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Exceptional Sets of Slices for Functions From the Bergman Space in the Ball

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Abstract. Let B_N be the unit ball in $\mathbb{C}^N$ and let f be a function holomorphic and L^2-integrable in B_N. Denote by $E(B_N, f)$ the set of all slices of the form $\Pi = L \cap B_N$, where L is a complex one-dimensional subspace of $\mathbb{C}^N$, for which $f|_\Pi$ is not L^2-integrable (with respect to the Lebesgue measure on L). Call this set the exceptional set for f. We give a characterization of exceptional sets which are closed in the natural topology of slices.

1 Introduction

Let B_N be the unit ball in $\mathbb{C}^N$. We have proved in [3] that there exists a function f holomorphic in B_N such that for every complex subspace L of $\mathbb{C}^N$, $f|_{L\cap B_N} \notin L^2(L \cap B_N)$ (where the space $L^2(L \cap B_N)$ is considered with respect to the Lebesgue measure in $L \cap B_N$). In this note we are interested in another problem: Let E be a subset of the slices of the form $\Pi = L \cap B_N$, where L is a complex one-dimensional subspace of $\mathbb{C}^N$. We are interested in determining those E for which there exists a function f holomorphic in B_N and L^2-integrable with respect to the Lebesgue measure (we write $f \in L^2H(B_N)$) such that for every one-dimensional complex subspace L of $\mathbb{C}^N$, $f|_{L\cap B_N} \notin L^2(L \cap B_N)$(with respect to the Lebesgue measure in L) iff $L \cap B_N \in E$. Let $\tilde{E} = \bigcup\{L \cap \partial B_N \mid L \cap B_N \in E\}$. Denote by ν the surface measure on ∂B_N. If a function f with the above described properties exists then, by Fubini's theorem, $\nu(\tilde{E}) = 0$.

We can identify E with a subset $\hat{E}$ of the complex projective space $\mathbb{CP}^N$. Similarly to [2] one can prove that $\hat{E}$ must be a G_δ-set in the natural topology of $\mathbb{CP}^N$: this is equivalent to say that $\tilde{E}$ is a G_δ-subset of ∂B_N. Following [1] or [2] we will call the set E the exceptional set of complex slices for f, and denote it by $E(B_N, f)$.

We will prove the following theorem:

Theorem 1 *Let E be a subset of one-dimensional complex slices such that $\nu(\tilde{E}) = 0$, and $\hat{E}$ is closed in $\mathbb{CP}^N$ (this is equivalent to assume that $\tilde{E}$ is closed in ∂B_N). Then there exists a function $f \in L^2H(B_N)$ such that $E(B_N, f) = E$.*

A weaker result would be the following: Given a set E of one-dimensional complex slices with $\nu(\tilde{E}) = 0$ and $\hat{E}$ closed in $\mathbb{CP}^N$, find a bounded domain of holomorphy C with $0 \in C$ and a function $f \in L^2H(C)$ such that for every one-dimensional complex

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subspace L of $\mathbb{C}^N$, f is not L^2-integrable on $L \cap C$ if and only if $L \cap B_N \in E$. (In this case, we will write $E = E(C, f)$).

We begin with such a weaker result, *i.e.*, we prove the following:

Theorem 2 *Let E be as in Theorem 1. Then there exists a strictly convex and balanced domain C in $\mathbb{C}^N$ and a function $f \in L^2H(C)$ such that $E(C, f) = E$.*

(We recall that a domain $C \subset \mathbb{C}^N$ is called *balanced* if for every $z \in C$ and every $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$, $\lambda z \in C$).

The reason to prove first Theorem 2, which is weaker than Theorem 1 is because of the clarity of the construction. One of the main ingredients of the proof of Theorem 2 is the following result by Wojtaszczyk:

Theorem 3 **([4], Theorem 1)** *There exists an integer $K = K(N)$ and a sequence $\{p_n\}$ of homogeneous polynomials in $\mathbb{C}^N$ of degree n (for n large enough, say $n \geq N_0$) such that*

$$|p_n(z)| \leq 2 \quad \text{for all } z \in \partial B_N; \tag{1}$$

$$\text{for each } s \text{ large enough, say } s \geq S_0, \quad \sum_{n=Ks}^{K(s+1)-1} |p_n(z)| \geq 0,5 \quad \text{for all } z \in \partial B_N. \tag{2}$$

In the proof of Theorem 2 we use this result exactly in the form stated in Theorem 3; in order to prove Theorem 1 we need first to show that the assertion of Theorem 3 holds also for strictly convex and balanced domains which are in some sense not too far from the unit ball; this requires further explanations, which might obscure the main proof.

In the sequel, we will denote by $B_N(z, r)$ the ball with center $z \in \mathbb{C}^N$ and of radius r, and $D(w, r)$ will denote the disc in the complex plane, centered at $w \in \mathbb{C}$, and of radius r. Also, we set U to be the unit disc in $\mathbb{C}$.

If D is a domain in $\mathbb{C}^N$, and $h \in L^2(D)$, we will denote by $\|h\|_D$ the L^2-norm of h in D. The Lebesgue measure (of arbitrary dimension) in a subset of $\mathbb{C}^N$ or of a subspace of $\mathbb{C}^N$ will be denoted by m.

2 The Exceptional Sets of Complex Planes in $\mathbb{C}^N$

In this section we will prove Theorem 2. We will begin with the result which is rather obvious, and can be proved by standard methods:

Lemma 4 *Let E be a closed subset of $\mathbb{CP}^N$. Then there exists a strictly convex domain $C \subset \mathbb{C}^N$ such that $C \subset B_N$, $\partial C \cap \partial B_N = E$, $\partial C \setminus \partial B_N \subset B_N$, and C is balanced. Moreover, there exists a function σ, which is strictly convex and smooth in $\mathbb{C}^N$, is non-negatively homogeneous (i.e., $\sigma(\lambda z) = |\lambda|\sigma(z)$ for $z \in \mathbb{C}^N$ and $\lambda \in \mathbb{C}$), and which is a defining function for C (i.e., $C = \{z \in \mathbb{C}^N \mid \sigma(z) < 1\}$ and* $\operatorname{grad} \sigma(w) \neq 0$ *for $w \in \partial C$).*

Let σ be a defining function for C, with the properties listed in Lemma 4. Given $w \in \mathbb{C}^N$, $w \neq 0$, denote by $[w]$ the class of w in $\mathbb{CP}^N$. For $[w] \in \mathbb{CP}^N$, set $\tilde{\sigma}([w]) =$

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$\sigma(\frac{w}{\|w\|})$. Then $\tilde{\sigma}$ is well-defined and smooth in $\mathbb{CP}^N$ (where $\mathbb{CP}^N$ is considered as the complex manifold), $\tilde{\sigma} \geq 1$, and $\tilde{\sigma}([w]) = 1$ precisely when $[w] \in \hat{E}$. Moreover, let $\tilde{\nu}$ be the measure in $\mathbb{CP}^N$ induced in the natural way in $\mathbb{CP}^N$ from ∂B_N.

Let $\psi\colon [0,1) \to \mathbb{R}$ be a function such that:

$$\psi > 0, \quad \psi \text{ is increasing, and } \lim_{t\to 1^-} \psi(t) = +\infty. \tag{3}$$

Suppose also that $f \in \mathcal{O}(B_N)$ (the space of functions holomorphic in B_N) is such that for every $z \in \mathbb{C}^N$ with $\|z\| = 1$, for every $0 < r < 1$,

$$\|f\|^2_{\{\lambda z \mid |\lambda| < r\}} =: \int_{D(0,r)} |f(\lambda z)|^2 \, dm(\lambda) \leq \psi(r).$$

Then there exists a constant $c > 0$, independent of f, such that

$$\begin{aligned}\int_C |f(z)|^2 \, dm(z) &\leq c \int_{\mathbb{CP}^N} \Big(\int_{D(0, \frac{1}{\tilde{\sigma}([w])})} |f(\lambda w)|^2 \, dm(\lambda) \Big) \, d\tilde{\nu}([w]) \\ &\leq c \int_{\mathbb{CP}^N} \psi\Big(\frac{1}{\tilde{\sigma}([w])} \Big) \, d\tilde{\nu}([w]).\end{aligned} \tag{4}$$

Lemma 5 *Suppose that E is as in Theorems 1 or 2. Let C be a strictly convex and balanced domain in $\mathbb{C}^N$, constructed with respect to E according to Lemma 4. Then there exists ψ satisfying (3), and such that*

$$\int_{\mathbb{CP}^N} \psi\Big(\frac{1}{\tilde{\sigma}([w])} \Big) \, d\tilde{\nu}([w]) < +\infty. \tag{5}$$

Proof of Lemma 5 Since $\tilde{\nu}(\mathbb{CP}^N) < +\infty$, $\tilde{\nu}(\hat{E}) = \tilde{\nu}\big(\{[w] \in \mathbb{CP}^N \mid \tilde{\sigma}([w]) = 1\}\big) = 0$, for every $[w] \in \mathbb{CP}^N$, $\tilde{\sigma}([w]) \geq 1$, and $\tilde{\sigma}$ is continuous, there exists a sequence $\{t_n\}_{n=1}^{\infty}$, $0 < t_1 < t_2 < \cdots < 1$, with $\lim_{n\to\infty} t_n = 1$, and such that

$$\tilde{\nu}\Big(\Big\{ [w] \in \mathbb{CP}^N \;\Big|\; \frac{1}{t_{n+1}} < \tilde{\sigma}([w]) \leq \frac{1}{t_n} \Big\} \Big) < \frac{1}{n^3}.$$

Define the function χ by $\chi(t) = n+1$ for $t \in [t_n, t_{n+1})$, $n = 1, 2, \ldots$, and $\chi(t) = 1$ for $t \in [0, t_1)$. Then

$$\begin{aligned}&\int_{\mathbb{CP}^N} \chi\Big(\frac{1}{\tilde{\sigma}([w])} \Big) \, d\tilde{\nu}([w]) \\ &\quad\leq \tilde{\nu}(\mathbb{CP}^N) + \sum_{n=1}^{\infty} (n+1) \tilde{\nu}\Big(\Big\{ [w] \in \mathbb{CP}^N \;\Big|\; \frac{1}{t_{n+1}} < \tilde{\sigma}([w]) \leq \frac{1}{t_n} \Big\} \Big) \\ &\quad\leq \tilde{\nu}(\mathbb{CP}^N) + \sum_{n=1}^{\infty} \frac{n+1}{n^3} < +\infty.\end{aligned}$$

Then it is sufficient to take ψ satisfying (3) and such that $\psi \le \chi$ on $[0, 1)$. ■

Lemma 6 *Given ψ satisfying (3), there exists a function $f \in \mathcal{O}(B_N)$ such that for every one-dimensional complex subspace L of $\mathbb{C}^N$ and every $0 < r < 1$, $f|_{L\cap B_N} \notin L^2(L \cap B_N)$, and*

$$\|f\|^2_{\{\lambda z||\lambda|<r|\}} = \|f\|^2_{L\cap B_N(0,r)} \le \psi(r).$$

Suppose for a moment that Lemma 6 is proved. Let E, C, σ and $\tilde{\sigma}$ be as before, and choose ψ to $\tilde{\sigma}$ according to Lemma 5. Construct f with respect to ψ like in Lemma 6. Then by (4) and (5), $f \in L^2H(C)$. Moreover, by Lemmas 6 and 4, for every one-dimensional complex subspace L of $\mathbb{C}^N$,

$$f|_{L\cap C} \notin L^2(L \cap C) \quad \text{iff } L \cap B_N \in E.$$

This gives the desired domain C and the function f, and ends the proof of Theorem 2. ■

Therefore in order to prove Theorem 2, it remains to prove Lemma 6.

Proof of Lemma 6 We prove first an auxiliary lemma:

Lemma 7 *Let ψ be a function satisfying (3). Then there exists a function h holomorphic in the unit disc U in $\mathbb{C}$ such that for every $0 < r < 1$,*

$$\|h\|^2_{D(0,r)} = \int_{D(0,r)} |h(w)|^2 \, dm(w) \le \psi(r)$$

and

$$\int_U |h(w)|^2 \, dm(w) = +\infty.$$

Proof of Lemma 7 Shrinking ψ if necessary we may assume that ψ is continuous. If

$$h(w) = \sum_{n=0}^{\infty} a_n w^n \tag{6}$$

is holomorphic in U, then

$$\int_{D(0,r)} |h(w)|^2 \, dm(w) = \pi \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} r^{2(n+1)},$$

and

$$\int_U |h(w)|^2 \, dm(w) = \pi \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}.$$

Therefore it is sufficient to choose non-negative numbers $\{a_n\}_{n=0}^{\infty}$ such that

$$\pi \sum_{n=0}^{\infty} \frac{a_n^2}{n+1} = +\infty, \tag{7}$$

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and for every r with $0 < r < 1$,

(8) $$\pi \sum_{n=0}^{\infty} \frac{a_n^2}{n+1} r^{2(n+1)} \leq \psi(r).$$

(Note that if the numbers $\{a_n\}$ satisfy (8), then the series $\sum_{n=0}^{\infty} a_n w^n$ is convergent uniformly on compact subsets of U, so it defines a holomorphic function in U).

Denote further

(9) $$b_n = \frac{\pi a_n^2}{n+1},$$

$n = 0, 1, 2, \ldots$. If we can choose $\{b_n\}_{n=0}^{\infty}$ such that $b_n \geq 0$, $n = 0, 1, \ldots,$

(10) $$\sum_{n=0}^{\infty} b_n = +\infty,$$

and for every r with $0 < r < 1$,

(11) $$\sum_{n=0}^{\infty} b_n r^{2(n+1)} \leq \psi(r),$$

and then we compute a_n by means of b_n according to (9), we get the desired coefficients $\{a_n\}_{n=0}^{\infty}$.

We claim that we can choose b_n satisfying (10) and (11), and it is sufficient to allow b_n to assume only the values 0 or 1 for convenient n. We do this inductively. Choose a positive integer k_1 so large that

$$r^{2(k_1+1)} < \psi(r) \quad \text{for } 0 \leq r < 1$$

(this is possible because of the assumptions on ψ). Set $b_{k_1} = 1$. The function $\psi_1(r) =: \psi(r) - r^{2(k_1+1)}$, $0 \leq r < 1$, is positive, continuous, and $\lim_{r\to 1^-} \psi_1(r) = +\infty$. There exists k_2 so large that $k_2 > k_1$, and

$$r^{2(k_2+1)} < \psi_1(r) \quad \text{for } 0 \leq r < 1.$$

Set $b_{k_2} = 1$. Similarly, the function $\psi_2(r) =: \psi_1(r) - r^{2(k_2+1)}$ is positive, continuous, and $\lim_{r\to 1^-} \psi_2(r) = +\infty$. Then there exists k_3 so large that $k_3 > k_2$, and

$$r^{2(k_3+1)} < \psi_2(r) \quad \text{for } 0 \leq r < 1.$$

We set $b_{k_3} = 1$, $\psi_3(r) =: \psi_2(r) - r^{2(k_3+1)}$, and choose the integer k_4, and so on. In this way we have defined $b_k = 1$ for $k = k_i$, $i = 1, 2, \ldots$. For other values of k we set $b_k = 0$.

Note that the condition (11) is satisfied by the construction. Moreover, since infinitely many b_k's are equal to 1, the condition (10) is also satisfied. ■

Consider the constant $K = K(N)$ from the assertion of Theorem 3. Note that we can assume that the numbers $k_1, k_2, \ldots$ in the proof of Lemma 7 can be chosen so that for all $l = 1, 2, \ldots,$

$$k_{l+1} - k_l > K = K(N), \tag{12}$$

and for each l there exists a positive integer s_l such that

$$k_l = K(N)s_l. \tag{13}$$

We need also further modification of the function h obtained in Lemma 7. For every $l = 1, 2, \ldots,$ consider the number k_l, where $\{k_l\}_{l=1}^{\infty}$ are chosen according to the proof of Lemma 7, and satisfy (12) and (13). Then, because of (9), (12), and the choice of the numbers b_k, we have $a_{k_l} > 0$ and $a_{k_l+1} = \cdots = a_{k_l+K(N)-1} = 0$. Define $c_{k_l}, c_{k_l+1}, \ldots, c_{k_l+K(N)-1}$ by

$$\frac{c_{k_l}^2}{k_l+1} = \frac{c_{k_l+1}^2}{k_l+2} = \cdots = \frac{c_{k_l+K(N)-1}^2}{k_l+K(N)} = \frac{1}{K(N)}\frac{a_{k_l}^2}{k_l+1}, \quad l = 1, 2, \ldots. \tag{14}$$

This gives the numbers c_n for some values of n. For other n, set $c_n = 0$. Note that because of (12), the definition of c_n is correct. Set

$$g(w) = \sum_{n=0}^{\infty} c_n w^n.$$

Then g is holomorphic in U, and by (14),

$$\sum_{n=0}^{\infty} \frac{c_n^2}{n+1} = \sum_{n=0}^{\infty} \frac{a_n^2}{n+1} = +\infty.$$

Moreover, for every r with $0 < r < 1$, and every $l = 1, 2, \ldots,$ we have by (14)

$$\begin{aligned}\frac{\pi a_{k_l}^2}{k_l+1} r^{2(k_l+1)} &= \left(\frac{\pi c_{k_l}^2}{k_l+1} + \cdots + \frac{\pi c_{k_l+K(N)-1}^2}{k_l+K(N)}\right) r^{2(k_l+1)} \\ &\geq \frac{\pi c_{k_l}^2}{k_l+1} r^{2(k_l+1)} + \frac{\pi c_{k_l+1}^2}{k_l+2} r^{2(k_l+2)} + \cdots + \frac{\pi c_{k_l+K(N)-1}^2}{k_l+K(N)} r^{2(k_l+K(N))},\end{aligned}$$

and so, for every $0 < r < 1$,

$$\sum_{n=0}^{\infty} \frac{\pi c_n^2}{n+1} r^{2(n+1)} \leq \sum_{n=0}^{\infty} \frac{\pi a_n^2}{n+1} r^{2(n+1)} \leq \psi(r). \tag{15}$$

Hence the function $g(w)$ also satisfies the assertions of Lemma 7, but the coefficients c_n satisfy further properties, which we need later.

Define now for $z \in B_N$,

$$F(z) = \sum_{n=N_0}^{\infty} c_n p_n(z), \tag{16}$$

where $\{p_n\}$ are polynomials from Theorem 3. Since $|p_n(z)| \leq 2$ for $z \in \partial B_N$, and c_n grow to infinity at most like Cn for some $C > 0$, it is not difficult to show that the series on the right-hand side of (16) converges in all of B_N to a function holomorphic in B_N.

Now fix $z \in \partial B_N$. Consider the function

$$F_z \colon U \ni w \to F(wz). \tag{17}$$

(We recall that U denotes the unit disc in $\mathbb{C}$). Then for $0 < r < 1$ we have by (1) and (15)

$$\begin{aligned}\|F_z\|_{D(0,r)} &= \int_{D(0,r)} |F(wz)|^2 \, dm(w) \\ &= \sum_{n=N_0}^{\infty} c_n^2 \int_{D(0,r)} |p_n(wz)|^2 \, dm(w) = \sum_{n=N_0}^{\infty} c_n^2 \int_{D(0,r)} |p_n(z)|^2 |w|^{2n} \, dm(w) \qquad (18)\\ &= \pi \sum_{n=N_0}^{\infty} \frac{c_n^2}{n+1} |p_n(z)|^2 r^{2(n+1)} \leq 4\pi \sum_{n=N_0}^{\infty} \frac{c_n^2}{n+1} r^{2(n+1)} \leq 4\psi(r).\end{aligned}$$

Moreover, similarly as above, and by the choice of coefficients c_n, in particular by (13) and (14), we conclude that there exist positive integers L_0 and M_0, depending only on N_0 and S_0 from Theorem 3, and a number $c > 0$ which depends only on $K = K(N)$ from Theorem 3 (in particular, L_0, M_0 and c do not depend on $z \in \partial B_N$) such that the following estimate holds:

$$\begin{aligned}\int_U |F(wz)|^2 \, dm_2(w) &= \pi \sum_{n=N_0}^{\infty} \frac{c_n^2}{n+1} |p_n(z)|^2 \\ &\geq \pi \sum_{l=L_0}^{\infty} \Big(\frac{c_{k_l}^2}{k_l+1} |p_{k_l}(z)|^2 + \cdots + \frac{c_{k_l+K(N)-1}^2}{k_l+K(N)} |p_{k_l+K(N)-1}(z)|^2 \Big) \\ &= \pi \sum_{l=L_0}^{\infty} \frac{a_{k_l}^2}{K(N)(k_l+1)} \big(|p_{k_l}(z)|^2 + \cdots + |p_{k_l+K(N)-1}(z)|^2 \big) \\ &= \frac{\pi}{K(N)} \sum_{l=L_0}^{\infty} \frac{a_{K(N)s_l}^2}{K(N)s_l+1} \Big(\sum_{n=K(N)s_l}^{K(N)(s_l+1)-1} |p_n(z)|^2 \Big)\end{aligned}$$

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$$(19) \qquad \geq \frac{c\pi}{K(N)} \sum_{l=L_0}^{\infty} \frac{a^2_{K(N)s_l}}{K(N)s_l+1} \Big(\sum_{n=K(N)s_l}^{K(N)(s_l+1)-1} |p_n(z)| \Big)^2$$

$$\geq \frac{1}{4} \frac{c\pi}{K(N)} \sum_{l=L_0}^{\infty} \frac{a^2_{K(N)s_l}}{K(N)s_l+1} = \frac{1}{4} \frac{c\pi}{K(N)} \sum_{n=M_0}^{\infty} \frac{a_n^2}{n+1} = +\infty$$

(the last inequality follows from (2)). In virtue of (18) and (19), it is sufficient to set $f = \frac{1}{4}F$ in order to obtain the function f satisfying the assertion of Lemma 6. This ends the proof of Lemma 6. ∎

We give now the outline of the proof of Theorem 1. Take any strictly convex and balanced domain C in $\mathbb{C}^N$ such that $B_N \subset C$, $\partial B_N \cap \partial C = E$, $\partial B_N \setminus \partial C \subset C$. As in Lemma 1 there exists a strictly convex, smooth and non-negatively homogeneous defining function σ for C. Since C is balanced, the homogeneous polynomials of different orders are mutually orthogonal in C with respect to the standard Lebesgue measure in $\mathbb{C}^N$. Looking at the proof of Theorem 2 we see that the main ingredient of the proof of the present theorem would be the following generalization of Wojtaszczyk's result:

Lemma 8 *Suppose that C is not far away from B_N (in the sense that the strictly convex, smooth, and non-negatively homogeneous definig function σ for C does not differ too much from the defining function for B_N, together with derivatives up to order three, in the uniform norm on some open set $W \supset \partial B_N \cup \partial C$). Then there exists an integer $K = K(N)$ and a sequence $\{p_n\}$ of homogeneous polynomials in $\mathbb{C}^N$ of degree n (for n large enough, say $n \geq N_0$) such that*

$$(20) \qquad |p_n(z)| \leq 2 \quad \textit{for all } z \in \partial C;$$

$$(21) \qquad \textit{for each } s \textit{ large enough, say } s \geq S_0, \sum_{n=Ks}^{K(s+1)-1} |p_n(z)| \geq 0,5 \quad \textit{for all } z \in \partial C.$$

Note We do not know whether the assertion of Lemma 8 is true for *all* strictly convex and balanced domains in $\mathbb{C}^N$.

Sketch of the proof of Lemma 8 Consider the proof of [4], Proposition 1. Let $\{\zeta_1, \ldots, \zeta_s\}$ be a $d/\sqrt{N}$-separated subset of the unit sphere $\mathbb{S}$ (for definition, see [4]). Set

$$p(z) =: \sum_{j=1}^{s} \frac{1}{\|\zeta_j\|^{2k}} \langle z, \zeta_j \rangle^k,$$

where $\|\ \|$ and $\langle\ ,\ \rangle$ denote the usual Euclidean norm and scalar product in $\mathbb{C}^N$. Fix j_0 with $1 \leq j_0 \leq s$. For $z \in \partial C$, let α denote the angle between z and ζ_{j_0} (treated as the vectors in $\mathbb{C}^N = \mathbb{R}^{2N}$). Then, for $z \in \partial C$ near ζ_{j_0}, we have

$$\|z - \zeta_{j_0}\| \approx \alpha,$$

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and

$$\left|\frac{1}{\|\zeta_{j_0}\|^{2k}}\langle z,\zeta_{j_0}\rangle^k\right| = \left(\frac{\|z\|}{\|\zeta_{j_0}\|}\right)^k \left|\left\langle \frac{z}{\|z\|}, \frac{\zeta_{j_0}}{\|\zeta_{j_0}\|}\right\rangle\right|^k.$$

Moreover,

$$\left|\left\langle \frac{z}{\|z\|}, \frac{\zeta_{j_0}}{\|\zeta_{j_0}\|}\right\rangle\right| = \cos\alpha \leq 1 - \frac{\alpha^2}{4}$$

for α small, and if ∂C is sufficiently near to ∂B_N, we have

$$\frac{\|z\|}{\|\zeta_{j_0}\|} \approx 1 + c\alpha^2$$

for some $c > 0$ independent of ζ_{j_0} and z, and this number c can be chosen arbitrarily close to zero. (This is the estimate to which we use the fact that ∂C is near to ∂B_N). Hence

$$\left|\frac{1}{\|\zeta_{j_0}\|^{2k}}\langle z,\zeta_{j_0}\rangle^k\right| \leq \left(1-\frac{1}{4}\alpha^2\right)^k (1+c\alpha^2)^k \leq \left(1-\frac{1}{8}\alpha^2\right)^k \tag{22}$$

for α small (*i.e.*, for z near ζ_{j_0}). Moreover, assume that $C \subset B(0, \frac{e}{2})$. Then, for other values of j, and $z \in \partial C$ still near to ζ_{j_0}, the following estimate holds for $N \leq k \leq 2N$:

$$\frac{1}{\|\zeta_j\|^{2k}}|\langle z,\zeta_j\rangle|^k = \frac{1}{\|\zeta_j\|^k}\|z\|^k\left|\left\langle \frac{z}{\|z\|}, \frac{\zeta_j}{\|\zeta_j\|}\right\rangle\right|^k \leq \left(\frac{e}{2}\right)^k e^{-\frac{c^2 k}{N}}. \tag{23}$$

This estimate is similar to [4], formula (5). Then, like in the proof of the estimates following [4], formula (5), we have by (22) and (23),

$$|p(z)| \leq 1 + \sum_{k=1}^{\infty}\left(\frac{e}{2}\right)^k e^{-(\frac{kd}{2})^2} 2^{N-1}(k+2)^{2N-2}.$$

The last sum can be chosen to be $\leq 0,1$ if $d > 0,5$ was chosen sufficiently large; this would give the convenient modification of [4], Proposition 1. The rest of the proof of Lemma 8 follows the proof of [4], Theorem 1. ■

Having proved Lemma 8, we can repeat the proof of Theorem 2, beginning with the formula (12), in order to end the proof of Theorem 1. ■

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