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UNICITY THEOREMS FOR MEROMORPHIC OR ENTIRE FUNCTIONS

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In this paper, we prove that there exist three finite sets S_j $(j = 1, 2, 3)$ such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2, 3$ must be identical. As a particular case of the above result, we obtain that there exist two finite sets S_j $(j = 1, 2)$ such that any two non-constant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical, which answers a question posed by Gross.

1. INTRODUCTION

By a meromorphic function we shall always mean a meromorphic function in the complex plane. We use the usual notation of Nevanlinna theory of meromorphic functions as explained in [5]. We use E to denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ $(r \to \infty, \quad r \notin E)$.

For any set S and any meromorphic function f let

$$E_f(S) = \bigcup_{a \in S} \{z \mid f(z) - a = 0\},$$

where each zero of $f - a$ with multiplicity m is repeated m times in $E_f(S)$ (see [1]).

Nevanlinna proved the following well-known theorem.

THEOREM A. (See [6, 4].) *Let $S_j = \{a_j\}$ $(j = 1, 2, 3, 4)$, where a_1, a_2, a_3 and a_4 are four distinct complex numbers ($a_j = \infty$ is allowed). Suppose that f and g are non-constant meromorphic functions satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2, 3, 4$. Then either $f \equiv g$, or f is a linear fractional transformation of g, two of the values, say a_1 and a_2, must be Picard values, and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.*

Using Theorem A, the present author proved the following results.

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THEOREM B. (See [7].) *If, in addition to the assumptions of Theorem A,*

$$\frac{(2a_3 - a_1 - a_2)a_4 + (2a_1a_2 - a_1a_3 - a_2a_3)}{(a_2 - a_1)(a_4 - a_3)} \neq -3, 0, 3,$$

then $f = g$.

THEOREM C. (See [7].) *Let* $S_j = \{a_j\}$ $(j = 1, 2, 3)$, *where* a_1, a_2 *and* a_3 *are three distinct finite complex numbers. Suppose that* f *and* g *are two non-constant entire functions satisfying* $E_f(S_j) = E_g(S_j)$ *for* $j = 1, 2, 3$. *If*

$$\frac{2a_3 - a_1 - a_2}{a_2 - a_1} \neq -3, 0, 3,$$

then $f = g$.

In [3] Gross also proved that there exist three finite sets S_j $(j = 1, 2, 3)$ such that any two non-constant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2, 3$ must be identical, and asked the following question (see [3, Question 6]): Can one find two finite sets S_j $(j = 1, 2)$ such that any two non-constant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical? Now it is natural to ask the following question: Can one find three finite sets S_j $(j = 1, 2, 3)$ such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2, 3$ must be identical?

Throughout this paper we shall use w to denote the constant $\exp((2\pi i)/(n))$, where n is a positive integer and $n > 6$.

In this paper we answer the above questions. In fact, we prove more generally the following theorems.

THEOREM 1. *Let* $S_1 = \{1, w, w^2, \ldots, w^{n-1}\}$, $S_2 = \{a, b\}$ *and* $S_3 = \{0\}$, *where* a *and* b *are constants such that* $ab \neq 0$, $a^n \neq b^n$, $a^{2n} \neq 1$, $b^{2n} \neq 1$ *and* $a^n b^n \neq 1$. *Suppose that* f *and* g *are non-constant meromorphic functions satisfying* $E_f(S_j) = E_g(S_j)$ *for* $j = 1, 2, 3$. *Then* $f = g$.

THEOREM 2. *Let* S_1 *and* S_2 *be defined as in Theorem 1, and let* $S_3 = \{\infty\}$. *Suppose that* f *and* g *are non-constant meromorphic functions satisfying* $E_f(S_j) = E_g(S_j)$ *for* $j = 1, 2, 3$. *Then* $f = g$.

From Theorem 2 we immediately obtain the following result, which answers the question posed by Goss.

THEOREM 3. *Let* S_1 *and* S_2 *be defined as in Theorem 1. Suppose that* f *and* g *are non-constant entire functions satisfying* $E_f(S_j) = E_g(S_j)$ *for* $j = 1, 2$. *Then* $f = g$.

The following interesting result will be needed in the proof of our theorems.

THEOREM 4. *Let* S_1 *and* S_3 *be defined as in Theorem 2. Suppose that* f *and* g *are non-constant meromorphic functions satisfying* $E_f(S_j) = E_g(S_j)$ *for* $j = 1, 3$. *Then either* $f = cg$, *where* $c^n = 1$, *or* $fg = d$, *where* $d^n = 1$.

2. Some Lemmas

LEMMA 1. (See [8].) *Let* f *and* g *be two non-constant meromorphic functions, and let* c_1, c_2 *and* c_3 *be three non-zero constants. If* $c_1 f + c_2 g = c_3$, *then*

$$T(r, f) < \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r, f) + S(r, f).$$

LEMMA 2. (See [6, 2].) *Let* $f_1, f_2, \ldots, f_m$ *be linearly independent meromorphic functions satisfying* $\sum_{j=1}^{m} f_j = 1$. *Then for* $k = 1, 2, \ldots, m$ *we have*

$$T(r, f_k) < \sum_{j=1}^{m} N\left(r, \frac{1}{f_j}\right) + N(r, f_k) + N(r, D) - \sum_{j=1}^{m} N(r, f_j) - N\left(r, \frac{1}{D}\right) + o(T(r)) \quad (r \notin E),$$

where D *denotes the Wronskain*

$$D = \begin{vmatrix} f_1 & f_2 & \cdots & f_m \\ f_1' & f_2' & \cdots & f_m' \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{(m-1)} & f_2^{(m-1)} & \cdots & f_m^{(m-1)} \end{vmatrix}$$

and $T(r)$ *denotes the maximum of* $T(r, f_j)$, $j = 1, 2, \ldots, m$.

LEMMA 3. (See [9].) *Let* f_1, f_2 *and* f_3 *be three meromorphic functions satisfying* $\sum_{j=1}^{3} f_j = 1$, *and let* $g_1 = -f_3/f_2$, $g_2 = 1/f_2$ *and* $g_3 = -f_1/f_2$. *If* f_1, f_2 *and* f_3 *are linearly independent, then* g_1, g_2 *and* g_3 *are linearly independent.*

3. Proof of Theorem 4

By the assumption, from Nevanlinna's second fundamental theorem, we have

$$\begin{aligned} (1) \qquad (n-1)T(r, g) &< \sum_{k=0}^{n-1} N\left(r, \frac{1}{g - w^k}\right) + N(r, g) + S(r, g) \\ &= \sum_{k=0}^{n-1} N\left(r, \frac{1}{f - w^k}\right) + N(r, f) + S(r, g) \\ &< (n+1)T(r, f) + S(r, g). \end{aligned}$$

Thus

$$T(r, g) = O(T(r, f)) \quad (r \notin E). \tag{2}$$

Again by the assumption, we obtain

$$f^n - 1 = e^h(g^n - 1), \tag{3}$$

where h is an entire function. From (1) and (3), we have

$$\begin{aligned} T(r, e^h) &= T\left(r, \frac{f^n - 1}{g^n - 1}\right) \\ &< T(r, f^n) + T(r, g^n) + O(1) \\ &= nT(r, f) + nT(r, g) + O(1) \\ &< \left(n + \frac{n(n+1)}{n-1}\right)T(r, f) + S(r, f). \end{aligned}$$

Thus

$$T(r, e^h) = O(T(r, f)) \quad (r \notin E). \tag{4}$$

Let us put $f_1 = f^n$, $f_2 = e^h$, $f_3 = -e^h g^n$, and let $T(r)$ denote the maximum of $T(r, f_j)$, $j = 1, 2, 3$. From (2), (3) and (4), we obtain

$$\sum_{j=1}^{3} f_j = 1 \tag{5}$$

and

$$T(r) = O(T(r, f)) \quad (r \notin E). \tag{6}$$

Suppose that f_1, f_2 and f_3 are linearly independent. Applying Lemma 2 to the functions f_j $(j = 1, 2, 3)$, from (5) and (6) we have

$$\begin{aligned} T(r, f_1) < \sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right) - N\left(r, \frac{1}{D}\right) + N(r, D) - N(r, f_2) - N(r, f_3) \\ + o(T(r, f)) \quad (r \notin E), \end{aligned} \tag{7}$$

where

$$D = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}. \tag{8}$$

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We note that

$$(9)\qquad \sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right) = nN\left(r, \frac{1}{f}\right) + nN\left(r, \frac{1}{g}\right)$$

and

$$(10)\qquad N\left(r, \frac{1}{D}\right) \geqslant nN\left(r, \frac{1}{f}\right) - 2\overline{N}\left(r, \frac{1}{f}\right) + nN\left(r, \frac{1}{g}\right) - 2\overline{N}\left(r, \frac{1}{g}\right).$$

From (5) and (8) we get

$$D = \begin{vmatrix} f_2' & f_3' \\ f_2'' & f_3'' \end{vmatrix}$$

and hence

$$(11)\qquad \begin{aligned} N(r, D) - N(r, f_2) - N(r, f_3) &\leqslant N\left(r, (g^n)''\right) - N(r, g^n) \\ &= 2\overline{N}(r, g) = 2\overline{N}(r, f). \end{aligned}$$

From (7), (9), (10) and (11) we deduce

$$(12)\qquad \begin{aligned} nT(r, f) &< 2\overline{N}\left(r, \frac{1}{f}\right) + 2\overline{N}\left(r, \frac{1}{g}\right) + 2\overline{N}(r, g) + o(T(r, f)) \\ &< 2T(r, f) + 4T(r, g) + o(T(r, f)) \quad (r \notin E). \end{aligned}$$

Let $g_1 = -f_3/f_2 = g^n$, $g_2 = 1/f_2 = e^{-h}$ and $g_3 = -f_1/f_2 = -e^{-h}f^n$. From (5) we obtain

$$\sum_{j=1}^{3} g_j = 1.$$

By Lemma 3 we know that g_1, g_2 and g_3 are linearly independent. In the same manner as above, we have

$$(13)\qquad nT(r, g) < 4T(r, f) + 2T(r, g) + o(T(r, f)) \quad (r \notin E).$$

Combining (12) and (13) we get

$$(14)\qquad (n-6)T(r, f) + (n-6)T(r, g) < o(T(r, f)) \quad (r \notin E).$$

Since $n > 6$, (14) is absurd. Hence f_1, f_2 and f_3 are linearly dependent. Then, there exist three constants $(c_1, c_2, c_3) \neq (0, 0, 0)$ such that

$$(15)\qquad c_1 f_1 + c_2 f_2 + c_3 f_3 = 0.$$

If $c_1 = 0$, from (15) we have $c_2 \neq 0$, $c_3 \neq 0$ and

$$f_3 = -\frac{c_2}{c_3} f_2$$

and hence

$$g^n = \frac{c_2}{c_3},$$

which is impossible. Thus $c_1 \neq 0$ and

$$f_1 = -\frac{c_2}{c_1} f_2 - \frac{c_3}{c_1} f_3. \tag{16}$$

Now combining (5) and (16) we get

$$\left(1 - \frac{c_2}{c_1}\right) f_2 + \left(1 - \frac{c_3}{c_1}\right) f_3 = 1. \tag{17}$$

We discuss the following three cases.

(a) Assume $c_1 \neq c_2$ and $c_1 \neq c_3$. From (17) we have

$$\left(1 - \frac{c_3}{c_1}\right) g^n + e^{-h} = 1 - \frac{c_2}{c_1}. \tag{18}$$

By Lemma 1 and (18) we obtain

$$\begin{aligned} nT(r, g) &< \overline{N}\left(r, \frac{1}{g}\right) + S(r, g) \\ &< T(r, g) + S(r, g), \end{aligned}$$

which is impossible.

(b) Assume $c_1 = c_3$. From (17) we have $c_1 \neq c_2$ and

$$f_2 = \frac{c_1}{c_1 - c_2},$$

that is,

$$e^h = \frac{c_1}{c_1 - c_2}. \tag{19}$$

From (5) and (19) we get

$$f^n - \frac{c_1}{c_1 - c_2} g^n = -\frac{c_2}{c_1 - c_2}. \tag{20}$$

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If $c_2 \neq 0$, by Lemma 1 we have

$$\begin{aligned} nT(r, f) &< \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r, f) + S(r, f) \\ &< 2T(r, f) + T(r, g) + S(r, f), \end{aligned}$$

and

$$\begin{aligned} nT(r, g) &< \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r, g) + S(r, g) \\ &< T(r, f) + 2T(r, g) + S(r, g). \end{aligned}$$

Hence,

$$(n-3)T(r, f) + (n-3)T(r, g) < S(r, f) + S(r, g),$$

which is impossible. Thus $c_2 = 0$. From (20) we deduce $f^n = g^n$ and $f = cg$, where $c^n = 1$.

(c) Assume $c_1 = c_2$. From (17) we have $c_1 \neq c_3$ and

$$f_3 = \frac{c_1}{c_1 - c_3}. \tag{21}$$

From (5) and (21) we get

$$f_1 + f_2 = -\frac{c_3}{c_1 - c_3},$$

that is

$$f^n + e^h = -\frac{c_3}{c_1 - c_3}. \tag{22}$$

If $c_3 \neq 0$, applying Lemma 1 to (22), we have

$$\begin{aligned} nT(r, f) &< \overline{N}\left(r, \frac{1}{f}\right) + S(r, f) \\ &< T(r, f) + S(r, f), \end{aligned}$$

which is impossible. Thus $c_3 = 0$. By (21) and (22) we deduce $f^n = -e^h$, $g^n = -e^{-h}$ and $f^n g^n = 1$. Thus $fg = d$, where $d^n = 1$.

This completes the proof of Theorem 4. ∎

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4. Proof of Theorem 2

By the assumption $E_f(S_j) = E_g(S_j)$ $(j = 1, 3)$, we have from Theorem 4 that $f = cg$, where $c^n = 1$, or $fg = d$, where $d^n = 1$. We discuss the following two cases.

(a) Suppose that

$$f = cg, \tag{23}$$

where $c^n = 1$.

We discuss the following three subcases.

(a_1) Assume that a is not a Picard value of f, then there exists z_0 such that $f(z_0) = a$. By $E_f(S_2) = E_g(S_2)$, we obtain $g(z_0) = a$ or $g(z_0) = b$.

If $g(z_0) = a$, by (23) we have $a = ca$. Thus $c = 1$, and $f = g$.

If $g(z_0) = b$, by (23) we have $a = cb$. Thus $a^n = c^n b^n = b^n$, which contradicts the assumption.

(a_2) Assume that b is not a Picard value of f. In the same manner as above, we have $f = g$.

(a_3) Assume that a and b are Picard values of f. By $E_f(S_2) = E_g(S_2)$, we know that a and b are Picard values of g. Again by (23), we know that ca and cb are Picard values of f. Since a meromorphic function has at most two Picard values, then $a = ca$ or $a = cb$.

If $a = ca$, then $c = 1$, and $f = g$. If $a = cb$, then $a^n = c^n b^n = b^n$, which contradicts the assumption.

(b) Suppose that

$$fg = d, \tag{24}$$

where $d^n = 1$.

By the proof of Theorem 4, it is easy to see that 0 and ∞ are Picard values of f. Since a meromorphic function has at most two Picard values, then a and b are not Picard values of f. Thus there exists z_0 such that $f(z_0) = a$. By $E_f(S_2) = E_g(S_2)$, we obtain $g(z_0) = a$ or $g(z_0) = b$.

If $g(z_0) = a$, by (24) we have $a^2 = d$. Thus $a^{2n} = d^n = 1$, which contradicts the assumption.

If $g(z_0) = b$, by (24) we have $ab = d$. Thus $a^n b^n = d^n = 1$, which is also a contradiction.

This completes the proof of Theorem 2. □

5. Proof of Theorem 1

Let $S_4 = \{c, d\}$ and $S_5 = \{\infty\}$, where $c = 1/a$ and $d = 1/b$. By the assumption, it is easy to see that $cd \neq 0$, $c^n \neq d^n$, $c^{2n} \neq 1$, $d^{2n} \neq 1$ and $c^n d^n \neq 1$.

Let $F = 1/f$ and $G = 1/g$. By $E_f(S_j) = E_g(S_j)$ $(j = 1, 2, 3)$, we obtain $E_F(S_j) = E_G(S_j)$ $(j = 1, 4, 5)$. Applying Theorem 2 to the meromorphic functions F and G, we have $F = G$. Thus $f = g$, which proves Theorem 1. $\square$

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