results for equation (1) analogous to Theorem 2 and its corollary.

(3) Readers who are familiar with abstract harmonic analysis and integration of Banach space valued functions will note that Theorem 2 can be generalized as follows with no essential change in the proof other than replacing Lebesgue measure by a Haar measure and interpreting measurability of an X-valued function in the sense of Lusin (see e.g. [3]): if G is a locally compact abelian group, X is a Banach space, $\delta > 0$, $f\colon G \times G \longrightarrow X$ is measurable and such that

$$\|f(x+h,y+h)-f(x+h,y)-f(x,y+h)+f(x,y)\| \leq \delta \quad \text{for all } x,y,h \in G$$

then there exist measurable functions $\varphi, \psi : G \longrightarrow X$ such that

$$\|f(x,y)-\{\varphi(x)+\psi(y)\}\| \leq 60\delta \quad \text{for all } x,y \in G.$$

The corollary to Theorem 2 can be generalized along the same lines.

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REFERENCES

1. Michael Albert and John A. Baker *Functions with bounded n^{th} differences*, Ann. Polon. Math. **XLIII**, (1983), 93–103.

2. John A. Baker *An analogue of the wave equation and certain related functional equations* Canad. Math. Bull. Vol. **12**, No 6, (1969), 837–846.

3. John A. Baker *Regularity properties of functional equations*, Aequationes Math., **6**(2/3), (1971), 243–248.

4. I. Fenyö *Remark on a paper of J. A. Baker*, Aequationes Math., **8**(1/2), (1972), 103–108.

5. H. Haruki *On the functional equation $f(x+t,y)+f(x-t,y) = f(x,y+t)+f(x,y-t)$*, Aequationes Math. **5**(1), (1970), 118–119.

6. D. H. Hyers *The stability of homomorphisms and related topics*, Global analysis - analysis on manifolds, Teubner-Texte zur Math. **57**, Teubner, Leipzig, (1983), 140–153.

7. M. A. McKiernan *The general solution of some finite difference equations analogous to the wave equation*, Aequationes Math. **8**(3), (1972), 263–266.

8. A. Ostrowski *Mathematische Miszellen, XIV: Über die Funktionalgleichung der Exponential-function und verwandte Functionalgleichungen*, Jber. Deutsch. Math.-Verein **38**, (1929), 54–62.

9. Steinhaus *Sur les distances des points dans les ensembles de mesure positive*, Fund. Math. **1**, (1920), 93–104.

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