MIXED HILBERT MODULAR FORMS AND FAMILIES OF ABELIAN VARIETIES

by MIN HO LEE

(Received 26 September, 1995)

1. Introduction. In **[18]** Shioda proved that the space of holomorphic 2-forms on a certain type of elliptic surface is canonically isomorphic to the space of modular forms of weight three for the associated Fuchsian group. Later, Hunt and Meyer **[6]** made an observation that the holomorphic 2-forms on a more general elliptic surface should in fact be identified with mixed automorphic forms associated to an automorphy factor of the form

$$j(g, z) = (cz + d)^2(c_\chi \omega(z) + d_\chi)$$

for z in the Poincaré upper half plane $\mathcal{H}$, $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\chi(g) = \begin{bmatrix} a_\chi & b_\chi \\ c_\chi & d_\chi \end{bmatrix}$, where g is an element of the fundamental group $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ of the base space of the elliptic fibration, $\chi : \Gamma \to \mathrm{SL}(2, \mathbb{R})$ the monodromy representation, and $\omega : \mathcal{H} \to \mathcal{H}$ the lifting of the period map of the elliptic surface.

Mixed automorphic forms of higher weight can also be defined using automorphy factors of the form

$$j(\gamma, z) = (cz + d)^k(c_\chi \omega(z) + d_\chi)^l$$

for nonnegative integers k and l, and certain types of such automorphic forms can be realized as the holomorphic forms of the highest degree on an elliptic variety, which is a fiber variety over a Riemann surface whose generic fiber is a product of a finite number of elliptic curves (see **[8]**, **[10]**). Certain aspects of mixed automorphic forms of several variables have also been investigated in **[11]** and **[12]**.

The purpose of this paper is to discuss a mixed version of Hilbert modular forms. More specifically, we introduce mixed Hilbert modular forms, describe some of their properties and show that the space of certain mixed Hilbert modular forms of type $(2, 2\nu)$ can be realized as holomorphic forms of the highest degree on a family of abelian varieties parametrized by a Hilbert modular variety.

I would like to thank the referee for various helpful suggestions.

2. Mixed Hilbert modular forms. In this section we define mixed Hilbert modular forms and discuss some of their properties. Let $\mathcal{H}^n = \mathcal{H} \times \ldots \times \mathcal{H}$ be the nth power of the Poincaré upper half plane

$$\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}.$$

The usual operation of $\mathrm{SL}(2, \mathbb{R})$ on $\mathcal{H}$ by linear fractional transformations induces an action of the nth power $\mathrm{SL}(2, \mathbb{R})^n$ of $\mathrm{SL}(2, \mathbb{R})$ on $\mathcal{H}^n$. Let F be a totally real number field with $[F : \mathbb{Q}] = n$. Thus there are n embeddings $F \hookrightarrow \mathbb{R}$, $a \mapsto a_j$ of F into $\mathbb{R}$, which induce an embedding

$$\mathrm{SL}(2, F) \hookrightarrow \mathrm{SL}(2, \mathbb{R}), \qquad M \mapsto (M_1, \ldots, M_n)$$

of the group $\mathrm{SL}(2, F)$ into $\mathrm{SL}(2, \mathbb{R})^n$, where

$$M_j = \begin{bmatrix} a_j & b_j \\ c_j & d_j \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R}) \quad \text{for} \quad M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, F)$$

and $j = 1, \ldots, n$. Throughout this paper we shall identify $\mathrm{SL}(2, F)$ with its embedded image in $\mathrm{SL}(2, \mathbb{R})^n$ under this embedding.

Let $\Gamma \subset \mathrm{SL}(2, F)$ be a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})^n$, $\chi : \Gamma \to \mathrm{SL}(2, F)$ a homomorphism, and $\omega : \mathscr{H}^n \to \mathscr{H}^n$ a holomorphic map such that

$$\omega(gz) = \chi(g)\omega(z)$$

for all $g \in \Gamma$ and $z \in \mathscr{H}^n$. We assume that the image of a parabolic element in Γ is a parabolic element in $\chi(\Gamma)$ and that the image of an element of the form $\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$ in Γ is of the form $\begin{bmatrix} 1 & \lambda_\chi \\ 0 & 1 \end{bmatrix}$ for some $\lambda_\chi \in F$. If $g \in \Gamma \subset \mathrm{SL}(2, F)$ and $z \in \mathscr{H}^n$, we set

$$\begin{aligned} J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(g, z) &= N(cz + d)^{2\mathbf{k}} \,.\, N(c_\chi \omega(z) + d_\chi)^{2\mathbf{l}} \\ &= \prod_{j=1}^{n} (c_j z_j + d_j)^{2k_j} \,.\, \prod_{j=1}^{n} (c_{\chi,j}\omega(z)_j + d_{\chi,j})^{2l_j} \end{aligned}$$

for $\mathbf{k} = (k_1, \ldots, k_n)$, $\mathbf{l} = (l_1, \ldots, l_n) \in \mathbb{Z}^n$ with each l_i, k_i nonnegative and

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma, \qquad \chi(g) = \begin{bmatrix} a_\chi & b_\chi \\ c_\chi & d_\chi \end{bmatrix} \in \mathrm{SL}(2, F).$$

Then $J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi} : \Gamma \times \mathscr{H}^n \to \mathbb{C}$ is an automorphy factor, i.e., it satisfies the relation

$$J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(gh, z) = J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(g, hz) \,.\, J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(h, z)$$

for all g, $h \in \Gamma$ and $z \in \mathscr{H}^n$. If $\mathbf{k} = (k, \ldots, k)$ and $\mathbf{l} = (l, \ldots, l)$ for some nonnegative integers k and l, then $J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}$ will also be denoted simply by $J^{2k,2l}_{\Gamma,\omega,\chi}$.

In order to discuss Fourier expansions we assume that $f : \mathscr{H}^n \to \mathbb{C}$ is a function that satisfies the functional equation

$$f(gz) = J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(g, z) f(z)$$

for all $g \in \Gamma$ and $z \in \mathscr{H}^n$. Then we can consider the Fourier expansion of f at the cusps of Γ as follows. Suppose first that ∞ is a cusp of Γ. We set

$$\Lambda = \Lambda(\Gamma) = \left\{ \lambda \in F \;\middle|\; \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \in \Gamma \right\},$$

and identify it with a subgroup of $\mathbb{R}^n$ via the natural embedding $F \hookrightarrow \mathbb{R}^n : \lambda \mapsto (\lambda_1, \ldots, \lambda_n)$. From our assumption on the homomorphism χ, for each $\lambda \in \Lambda$, we have

$$\chi \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \lambda_\chi \\ 0 & 1 \end{bmatrix}$$

for some $\lambda_\chi \in F$, and therefore we obtain

$$J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}\left(\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, z \right) = 1.$$

Thus f is periodic with $f(z+\lambda)=f(z)$ for all $z \in \mathscr{H}$ and $\lambda \in \Lambda$, and hence it has a Fourier expansion. Let Λ^* denote the dual lattice given by

$$\Lambda^* = \{\xi \in F \mid T(\xi\lambda) \in \mathbb{Z} \text{ for all } \lambda \in \Lambda\},$$

where $T(\xi\lambda) = \sum_{j=1}^{n} \xi_j \lambda_j$. Then the *Fourier expansion of f at ∞* is given by

$$f(z) = \sum_{\xi \in \Lambda^*} a_\xi e^{2\pi i T(\xi z)},$$

where $T(\xi z) = \sum_{j=1}^{n} \xi_j z_j$.

Now we consider an arbitrary cusp s of Γ. Let σ be an element of $SL(2,F) \subset SL(2,\mathbb{R})^n$ such that $\sigma(\infty)=s$. We assume that the homomorphism $\chi:\Gamma \to SL(2,F)$ can be extended to a mapping $\chi:\Gamma' \to SL(2,F)$, where

$$\Gamma' = \Gamma \cup \{\sigma \in SL(2,F) \mid \sigma(\infty)=s,\ s \text{ a cusp of } \Gamma\}.$$

We fix $\mathbf{k}, \mathbf{l} \in \mathbb{Z}^n$, and set

$$\Gamma^\sigma = \sigma^{-1}\Gamma\sigma,$$
$$(f\|[\sigma])(z) = J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(\sigma, z)^{-1} f(\sigma z).$$

LEMMA 2.1. *If f satisfies*

$$f(gz) = J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(g,z) f(z)$$

for all $g \in \Gamma$ and $z \in \mathscr{H}^n$, then the function $f\|[\sigma]: \mathscr{H}^n \to \mathbb{C}$ satisfies the functional equation

$$(f \mid [\sigma])(gz) = J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(g,z)(f \mid [\sigma])(z)$$

for all $g \in \Gamma^\sigma$ and $z \in \mathscr{H}^n$.

Proof. Let $g = \sigma^{-1}\gamma\sigma \in \Gamma^\sigma$ with $\gamma \in \Gamma$. Then we have

$$\begin{aligned}(f \mid [\sigma])(gz) &= J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(\sigma, \sigma^{-1}\gamma\sigma z)^{-1} f(\sigma\sigma^{-1}\gamma\sigma z)\\ &= J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(\sigma, \sigma^{-1}\gamma\sigma z)^{-1} J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(\gamma, \sigma z) f(\gamma\sigma z)\\ &= J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(\sigma^{-1}\gamma, \sigma z) f(\sigma z),\end{aligned}$$

since

$$J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(\gamma, \sigma z) = J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(\sigma\sigma^{-1}\gamma, \sigma z) = J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(\sigma, \sigma^{-1}\gamma\sigma z) J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(\sigma^{-1}\gamma, \sigma z).$$

However, we have

$$J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(\sigma^{-1}\gamma\sigma, z) = J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(\sigma^{-1}\gamma, \sigma z) J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(\sigma, z).$$

Thus we obtain

$$\begin{aligned}(f \mid [\sigma])(gz) &= J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(\sigma^{-1}\gamma\sigma, z) J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(\sigma, z)^{-1} f(\sigma z)\\ &= J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(\sigma^{-1}\gamma\sigma, z)(f \mid [\sigma])(\sigma z);\end{aligned}$$

hence the lemma follows.

Since ∞ is a cusp of Γ^σ, the function $f \mid [\sigma]$ has a Fourier expansion at ∞ of the form

$$(f \mid [\sigma])(z) = \sum_{\xi \in \Lambda^*} a_\xi e^{2\pi i T(\xi z)}.$$

This series is called a *Fourier expansion of f at the cusp s*, and the coefficients a_ξ are called the *Fourier coefficients of f at s.*

DEFINITION 2.2. Let $\Gamma \in SL(2, \mathbb{R})^n$ be a discrete subgroup with cusp s, and let $f: \mathscr{H}^n \to \mathbb{C}$ be a holomorphic function satisfying the relation

$$f(gz) = J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(g, z) f(z).$$

(i) The function f is *regular at s* if the Fourier coefficients of f at s satisfy the condition that $\xi \geq 0$ whenever $a_\xi \neq 0$.

(ii) The function f *vanishes at s* if the Fourier coefficients of f at s satisfy the condition that $\xi > 0$ whenever $a_\xi \neq 0$.

REMARK 2.3. Given a cusp s of Γ there may be more than one element $\sigma \in SL(2, F)$ such that $\sigma(\infty) = s$. However the above definition makes sense because of the next lemma.

LEMMA 2.4. *Let s be a cusp of Γ and assume that $\sigma(\infty) = \sigma'(\infty) = s$ for σ, $\sigma' \in SL(2, F)$. Then $f \mid [\sigma]$ is regular (resp. vanishes) at ∞ if and only if $f \mid [\sigma']$ is regular (resp. vanishes) at ∞.*

Proof. It is sufficient to prove the lemma for the case when σ' is the identity element in $SL(2, F)$ and $s = \infty$. Then we have $\sigma(\infty) = \infty$, and hence

$$\sigma = \begin{bmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

for some b, $\delta \in F$. Let $\Lambda_\sigma = \Lambda(\Gamma^\sigma) = \Lambda(\sigma^{-1}\Gamma\sigma)$. Then $\lambda \in \Lambda_\sigma$ if and only if

$$\sigma \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \sigma^{-1} = \begin{bmatrix} 1 & \delta^2\lambda \\ 0 & 1 \end{bmatrix} \in \Gamma;$$

hence we have $\Lambda_\sigma = \delta^{-2}\Lambda$. Therefore $\Lambda^*_\sigma = \delta^2\Lambda^*$, and we have the Fourier expansions

$$f(z) = \sum_{\xi \in \Lambda^*} a_\xi e^{2\pi i T(\xi z)}, \qquad (f \mid [\sigma])(z) = \sum_{\xi \in \Lambda^*_\sigma} a^\sigma_\xi e^{2\pi i T(\xi z)}.$$

On the other hand, we have

$$\begin{aligned} (f \mid [\sigma])(z) &= J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(\sigma, z)^{-1} f(\sigma z) = J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(\sigma, z)^{-1} f(\delta^2(z + b)) \\ &= J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(\sigma, z)^{-1} \sum_{\xi \in \Lambda^*} a_\xi e^{2\pi i T(\delta^2 b \xi)} e^{2\pi i T(\xi\delta^2 z)} \\ &= J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(\sigma, z)^{-1} \sum_{\xi \in \Lambda^*_\sigma} a_{\xi\delta^{-2}} e^{2\pi i T(b\xi)} e^{2\pi i T(\xi z)}. \end{aligned}$$

Thus we obtain

$$a^\sigma_\xi = J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(\sigma, z)^{-1} e^{2\pi i T(b\xi)} a_{\xi\delta^{-2}}$$

for all $\xi \in \Lambda^*_\sigma$. The lemma follows from this relation.

DEFINITION 2.5. Let Γ, χ, and ω be as above, and assume that the quotient space $\Gamma \backslash \mathscr{H}^n \cup \{\text{cusps}\}$ is compact. A *mixed Hilbert modular form of type* $(2\mathbf{k}, 2\mathbf{l})$ associated to Γ, χ and ω is a holomorphic function $f: \mathscr{H}^n \to \mathbb{C}$ satisfying the following conditions:

(i) $f(\gamma z) = J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}(\gamma, z) f(z)$ for all $\gamma \in \Gamma$;

(ii) f is regular at the cusps of Γ.

The holomorphic function f is a *mixed Hilbert modular cusp form* if (ii) is replaced with the following condition:

(ii)′ f vanishes at the cusps of Γ.

If $\mathbf{k}=(k,\dots,k)$ and $\mathbf{l}=(l,\dots,l)$ with nonnegative integers k and l, then a mixed Hilbert modular form of type $(2\mathbf{k},2\mathbf{l})$ will also be called a *mixed Hilbert modular form of type* $(2k,2l)$.

As in the case of the usual Hilbert modular forms, Koecher's principle also holds true in the mixed case as is described in the next proposition. Thus the condition (ii) is not necessary for $n\geq 2$.

Proposition 2.6. *If $n\geqslant 2$, then any holomorphic function $f:\mathscr{H}^n\to\mathbb{C}$ satisfying the condition* (i) *in Definition* 2.5 *is a mixed Hilbert modular form of type* $(2\mathbf{k},2\mathbf{l})$ *associated to* Γ, ω *and* χ.

Proof. Let ε be an element in F such that the transformation $z\mapsto\varepsilon z+b$ is contained in Γ for some b. Then we have

$$\begin{aligned} f(\varepsilon z+b) &= J^{2\mathbf{k},2\mathbf{l}}_{\Gamma,\omega,\chi}\left(\begin{bmatrix}\varepsilon^{1/2} & b\varepsilon^{-1/2}\\ 0 & \varepsilon^{-1/2}\end{bmatrix},z\right)f(z)\\ &= \varepsilon_1^{-k_1}\dots\varepsilon_n^{-k_n}\varepsilon_{\chi,1}^{-l_1}\dots\varepsilon_{\chi,n}^{-l}f(z)\\ &= N(\varepsilon^{-\mathbf{k}})N(\varepsilon_\chi^{-1})f(x)\end{aligned}$$

if

$$\chi\begin{bmatrix}\varepsilon^{1/2} & b\varepsilon^{-1/2}\\ 0 & \varepsilon^{-1/2}\end{bmatrix}=\begin{bmatrix}\varepsilon_\chi^{1/2} & d\varepsilon_\chi^{-1/2}\\ 0 & \varepsilon_\chi^{-1/2}\end{bmatrix}$$

for some elements ε_χ, $d\in F$ (note that the image of a parabolic element under χ is a parabolic element). Hence, if $f(z)=\sum_{\xi\in\Lambda^*}a_\xi e^{2\pi iT(\xi z)}$ is the Fourier expansion of $f(z)$ at ∞, then we have

$$a_{\varepsilon\xi}=a_\xi e^{2\pi iT(\xi b)}N(\varepsilon^{-\mathbf{k}})N(\varepsilon_\chi^{-1}).$$

Now suppose $\xi=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$ with $\xi_i<0$ for some i, and choose a unit $\varepsilon\gg 0$ such that $\varepsilon_i>1$ and $\varepsilon_j<1$ for $j\neq i$. Let c be any positive real number, and consider the subseries

$$\sum_{m=1}^{\infty}a_{\varepsilon^{2m}\xi}e^{2\pi iT(\varepsilon^{2m}\xi ic)}=a_\xi e^{2\pi iT(\xi b)}\sum_{m=1}^{\infty}N(\varepsilon^{-2m\mathbf{k}})N((\varepsilon^{2m})_\chi^{-1})e^{-2\pi cT(\varepsilon^{2m}\xi)}$$

of the Fourier series of $f(ic)$. Since we have

$$T(\varepsilon^{2m}\xi)=\varepsilon_i^{2m}\xi_i+\sum_{j\neq i}\varepsilon_j^{2m}\xi_j,$$

the above subseries cannot converge unless $a_\xi=0$. Therefore ξ is positive whenever $a_\xi\neq 0$.

3. Families of abelian varieties. In this section we discuss a relation between mixed Hilbert modular forms and holomorphic forms on a family of abelian varieties parameterized by a Hilbert modular variety. Let $\mathscr{H}^n$, Γ, ω and χ be as in Section 2. Thus $\Gamma\subset\mathrm{SL}(2,F)$ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})^n$, $\chi:\Gamma\to\mathrm{SL}(2,F)\subset\mathrm{SL}(2,\mathbb{R})^n$ is a homomorphism of groups, and $\omega:\mathscr{H}^n\to\mathscr{H}^n$ is a holomorphic map satisfying

$$\omega(gz)=\chi(g)\omega(z)$$

for all $g \in \Gamma$ and $z \in \mathcal{H}^n$. Throughout the rest of this paper, we shall assume that

$$\gamma . (\mathbb{Z} \times \mathbb{Z})^n \subset (\mathbb{Z} \times \mathbb{Z})^n \quad \text{for all} \quad \gamma \in \Gamma.$$

Consider the semidirect product $\Gamma \ltimes (\mathbb{Z} \times \mathbb{Z})^{mn}$ consisting of elements of the form

$$\begin{aligned}(g, (\mu, \nu)) &= (g_1, \dots, g_n; (\mu, \nu)_1, \dots, (\mu, \nu)_n) \\ &= (g_1, \dots, g_n; (\mu_1^1, \nu_1^1), \dots, (\mu_1^m, \nu_1^m); \dots; (\mu_n^1, \nu_n^1), \dots, (\mu_n^m, \nu_n^m))\end{aligned}$$

with its multiplication operation given by

$$(g, (\mu, \nu)) . (g', (\mu', \nu')) = (gg', (\mu, \nu)g' + (\mu', \nu')),$$

where

$$(\mu, \nu) = ((\mu, \nu)_1, \dots, (\mu, \nu)_n) = ((\mu_1^1, \nu_1^1), \dots, (\mu_1^m, \nu_1^m); \dots, (\mu_n^1, \nu_n^1), \dots, (\mu_n^m; \nu_n^m))$$

with $\mu_j^k, \nu_j^k \in \mathbb{Z}$ for $1 \le j \le n$ and $1 \le k \le m$, and

$$\begin{aligned}(\mu, \nu)g' &= ((\mu, \nu)_1 g_1', \dots, (\mu, \nu)_n g_n') \\ &= ((\mu_1^1, \nu_1^1)g_1', \dots, (\mu_1^m, \nu_1^m)g_1'; \dots; (\mu_n^1, \nu_n^1)g_n', \dots, (\mu_n^m, \nu_n^m)g_n')\end{aligned}$$

for $g' = (g_1', \dots, g_n') \in \Gamma \subset \mathrm{SL}(2, \mathbb{R})^n$. Then the discrete group $\Gamma \ltimes (\mathbb{Z} \times \mathbb{Z})^{mn}$ operates on $\mathcal{H}^n \times \mathbb{C}^{mn}$ by

$$\begin{aligned}(g, (\mu, \nu)) . (z, \zeta) = \Bigg(&\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \dots, \frac{a_n z_n + b_n}{c_n z_n + d_n}; \\ &\frac{\mu_1^1 \omega(z)_1 + \nu_1^1 + \zeta_1^1}{c_{\chi,1}\omega(z)_1 + d_{\chi,1}}, \dots, \frac{\mu_1^m \omega(z)_1 + \nu_1^m + \zeta_1^m}{c_{\chi,1}\omega(z)_1 + d_{\chi,1}}; \dots \\ &\dots; \frac{\mu_n^1 \omega(z)_n + \nu_n^1 + \zeta_n^1}{c_{\chi,n}\omega(z)_n + d_{\chi,n}}, \dots, \frac{\mu_n^m \omega(z)_n + \nu_n^m + \zeta_n^m}{c_{\chi,n}\omega(z)_n + d_{\chi,n}}\Bigg),\end{aligned}$$

where

$$g = (g_1, \dots, g_n) \in \Gamma \quad \text{with} \quad g_j = \begin{bmatrix} a_j & b_j \\ c_j & d_j \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R}) \quad \text{for } 1 \le j \le n,$$

$$(\mu, \nu) = ((\mu_1^1, \nu_1^1), \dots, (\mu_1^m, \nu_1^m); \dots; (\mu_n^1, \nu_n^1), \dots, (\mu_n^m, \nu_n^m)) \in (\mathbb{Z} \times \mathbb{Z})^{mn},$$

$$(z, \zeta) = (z_1, \dots, z_n; \zeta_1^1, \dots, \zeta_1^m; \dots, \zeta_n^1, \dots, \zeta_n^m) \in \mathcal{H}^n \times \mathbb{C}^{mn},$$

$$\chi(g) = (\chi(g)_1, \dots, \chi(g)_n) \quad \text{with} \quad \chi(g)_j = \begin{bmatrix} a_{\chi,j} & b_{\chi,j} \\ c_{\chi,j} & d_{\chi,j} \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R}) \quad \text{for} \quad 1 \le j \le n,$$

$$\omega(z) = (\omega(z)_1, \dots, \omega(z)_n) \in \mathcal{H}^n.$$

Now we assume that Γ does not contain elements of finite order, and set

$$E_{\Gamma,\omega,\chi}^{m,n} = \Gamma \ltimes (\mathbb{Z} \times \mathbb{Z})^{mn} \backslash \mathcal{H}^n \times \mathbb{C}^{mn},$$

where the quotient is taken with respect to the operation of $\Gamma \ltimes (\mathbb{Z} \times \mathbb{Z})^{mn}$ on $\mathcal{H}^n \times \mathbb{C}^{mn}$ described above. If X_Γ denotes the Hilbert modular variety $\Gamma \backslash \mathcal{H}^n$, then the canonical projection $\mathcal{H}^n \times \mathbb{C}^{mn} \to \mathcal{H}^n$ determines the mapping $\pi_\Gamma^{m,n}: E_{\Gamma,\omega,\chi}^{m,n} \to X_\Gamma$.

PROPOSITION 3.1. (i) *When* $m=1$, *the corresponding map* $\pi_\Gamma^{1,n}:E_{\Gamma,\omega,\chi}^{1,n}\to X_\Gamma$ *is a fiber variety over the Hilbert modular variety* $X_\Gamma=\Gamma\backslash\mathcal{H}^n$ *whose fiber* $\mathbb{C}^n/(\mathbb{Z}\times\mathbb{Z})^n$ *has a canonical structure of an abelian variety.*

(ii) *The space* $E_{\Gamma,\omega,\chi}^{m,n}$ *is an* m*-fold fiber power of* $E_{\Gamma,\omega,\chi}^{1,n}$ *in* (i) *over* X_Γ.

Proof. These statements are proved in [**14**, Proposition 7.4] for the case of $E_{\chi(\Gamma),\mathrm{id},\mathrm{id}}^{m,n}$. The proof for the general case follows from the observation that $E_{\Gamma,\omega,\chi}^{m,n}$ can be obtained by pulling back the fiber bundle $E_{\chi(\Gamma),\mathrm{id},\mathrm{id}}^{m,n}$ over $X_{\chi(\Gamma)}=\chi(\Gamma)\backslash\mathcal{H}^n$ via the natural map $X_\Gamma\to X_{\chi(\Gamma)}$ induced by $\omega:\mathcal{H}^n\to\mathcal{H}^n$ so that the diagram

$$\begin{array}{ccc} E_{\Gamma,\omega,\chi}^{m,n} & \to & E_{\chi(\Gamma),\mathrm{id},\mathrm{id}}^{m,n} \\ \downarrow & & \downarrow \\ X_\Gamma & \longrightarrow & X_{\chi(\Gamma)} \end{array}$$

is commutative (see also [**7**], [**9**], [**13**], [**17**, Chapter IV]).

Given a nonnegative integer ν, let $J_{\Gamma,\omega,\chi}^{2,2\nu}:\Gamma\times\mathcal{H}^n\to\mathbb{C}$ be the automorphy factor described in Section 2, that is, the automorphy factor $J_{\Gamma,\omega,\chi}^{2\mathbf{k},2\mathbf{l}}$ for $\mathbf{k}=(1,\ldots,1)$ and $\mathbf{l}=(\nu,\ldots,\nu)$. Then the discrete subgroup $\Gamma\subset G$ operates on $\mathcal{H}^n\times\mathbb{C}$ by

$$g\cdot(z,\zeta)=(gz,J_{\Gamma,\omega,\chi}^{2,2\nu}(g,z)\zeta)$$

for all $g\in\Gamma$ and $(z,\zeta)\in\mathcal{H}^n\times\mathbb{C}$. We set

$$\mathcal{L}_{\Gamma,\omega,\chi}^{2,2\nu}=\Gamma\backslash\mathcal{H}^n\times\mathbb{C},$$

where the quotient is taken with respect to the operation described above. Then the natural projection $\mathcal{H}^n\times\mathbb{C}\to\mathcal{H}^n$ induces on $\mathcal{L}_{\Gamma,\omega,\chi}^{2,2\nu}$ the structure of a line bundle over the arithmetic variety $X_\Gamma=\Gamma\backslash\mathcal{H}^n$, and the sections of this bundle can be identified with functions $f:\mathcal{H}^n\to\mathbb{C}$ satisfying

$$f(gz)=J_{\Gamma,\omega,\chi}^{2,2\nu}(g,z)f(z)$$

for all $g\in\Gamma$ and $z\in\mathcal{H}^n$.

THEOREM 3.2. *Let* $\Omega^{(2\nu+1)n}$ *be the sheaf of holomorphic* $(2\nu+1)n$*-forms on* $E_{\Gamma,\omega,\chi}^{2,2\nu}$. *Then the space of sections of the line bundle* $\mathcal{L}_{\Gamma,\omega,\chi}^{2,2\nu}$ *over* X_Γ *is canonically isomorphic to the space* $H^0(E_{\Gamma,\omega,\chi}^{2,2\nu},\Omega^{(2\nu+1)n})$ *of holomorphic* $(2\nu+1)n$*-forms on* $E_{\Gamma,\omega,\chi}^{2,2\nu}$.

Proof. From the construction of $E_{\Gamma,\omega,\chi}^{m,n}$ it follows that a holomorphic $(2\nu+1)n$-form on $E_{\Gamma,\omega,\chi}^{m,n}$ can be regarded as a holomorphic $(2\nu+1)n$-form on $\mathcal{H}^n\times\mathbb{C}^{2\nu n}$ that is invariant under the operation of $\Gamma\ltimes(\mathbb{Z}\times\mathbb{Z})^{2\nu n}$. Since $(2\nu+1)n$ is the complex dimension of the space $\mathcal{H}^n\times\mathbb{C}^{2\nu n}$, a holomorphic $(2\nu+1)n$-form on $\mathcal{H}^n\times\mathbb{C}^{2\nu n}$ is of the form

$$\theta=\hat{f}(z,\zeta)dz\wedge d\zeta,$$

where

$$(z,\zeta)=(z_1,\ldots,z_n;\,\zeta_1^1,\ldots,\zeta_1^{2\nu};\ldots;\zeta_n^1,\ldots,\zeta_n^{2\nu})\in\mathcal{H}^n\times\mathbb{C}^{2\nu n},$$
$$dz=dz_1\wedge\ldots\wedge dz_n,$$
$$d\zeta=d\zeta_1^1\wedge\ldots\wedge d\zeta_1^{2\nu}\wedge\ldots\wedge d\zeta_n^1\wedge\ldots\wedge d\zeta_n^{2\nu},$$

and $\hat{f}$ is a holomorphic function. Given a fixed point $z_0\in\mathcal{H}^n$, the holomorphic form θ descends to a holomorphic $2\nu n$-form on the corresponding fiber of the fiber bundle

$E^{m,n}_{\Gamma,\omega,\chi} \to X_\Gamma = \Gamma\backslash\mathscr{H}^n$. Since the complex dimension of the fiber is $2\nu n$, the dimension of the space of holomorphic $2\nu n$-forms is one. Thus the mapping $\zeta \mapsto \tilde{f}(z, \zeta)$ is a holomorphic $2\nu n$-variable function with $2\nu n$ independent variables, and therefore must be constant. Hence the function $\tilde{f}(z, \zeta)$ depends only on z, and $\tilde{f}(z, \zeta) = f(z)$ where f is a holomorphic function on $\mathscr{H}^n$. Given $(g, (\mu, \nu)) \in \Gamma \ltimes (\mathbb{Z} \times \mathbb{Z})^{2\nu n}$ as above, we have the operations

$$dz_j \mid (g, (\mu, \nu)) = (c_j z_j + d_j)^{-2} dz_j, \qquad 1 \le j \le n,$$

$$d\zeta^k_j \mid (g, (\mu, \nu)) = (c_{\chi,j}\omega(z)_j + d_{\chi,j})^{-1} d\zeta^k_j + \sum_{i=1}^n F_i(z, \zeta) dz_i$$

for $1 \le k \le 2\nu$, $1 \le j \le n$, and some functions $F_i(z, \zeta)$. Thus the operation of $(g, (\mu, \nu))$ on θ is given by

$$\theta \mid (g, (\mu, \nu)) = f(gz) \prod_{j=1}^n (c_j z_j + d_j)^{-2} (c_{\chi,j}\omega(z)_j + d_{\chi,j})^{-2\nu} dz \wedge d\zeta.$$

Hence it follows that

$$f(gz) = f(z) \prod_{j=1}^n (c_j z_j + d_j)^2 (c_{\chi,j}\omega(z)_j + d_{\chi,j})^{2\nu} = f(z) J^{2,2\nu}_{\Gamma,\omega,\chi}(g, z),$$

and therefore f can be identified with a section of $\mathscr{L}^{2,2\nu}_{\Gamma,\omega,\chi}$.

COROLLARY 3.3. *Let $\mathscr{A}_{2,2\nu}(\Gamma, \omega, \chi)$ be the space of mixed Hilbert modular forms of type $(2, 2\nu)$ associated to Γ, ω and χ. If $n \geqslant 2$, then there is a canonical isomorphism*

$$\mathscr{A}_{2,2\nu}(\Gamma, \omega, \chi) \cong H^0(E^{2,2\nu}_{\Gamma,\omega,\chi}, \Omega^{(2\nu+1)n}).$$

Proof. The corollary follows from Theorem 3.2 and Proposition 2.6.

4. Compactifications. Arithmetic varieties such as the Hilbert modular variety $X_\Gamma = \Gamma\backslash\mathscr{H}^n$ considered in Section 3 can be regarded as connected components of Shimura varieties [**3**]). Mixed Shimura varieties generalize Shimura varieties, and they play an essential role in the theory of compactifications of Shimura varieties ([**1**], [**4**], [**5**]). A typical mixed Shimura variety is essentially a torus bundle over a family of abelian varieties parametrized by a Shimura variety (see [**15**], [**16**]). A Shimura variety and a family of abelian varieties which it parametrizes can also be considered as special cases of mixed Shimura varieties. In this section we discuss extensions of the results obtained in Section 3 to the compactifications of families of abelian varieties using the theory of toroidal compactifications of mixed Shimura varieties developed in [**1**] (see also [**4**]).

Let $\pi_\Gamma : E^{2,2\nu}_{\Gamma,\omega,\chi} \to X_\Gamma$ be the family of abelian varieties parametrized by an arithmetic variety described in Section 3. Using the language of Shimura varieties, $E^{2,2\nu}_{\Gamma,\omega,\chi}$ can be regarded as the mixed Shimura variety $M^{K_f}(P, \mathscr{X})(\mathbb{C})$ associated to the group

$$P = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{SL}(2, F) \ltimes V_{4\nu n}$$

and the subgroup $K_f \subset P(\mathbb{A}_f)$ with $K_f \cap P(\mathbb{Q}) = \Gamma \ltimes (\mathbb{Z} \times \mathbb{Z})^{2\nu n}$, where Res is Weil's restriction map and $V_{4\nu n}$ is a $\mathbb{Q}$-vector space of dimension $4\nu n$. Thus $\mathscr{X}$ is a left homogeneous space under the subgroup $P(\mathbb{R}) . U(\mathbb{C}) \subset P(\mathbb{C})$, where U is a subgroup of the unipotent radical W of P, and $M^{K_f}(P,\mathscr{X})(\mathbb{C}) = P(\mathbb{Q})\backslash\mathscr{X}\times(P(\mathbb{A}_f)/K_f)$, where the operation of $P(Q)$ on $\mathscr{X}$ is via χ and ω (see **[15]**, **[16]** for details). The arithmetic variety X_Γ is the mixed Shimura variety $M^{K_f}((P,\chi)/W)(\mathbb{C})$, which is in fact a pure Shimura variety. Furthermore, the mapping π_Γ can be considered as the natural projection map

$$M^{K_f}(P,\mathscr{X})(\mathbb{C}) \to M^{K_f}((P,\mathscr{X})/W)(\mathbb{C}).$$

There are a number of ways of compactifying Shimura varieties. Among those are Baily–Borel compactifications (see **[2]**) and toroidal compactifications. The toroidal compactifications of mixed Shimura varieties were constructed by R. Pink in **[16]**. Let $\bar{X}_\Gamma$ be the Baily–Borel compactification of X_Γ, and denote by

$$\tilde{E}^{2,2\nu}_{\Gamma,\omega,\chi} = M^{k_f}(P,\mathscr{X},\mathscr{S})(\mathbb{C})$$

the toroidal compactification of $E^{2,2\nu}_{\Gamma,\omega,\chi} = M^{K_f}(P,\mathscr{X})(\mathbb{C})$ associated to a K_f-admissible partial cone decomposition $\mathscr{S}$ for $(P,\mathscr{X})$. Then π_Γ induces the mapping $\bar{\pi}_\Gamma:\tilde{E}^{2,2\nu}_{\Gamma,\omega,\chi}\to\bar{X}_\Gamma$ of compactifications (see **[16]** for details).

THEOREM 4.1. *Let $\Omega^{(2\nu+1)n}(\log\partial\tilde{E})$ be the sheaf of holomorphic $(2\nu+1)n$-forms on $\tilde{E}^{2,2\nu}_{\Gamma,\omega,\chi}$ with logarithmic poles along the boundary*

$$\partial\tilde{E} = \tilde{E}^{2,2\nu}_{\Gamma,\omega,\chi} - E^{2,2\nu}_{\Gamma,\omega,\chi}.$$

Then there exists an extension $\bar{\mathscr{L}}^{2,2\nu}_{\Gamma,\omega,\chi}$ of $\mathscr{L}^{2,2\nu}_{\Gamma,\omega,\chi}$ to the Baily-Borel compactification $\bar{X}_\Gamma$ of X_Γ, which depends only on $(P,\mathscr{X})/U$ up to isomorphism, such that there is a canonical isomorphism

$$\Omega^{(2\nu+1)n}(\log\partial\tilde{E}) \cong \bar{\pi}^*_\Gamma\bar{\mathscr{L}}^{2,2\nu}_{\Gamma,\omega,\chi}$$

of sheaves, where the line bundle $\bar{\mathscr{L}}^{2,2\nu}_{\Gamma,\omega,\chi}$ is regarded as an invertible sheaf.

Proof. By Proposition 8.1 in **[16]**, there is an invertible sheaf $\mathscr{F}$ on the Baily–Borel compactification $\bar{X}_\Gamma$ of X such that there is a canonical isomorphism

$$\pi^*_\Gamma\mathscr{F} \cong \Omega^{(2\nu+1)n}(\log\partial\tilde{E}).$$

On the other hand, using Theorem 3.2 we obtain a canonical isomorphism $\pi^*_\Gamma\mathscr{L}^{2,2\nu}_{\Gamma,\omega,\chi}\cong\Omega^{(2\nu+1)n}$. Thus it follows that $\bar{\mathscr{L}}^{2,2\nu}_{\Gamma,\omega,\chi} = \mathscr{F}$ is the desired extension of $\mathscr{L}$.

REMARK 4.2. Let $\tilde{X}_\Gamma$ be the toroidal compactification of X_Γ, and let $\tilde{\mathscr{L}}^{2,2\nu}_{\Gamma,\omega,\chi}$ be the canonical extension of $\mathscr{L}^{2,2\nu}_{\Gamma,\omega,\chi}$ to $\tilde{X}_\Gamma$. Let $\iota: X_\Gamma\to\bar{X}_\Gamma$ be the canonical embedding of X_Γ into its Baily–Borel compactification $\bar{X}_\Gamma$. Then the image of the restriction map

$$H^0(\tilde{X}_\Gamma,\tilde{\mathscr{L}}^{2,2\nu}_{\Gamma,\omega,\chi}) \to H^0(X_\Gamma,\mathscr{L}^{2,2\nu}_{\Gamma,\omega,\chi}) \cong H^0(\bar{X}_{\Gamma,}\iota*\mathscr{L}^{2,2\nu}_{\Gamma,\omega,\chi})$$

is the subspace of sections regular at infinity, and hence it is the space of sections in $H^0(\bar{X}_{\Gamma,}\iota*\mathscr{L}^{2,2\nu}_{\Gamma,\omega,\chi})$ which vanish on $\bar{X}_\Gamma - X_\Gamma$, i.e., the space of mixed Hilbert modular cusp forms (see **[5**, p. 40], **[2**, Section 10]).

REFERENCES

1. A. Ash, D. Mumford, M. Rapoport and Y. Tai, *Smooth compactification of locally symmetric varieties. Lie groups: history, frontiers and applications IV*, (Math. Sci. Press, 1975).

2. W. Baily and A. Borel, Compactification of arithmetic quotients of bounded symmetric domains, *Ann. of Math.* **84** (1966), 442–528.

3. P. Deligne, Variétés de Shimura: interpretation modulaire, et techniques de construction de modèles canoniques, *Proc. Sympos. Pure Math.* **33** (1979), 247–290.

4. M. Harris, Functorial properties of toroidal compactification of locally symmetric varieties, *Proc. London Math. Soc.* (3) **59** (1989), 1–22.

5. M. Harris, Automorphic forms of $\bar{\partial}$-cohomology type as coherent cohomology classes, *J. Differential Geom.* **32** (1990), 1–63.

6. B. Hunt and W. Meyer, Mixed automorphic forms and invariants of elliptic surfaces, *Math. Ann.* **271** (1985), 53–80.

7. M. Kuga, Fiber varieties over a symmetric space whose fibers are abelian varieties I, II, Lect. Notes, Univ. Chicago (1963/64).

8. M. H. Lee, Mixed cusp forms and holomorphic forms on elliptic varieties, *Pacific J. Math.* **132** (1988), 363–370.

9. M. H. Lee, Conjugates of equivariant holomorphic maps of symmetric domains, *Pacific J. Math.* **149** (1991), 127–144.

10. M. H. Lee, Mixed cusp forms and Poincaré series, *Rocky Mountain J. Math.* **23** (1993), 1009–1022.

11. M. H. Lee, Mixed Siegel modular forms and Kuga fiber varieties, *Illinois J. Math.* **38** (1994), 692–700.

12. M. H. Lee, Mixed automorphic vector bundles on Shimura varieties, *Pacific J. Math.* **173** (1996), 105–126.

13. M. H. Lee, Hodge cycles on Kuga fiber varieties, *J. Austral. Math. Soc. Ser. A* **61** (1996), 1-13.

14. Y. Manin, Non-archimedian integration and Jacquet-Langlands p-adic L-functions, *Russian Math. Surveys* **31** (1976), 5–57.

15. J. Milne, Canonical models of (mixed) Shimura varieties and automorphic vector bundles, *Automorphic forms, Shimura varieties, and L-functions, vol. I* (Academic Press, 1990), 283–414.

16. R. Pink, *Arithmetical compactification of mixed Shimura varieties*, Bonner Math. Schriften, 209 (Universität Bonn, 1990).

17. I. Satake, *Algebraic structures of symmetric domains* (Princeton Univ. Press, 1980).

18. T. Shioda, On elliptic modular surfaces, *J. Math. Soc. Japan* **24** (1972), 20–59.

Department of Mathematics
University of Northern Iowa
Cedar Falls
IA 50614, U.S.A.
E-mail address: lee@math.uni.edu