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ON A PROBLEM OF P. TURÁN

BY
P. O. H. VÉRTESI (Budapest)*

1. **Introduction.** Let us consider the well-known Hermite-Fejér interpolating process on the interval $[-1, 1]$ i.e. let

(1.1) $$-1 \leq x_{nn} < x_{n-1,n} < \cdots < x_{1n} \leq 1$$

(sometimes we omit the second indices),

(1.2) $$\omega_n(x) = \omega(x) = c \prod_{j=1}^{n} (x - x_{jn})$$

then for a function $f(x)$ defined on $[-1, 1]$, the Hermite-Fejér interpolating polynomials of degree $\leq 2n-1$ are defined by

(1.3) $$H_n(f; x) = \sum_{k=1}^{n} f(x_{kn}) h_{kn}(x) + \sum_{k=1}^{n} y'_{kn} \hbar_{kn}(x) \qquad \text{where}$$

(1.4) $$\ell_{kn}(x) = \ell_k(x) = \frac{\omega(x)}{\omega'(x_k)(x - x_k)} \qquad (k = 1, 2, \ldots, n),$$

(1.5) $$h_{kn}(x) = h_k(x) = \left[1 - \frac{\omega''(x_k)}{\omega'(x_k)}(x - x_k)\right] \ell_k^2(x) \qquad (k = 1, 2, \ldots, n),$$

(1.6) $$\hbar_{kn}(x) = \hbar_k(x) = (x - x_k)\ell_k^2(x) \qquad (k = 1, 2, \ldots, n)$$

and y'_{nk} are prescribed real numbers.

It is well known that

(1.7) $$H_n(f; x_k) = f(x_k), \; H'_n(f; x_k) = y'_k \qquad (k = 1, 2, \ldots, n).$$

For the Chebysheff nodes $\omega_n(x) = \cos n\theta = T_n(x) \quad (x = \cos\theta)$,

(1.8) $$x_{kn} = \cos\theta_{kn} = \cos\frac{2k-1}{2n}\pi, \qquad (k = 1, 2, \ldots, n),$$

according to a classical result of Fejér [1], $H_n(f; x)$ converges uniformly to $f(x)$ whenever $f(x)$ is continuous and $|y'_{k,n}| \leq M$ $(k=1, 2, \ldots, n;\; n=1, 2, \ldots)$. Moreover, if $y'_{kn} \equiv 0$, then we have (Vértesi [3])

(1.9) $$|f(x) - H_n(f; x)| = 0(1)\left[\sum_{i=1}^{n} \frac{1}{i^2}\, \omega\left(f; \frac{i(1-x^2)^{1/2}}{n}\right) + \sum_{i=1}^{n} \frac{1}{i^2}\, \omega\left(f; \frac{i^2}{n^2}\right)\right],$$

where $\omega(f; t)$ is the modulus of continuity for $f(x)$.

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On the other hand the famous Faber theorem [4] states that for all nodes-systems (1.1) there is a continuous function $f_1(x)$ on $[-1, 1]$ such that if $L_n(f; x) = \sum_{k=1}^{n} f(x_{kn})\ell_{kn}(x)$ and $\|f\| = \max_{-1\leq x\leq 1} |f(x)|$, then

$$\limsup_{n\to\infty} \|L_n(f_1; x)\| = \infty \tag{1.10}$$

The problem then arises: why do the polynomials $H_n(f; x)$ behave better regarding convergence than the Lagrange interpolation polynomials $L_n(f; x)$? Following the ideas of E. Feldheim and P. Turán it may be conjectured that the prescribing of the derivatives at the nodes regulates the behavior of the H_n process. We could then expect that if we fail to prescribe the derivative values at "few" points, we do not spoil the good convergence property of the H_n process. The facts do not prove our expectations.

In his paper [2] P. Turán proved among others the following. If we investigate the behavior of the polynomials $H^*_{r(n),n}(f; x)$ of degree $\leq 2n-2$ coinciding with $f(x)$ for $x = x_1, x_2, \ldots, x_n$ and

$$\left(\frac{dH^*_{r(n),n}(f; x)}{dx}\right)_{x=x_j} = y'_{jn} \quad \text{for} \quad 1 \leq j \leq n, \qquad j \neq r(n). \tag{1.11}$$

then

THEOREM 1.1 (P. Turán). *Choosing for each n the "exceptional" fundamental point in* (1.8) *suitably near to* $\cos \pi/5$ *say, the corresponding polynomials* $H^*_{r(n),n}(f; x)$ *with*

$$y'_{jn} = 0 \qquad j \neq r(n) \tag{1.12}$$

are uniformly bounded if only $f(x)$ is bounded in $[-1, 1]$ *but for a suitable continuous function* $f_1(x)$, $\lim_{n\to\infty} H^*_{r(n),n}(f_1; x)_{x=\cos\pi/5}$ *does not exist.*

In this connection Professor Turán raised the following problem: When does the situation take a turn for the worse, i.e., how many derivatives have to be omitted "to ensure" that for a suitable x^* and for a continuous $f_2(x)$ we shall have $\limsup_{n\to\infty} |H^{**}_n(f_2; x^*)| = \infty$?

2. **Results.** Here we shall provide an answer for the above mentioned problem. We shall denote the "exceptional" points (where we do not prescribe the derivative values) by $\eta_1 \equiv \eta_{1(n)}$ and $\eta_2 \equiv \eta_{2(n)}$ and by $H^{**}_n(f; x) \equiv H^{**}_n(\eta_{1(n)}; \eta_{2(n)}; f; x)$ the Hermite-Fejér interpolating polynomials of degree $\leq 2n-3$ on the nodes (1.8) satisfying

$$H^{**}_n(f; x_{kn}) = f(x_{kn}) \qquad (k = 1, 2, \ldots, n), \tag{2.1}$$

$$\left(\frac{dH^{**}_n(f; x)}{dx}\right)_{x=x_{kn}} = 0 \qquad (x_{kn} \neq \eta_{1(n)}, \eta_{2(n)}). \tag{2.2}$$

We shall prove the following:

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THEOREM 2.1. *For any given* $x^* \in [-1, 1]$ *there exist a continuous function* $h(x)$ *and two "exceptional" point sequences* $\{\eta_{1(n)}\}$ *and* $\{\eta_{2(n)}\}$ *such that for suitable* $n_1, n_2, \ldots$ *we have*

$$H_n^{**}(h; x^*) - h(x^*) > n \qquad (n = n_1, n_2, \ldots). \tag{2.3}$$

Moreover, if $x^* = \pm 1$, *we have*

$$H_n^{**}(h; \pm 1) - h(\pm 1) > n^3 \qquad (n = n_1, n_2, \ldots). \tag{2.4}$$

On the other hand the following holds:

THEOREM 2.2. *There exist "exceptional" point sequences* $\{\eta_{1(n)}\}$ *and* $\{\eta_{2(n)}\}$ *such that for any given continuous function* $f(x)$ *and any given subinterval* $[a, b] \subset (-1, 1)$, *we have*

$$|H_n^{**}(f; x) - f(x)| = 0(1) \sum_{i=1}^{n} \frac{1}{i^2}\, \omega\left(f; \frac{i}{n}\right) \tag{2.5}$$

uniformly in $[a, b]$.

For the case when $\omega_n(x) = \Pi_n(x)$, where

$$\Pi_n(x) = \int_{-1}^{x} P_{n-1}(t)\, dt = (x - \xi_{1n})(x - \xi_{2n}) \cdots (x - \xi_{nn}) \tag{2.6}$$

and $P_{n-1}(x)$ is the Legendre-polynomial of degree $n-1$ with $P_{n-1}(1) = 1$, we have the following results:

THEOREM 2.3. *For any* $x^* \in (-1, 1)$ *there exist a continuous function* $h(x)$ *and two "exceptional" point sequences* $\{\eta_{1(n)}\}$ *and* $\{\eta_{2(n)}\}$ *such that for suitable* $0 < n_1 < n_2 < \cdots$ *we have*

$$H_n^{**}(h; x^*) - h(x^*) > n \qquad (n = n_1, n_2, \ldots), \tag{2.7}$$

where H_n^{**} *is defined by* (2.1) *and* (2.2) *for the nodes* $\xi_{1n} < \cdots < \xi_{nn}$ *of* (2.6).

THEOREM 2.4. *There exist "exceptional" point sequences such that for any continuous function* $f(x)$ *and any fixed* $[a, b] \subset (-1, 1)$, *we have*

$$|H_n^{**}(f; x) - f(x)| = o(1)$$

uniformly in $[a, b]$.

3. **Proof of Theorem 2.1.** Following the idea of the Turán's paper (2) we consider the polynomial (1.3) with (2.1) and (2.2) i.e.

$$H_n^{**}(f; x) = \sum_{k=1}^{n} f(x_{kn}) \left[1 - \frac{\omega''(x_k)}{\omega'(x_k)}(x - x_k)\right] \frac{\omega^2(x)}{\omega'(x_k)^2 (x - x_k)^2}$$
$$+ z_1'(x - \eta_1) \frac{\omega^2(x)}{\omega'(\eta_1)^2 (x - \eta_1)^2} + z_2'(x - \eta_2) \frac{\omega^2(x)}{\omega'(\eta_2)^2 (x - \eta_2)^2}.$$

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We wish to determine the $z_1'\equiv z_{1(n)}'$ and $z_2'\equiv z_{2(n)}'$ at η_1 and η_2 so that the coefficients of x^{2n-1} and x^{2n-2} in H_n^{**} should be equal to zero. We get the following linear system of equations for z_1' and z_2'

$$(3.1)\qquad -\sum_{j=1}^{n} f(x_j)\frac{\omega''(x_j)}{\omega'(x_j)^3}+\frac{z_1'}{\omega'(\eta_1)^2}+\frac{z_2'}{\omega'(\eta_2)^2}=0$$

$$(3.2)\qquad \begin{aligned}&\sum_{j=1}^{n}\left\{f(x_j)\left[\frac{1}{\omega'(x_j)^2}+x_j\frac{\omega''(x_j)}{\omega'(x_j)^3}\right]+2f(x_j)\frac{\omega''(x_j)}{\omega'(x_j)^3}\sum_{i\neq j}x_i\right\}\\ &\qquad -\frac{z_1'\eta_1}{\omega'(\eta_1)^2}-\frac{2z_1'}{\omega'(\eta_1)^2}\sum_{x_i\neq\eta_1}x_i-\frac{z_2'\eta_2}{\omega'(\eta_2)^2}-\frac{2z_2'}{\omega'(\eta_2)^2}\sum_{x_i\neq\eta}x_i=0.\end{aligned}$$

Using that

$$(3.3)\qquad \sum_{i=1}^{n} x_{in}=0\qquad (n=1,2,\ldots),$$

we get

$$(3.4)\qquad -\sum_{j=1}^{n} f(x_j)\frac{\omega''(x_j)}{\omega'(x_j)^3}+\frac{z_1'}{\omega'(\eta_1)^2}+\frac{z_2'}{\omega'(\eta_2)^2}=0,$$

$$(3.5)\qquad \sum_{j=1}^{n} f(x_j)\left[\frac{1}{\omega'(x_j)^2}-\frac{\omega''(x_j)}{\omega'(x_j)^3}x_j\right]+\frac{\eta_1 z_1'}{\omega'(\eta_1)^2}+\frac{\eta_2 z_2'}{\omega'(\eta_2)^2}=0.$$

We also have the following well-known equalities for the nodes (1.8):

$$(3.6)\qquad \frac{1}{\omega'(x_j)^2}=\frac{1-x_j^2}{n^2},\ \frac{\omega''(x_j)}{\omega'(x_j)}=\frac{x_j}{1-x_j^2}\qquad (j=1,2,\ldots n),$$

$$(3.7)\qquad \ell_{jn}(x)=\frac{(-1)^{j+1}(1-x_j^2)^{1/2}T_n(x)}{n(x-x_j)}\qquad (j=1,2,\ldots,n).$$

From (3.4), (3.5), and (3.6) we obtain

$$(3.8)\qquad (1-\eta_1^2)z_1'=\frac{1}{\eta_2-\eta_1}\left[\eta_2\sum_{j=1}^{n}f(x_j)x_j+\sum_{j=1}^{n}f(x_j)(1-2x_j^2)\right],$$

$$(3.9)\qquad (1-\eta_2^2)z_2'=\frac{1}{\eta_1-\eta_2}\left[\eta_1\sum_{j=1}^{n}f(x_j)x_j+\sum_{j=1}^{n}f(x_j)(1-2x_j^2)\right].$$

By (1.3)–(1.6), (1.11) and (3.7)–(3.9) we get

$$(3.10)\qquad \begin{aligned}H_n^{**}(f;x)-f(x)&=\sum_{j=1}^{n}f(x_{jn})h_{jn}(x)-f(x)\\ &\quad+\frac{T_n^2(x)}{n^2(x-\eta_1)(\eta_2-\eta_1)}\left[\eta_2\sum_{j=1}^{n}f(x_j)x_j+\sum_{j=1}^{n}f(x_j)(1-2x_j^2)\right]\\ &\quad+\frac{T_n^2(x)}{n^2(x-\eta_2)(\eta_1-\eta_2)}\left[\eta_1\sum_{j=1}^{n}f(x_j)x_j+\sum_{j=1}^{n}f(x_j)(1-2x_j^2)\right].\end{aligned}$$

For any $x^* \in [-1, 1]$ we can choose a subsequence $n_1, n_2, \ldots$ such that

$$|T_n(x^*)| > c > 0 \qquad (n = n_1, n_2, \ldots) \tag{3.11}$$

(see e.g. (3.7) in [5]). Furthermore if $x^* \in (-1, 1)$ we can choose $\eta_{1(n)}$ and $\eta_{2(n)}$ so that

$$\eta_{2(n)} < x^* < \eta_{1(n)}, \eta_{1(n)}-\eta_{2(n)} \sim \frac{1}{n}, |x^*-\eta_{i(n)}| \sim \frac{1}{n} \qquad (i = 1, 2), n = n_1, n_2, \ldots, \tag{3.12}$$

$$(f_n \sim g_n \quad \text{if} \quad f_n = 0(g_n) \quad \text{and} \quad g_n = 0(f_n)).$$

Hence with $h_1(x)=1-2x^2$ we get on using (3.10), (3.11), (3.12), (1.9) and the fact that $\sum_{j=1}^n (1-2x_j^2)^2 = \sum_{j=1}^n (\cos 2\theta_j)^2 \sim n$,

$$H_n^{**}(h_1; x^*)-h_1(x^*) > \frac{c^2}{n^2} n^2 \sum_{j=1}^n (1-2x_j^2)^2 \pm c_1 \frac{\log n}{n} > c_2 n \qquad (n = n_1, n_2, \ldots)$$

Setting $h=h_1/c_2$ we get (2.3).

If $x^*=1$, we choose η_1 and η_2 so that

$$\eta_2 < \eta_1 < 1, \eta_1-\eta_2 \sim \frac{1}{n^2}, 1-\eta_i \sim \frac{1}{n^2} \qquad (i = 1, 2) \quad (n = n_1, n_2, \ldots). \tag{3.13}$$

Using that

$$\left|\frac{1}{(1-\eta_1)(\eta_2-\eta_1)}+\frac{1}{(1-\eta_2)(\eta_1-\eta_2)}\right| \sim n^4 \tag{3.14}$$

we can deduce (2.4). A similar argument holds if $x^*=-1$.

Proof of Theorem 2.2. Set $\eta_1=x_1$, $\eta_2=x_n$. Then for any $x \in [a, b]$ we have

$$\left|\frac{T_n^2(x)}{n^2(x-\eta_1)(\eta_1-\eta_2)}\left[\eta_2 \sum_{j=1}^n f(x_j)x_j + \sum_{j=1}^n f(x_j)(1-2x_j^2)\right]\right| \le \frac{c^2}{n^2} c_1 n \le \frac{c_2}{n}.$$

Using (3.10) and (1.9) we get (2.5).

Proof of Theorems 2.3 and 2.4. In this case we have

$$\Pi_n'(1) = 1, \Pi_n'(-1) = (-1)^{n-1}, \Pi_n'(\xi_r) = P_{n-1}(\xi_r) \qquad (r = 2, 3, \ldots, n-1) \tag{3.15}$$

$$\Pi_n''(1) = \frac{n(n-1)}{n}, \Pi_n''(-1) = (-1)^n \frac{n(n-1)}{2}, \Pi_n''(\xi_r) = 0 \qquad (r = 2, 3, \ldots, n-1) \tag{3.16}$$

and if $x \in [a, b] \subset (-1, 1)$,

$$\Pi_n(\cos\theta) = B(\sin\theta)^{1/2} n^{-3/2} \cos\left\{(n-\tfrac{1}{2})\theta+\frac{3\pi}{4}\right\}+0\left(\frac{1}{n^2}\right) \qquad \text{(see [2]).} \tag{3.17}$$

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Using (3.3), (3.4) (3.15) and (3.16) we get

$$\frac{z_1'}{\Pi'(\eta_1)^2}+\frac{z_2'}{\Pi'(\eta_2)^2}=\frac{n(n-1)}{2}\,[f(1)-f(-1)],$$

$$\frac{\eta_1 z_1'}{\Pi'(\eta_1)^2}+\frac{\eta_2 z_2'}{\Pi'(\eta_2)^2}=\frac{n(n-1)}{2}\,[f(1)+f(-1)]-[f(1)+(-1)^{n-1}f(-1)]-\sum_{j\neq 1,n}\frac{f(x_j)}{\Pi'^2(x_j)}.$$

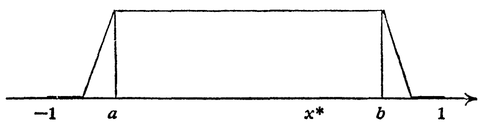
Solving for z_1', say

$$\frac{z_1'}{\Pi'(\eta_1)^2}=\frac{n(n-1)}{2(\eta_1-\eta_2)}\,[f(1)+f(-1)]-\frac{\eta_2 n(n-1)}{2(\eta_1-\eta_2)}\,[f(1)-f(-1)]$$

$$-\frac{1}{\eta_1-\eta_2}\,[f(1)+(-1)^{n-1}f(-1)]-\frac{1}{\eta_1-\eta_2}\sum_{j\neq 1,n}\frac{f(x_j)}{\Pi'^2(x_j)}.$$

If $x^*\in[a,b]\subset(-1,1)$ then by (3.12), (3.17) we get (2.7) for $h(x)$ defined by this figure

$h(x)$



(we used (3.15) and $P_{n-1}(\xi_r)\sim n^{-1/2}$ if $\xi_r\in[a,b]\subset(-1,1)$ ([6], (7.32.6).) We omit the details.

For the convergence theorem we let $\eta_1=\xi_2$, $\eta_2=\xi_{n-1}$. We can obtain (2.8), using that $H_n(f;x)\to f(x)$ uniformly on $[-1,1]$.

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Department of Mathematics
University of Alberta
Edmonton, Alberta
Canada T6G 2G1.

Mathematical Institute of the Hungarian
Academy of Sciences
1053 Budapest, Rea'ltanoda U. 13–15
Hungary