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A diagonal dominance criterion for exponential dichotomy

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Roughly speaking, a system of linear differential equations has an exponential dichotomy if it has a subspace of solutions shrinking exponentially and a complementary subspace of solutions growing exponentially. In the case of constant coefficients, this happens if and only if the eigenvalues of the coefficient matrix have nonzero real parts. In the general case, Lazer has shown that if the coefficient matrix function is bounded and satisfies a diagonal dominance condition (which, in the constant case, is a sufficient but not necessary condition that the eigenvalues have nonzero real parts) then the system has an exponential dichotomy. In this paper we prove the same result with a weaker diagonal dominance condition, thus generalizing a theorem of Nakajima.

1. Statement of the theorem

We consider a system of linear differential equations,

$$x' = A(t)x \ , \tag{1}$$

where $A(t) = [a_{ij}(t)]$ is a real $n \times n$ matrix function defined and continuous on $(-\infty, \infty)$. (1) is said to have an exponential dichotomy if it has a fundamental matrix $X(t)$ satisfying the inequalities,

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$$|X(t)PX^{-1}(s)| \le Ke^{-\alpha(t-s)} \quad (s \le t) ,$$

$$|X(t)(I-P)X^{-1}(s)| \ge Ke^{-\alpha(s-t)} \quad (s \le t) ,$$

where $|\cdot|$ denotes some matrix norm, P is a projection $(P^2 = P)$, and $K > 0$, $\alpha > 0$ are constants. ($|\cdot|$ denotes modulus when the argument is a scalar and denotes the norm $\sup_{i=1}^{n} |x_i|$ when x is a vector with components $x_1, x_2, \ldots, x_n$.)

$A(t)$ is said to be row dominant if there exists $\delta > 0$ such that

$$(2) \qquad |a_{ii}(t)| \ge \sum_{j=1}^{n}{}_{(j\ne i)} |a_{ij}(t)| + \delta$$

for $i = 1, 2, \ldots, n$ and all t and column dominant if

$$(3) \qquad |a_{ii}(t)| \ge \sum_{j=1}^{n}{}_{(j\ne i)} |a_{ji}(t)| + \delta .$$

Note that either (2) or (3) implies that $|\det A(t)| \ge \delta^n$ (see [4, p. 16]). It is a consequence of a result of Lazer [2] that if $A(t)$ is row or column dominant and bounded, then (1) has an exponential dichotomy.

We say that $A(t)$ is weakly row (column) dominant if $A(t)$ satisfies (2) (respectively (3)) with $\delta = 0$. In [3] Nakajima has proved that if

(i) $A(t)$ is bounded,

(ii) $\inf_{-\infty<t<\infty} |\det A(t)| > 0$,

(iii) $A(t)$ is weakly column dominant, and

(iv) $a_{ii}(t) \le 0$ for all t and $i = 1, 2, \ldots, n$,

then (1) has an exponential dichotomy with $P = I$. In this paper we prove the following theorem.

THEOREM. *If* (i), (ii), *and* (iii) *hold or if* (i), (ii), *and*

(iii)' $A(t)$ *is weakly row dominant*

hold, then (1) *has an exponential dichotomy.*

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2. Proof of the theorem

We only consider the row dominant case because the other one can be deduced from it by the method used in the proof of Corollary 2 in Berkey [1].

We, firstly, note a result which follows easily from Lemma 2 in [3].

LEMMA 1. *Let* $A = [a_{ij}]$ *be a real nonsingular* $n \times n$ *matrix such that*

$$(4) \qquad |a_{ii}| \geq \sum_{j=1 (j\neq i)}^{n} |a_{ij}| \quad \text{for} \quad i = 1, 2, \ldots, n .$$

Then all principal minors of A *are nonzero; that is,*

$$\det[a_{k_ik_j}] = \det\begin{bmatrix} a_{k_1k_1} & \cdots & a_{k_1k_p} \\ \vdots & & \vdots \\ a_{k_pk_1} & \cdots & a_{k_pk_p} \end{bmatrix} \neq 0$$

for $1 \leq k_1 < k_2 < \ldots < k_p \leq n$.

Let $A(t)$ satisfy (i), (ii), and (iii)'. Then Lemma 1 implies that $a_{ii}(t) \neq 0$ for all i and t . We define e_i as 1 if $a_{ii}(t) > 0$ and as -1 if $a_{ii}(t) < 0$.

Suppose for some $k_1, k_2, \ldots, k_p$ such that $1 \leq k_1 < k_2 < \ldots < k_p \leq n$,

$$\sum_{i=1}^{p} e_{k_i}\left\{\sum_{j=1}^{p} a_{k_ik_j}(t)\right\} = 0 .$$

Then since

$$e_{k_i}\left\{\sum_{j=1}^{p} a_{k_ik_j}(t)\right\} \geq |a_{k_ik_i}(t)| - \sum_{j=1 (j\neq i)}^{p} |a_{k_ik_j}(t)| \geq 0 ,$$

$$\sum_{j=1}^{p} a_{k_ik_j}(t) = 0 \quad \text{for} \quad i = 1, 2, \ldots, n .$$

This implies $\det[a_{k_ik_j}(t)] = 0$, contradicting Lemma 1. So we must have

$$\sum_{i=1}^{p} e_{k_i}\left\{\sum_{j=1}^{p} a_{k_ik_j}(t)\right\} > 0 \quad \text{for all } t .$$

Further, suppose there exists a sequence t_m such that

$$\sum_{i=1}^{p} e_{k_i}\left\{\sum_{j=1}^{p} a_{k_ik_j}(t_m)\right\} \to 0 \quad \text{as } m \to \infty .$$

By taking a subsequence if necessary, we may assume that $A(t_m) \to A = [a_{ij}]$. Then in A the principal minor $\det[a_{k_ik_j}] = 0$. This contradicts Lemma 1 since A must be nonsingular in view of (ii) and certainly satisfies (4).

So we have shown the following.

COROLLARY 1. *If* $A(t) = [a_{ij}(t)]$ *satisfies* (i), (ii), *and* (iii)' *then* $a_{ii}(t) \neq 0$ *for all* i *and* t *and there exists* $\Delta > 0$ *such that if* $1 \le k_1 < k_2 < \dots < k_p \le n$,

$$\sum_{i=1}^{p} e_{k_i}\left\{\sum_{j=1}^{p} a_{k_ik_j}(t)\right\} \ge \Delta \quad \textit{for all } t ,$$

where e_i *is* 1 *if* $a_{ii}(t) > 0$ *and is* -1 *if* $a_{ii}(t) < 0$.

After these preliminaries our first aim is to show that (1) has a subspace of solutions whose norms are strictly decreasing. We begin with the following

LEMMA 2. *Let* $x(t)$ *be a nontrivial solution of* (1). *Then for all real* t_0 *there exists* $\varepsilon > 0$ *so that* $|x(t)|$ *is either strictly decreasing in* $(t_0-\varepsilon, t_0]$ *or strictly increasing in* $[t_0, t_0+\varepsilon)$.

Proof. Let

$$I = \{i : |x(t_0)| = |x_i(t_0)|\} ,$$

$$I_1 = \{i : i \in I, a_{ii}(t) < 0\} ,$$

and

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$$I_2 = \{i : i \in I, a_{ii}(t) > 0\} .$$

If i is in I_1 , then

$$\frac{d}{dt} |x_i(t)| \Big|_{t=t_0}$$

$$(5) \qquad = |x_i(t_0)|^{-1}\left\{a_{ii}(t_0)|x_i(t_0)|^2 + \sum_{j=1 (j\neq i)}^{n} a_{ij}(t_0)x_i(t_0)x_j(t_0)\right\}$$

$$\leq a_{ii}(t_0)|x_i(t_0)| + \sum_{j=1 (j\neq i)}^{n} |a_{ij}(t_0)||x_j(t_0)|$$

$$(6) \qquad \leq \left\{a_{ii}(t_0) + \sum_{j=1 (j\neq i)}^{n} |a_{ij}(t_0)|\right\}|x_i(t_0)|$$

$$\leq 0 .$$

Similarly, if i is in I_2 , then

$$\frac{d}{dt} |x_i(t)| \Big|_{t=t_0} \geq 0 .$$

Now we define

$$I_3 = \left\{i : i \in I_1, \frac{d}{dt} |x_i(t)| \Big|_{t=t_0} < 0\right\}$$

and

$$I_4 = \left\{i : i \in I_2, \frac{d}{dt} |x_i(t)| \Big|_{t=t_0} > 0\right\} .$$

Suppose I_3 is nonempty. Then it is easy to see that if $t \leq t_0$ and sufficiently near t_0 ,

$$|x(t)| = \sup_{i\in I_3} |x_i(t)| .$$

But each of the $|x_i(t)|$ is a strictly decreasing function for $t \leq t_0$ and near t_0 and so $|x(t)|$ is also. Similarly, if I_4 is non-empty then we show that $|x(t)|$ is strictly increasing for $t \geq t_0$ and near

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t_0 .

Suppose now that I_3 and I_4 are empty. Then

$$\frac{d}{dt}|x_i(t)|\Big|_{t=t_0} = 0 \text{ for all } i \text{ in } I .$$

This implies that $a_{ij}(t_0) = 0$ when i is in I and j is not in I for otherwise we get strict inequality in (6) (or in the analogous inequality when i is in I_2). Fix a k in I . Then we can write $x_i(t_0) = \delta_i x_k(t_0)$, where $|\delta_i| = 1$, when i is in I and, from (5), we then have

$$\sum_{j\in I} a_{ij}(t_0)\delta_j = 0 \text{ for all } i \text{ in } I .$$

This means that the determinant of the matrix, formed by the elements $a_{ij}(t_0)$ in $A(t_0)$ both of whose indices belong to I , is zero, contradicting Lemma 1. Thus I_3 and I_4 cannot both be empty and the proof of the lemma is complete.

COROLLARY 2. *If* $x(t)$ *is a nontrivial solution of* (1), *then for all* t_0 , $|x(t)|$ *is either strictly decreasing on* $(-\infty, t_0]$ *or strictly increasing on* $[t_0, \infty)$.

Proof. By the lemma there exists $\varepsilon > 0$ such that $|x(t)|$ is strictly decreasing on $(t_0-\varepsilon, t_0]$ or strictly increasing on $[t_0, t_0+\varepsilon)$. Suppose the first possibility holds. Let t_1 be the least number less than t_0 such that $|x(t)|$ is strictly decreasing on $(t_1, t_0]$. Suppose $t_1 > -\infty$. Then at t_1 we can apply the lemma to deduce that there exists $\varepsilon_1 > 0$ such that $|x(t)|$ is strictly decreasing on $(t_1-\varepsilon_1, t_1]$ and hence on $(t_1-\varepsilon_1, t_0]$. This contradicts the definition of t_1 . So t_1 must be $-\infty$. Similarly, if the second possibility holds we deduce that $|x(t)|$ is strictly increasing on $[t_0, \infty)$. This completes the proof of the corollary.

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In real euclidean n-space R^n define the subspace,

$$S = \left\{x : x \in R^n,\ x_i = 0 \text{ if } a_{ii}(t) > 0\right\} .$$

Then the dimension of S is q, where q is the number of i's such that $a_{ii}(t) < 0$. Let $x(t)$ be a solution of (1) with $x(t_0) \neq 0$ and $x(t_0)$ in S. Then in the proof of Lemma 2, I_2 is empty and so I_3 must be nonempty. Hence $|x(t)|$ must be strictly decreasing on $(t_0-\varepsilon, t_0]$ and so, as in the proof of Corollary 2, on $(-\infty, t_0]$.

Now let $X(t)$ be the fundamental matrix of (1) with $X(0) = I$. For any positive integer m, if $x(t)$ is a nontrivial solution of (1) with $x(0)$ in $X^{-1}(m)S$, then $|x(t)|$ is strictly decreasing on $(-\infty, m]$. For each m choose an orthonormal basis $h_{1m}, h_{2m}, \ldots, h_{qm}$ for $X^{-1}(m)S$. Then there is a subsequence $h_{ij_m} \to h_i$ as $m \to \infty$ and $h_1, h_2, \ldots, h_q$ will be an orthonormal basis for a subspace V of dimension q. If $x(t)$ is a nontrivial solution of (1) with $x(0)$ in V, then $|x(t)|$ must be nonincreasing on $(-\infty, \infty)$, since $x(t)$ is the pointwise limit as $m \to \infty$ of solutions whose norms are strictly decreasing on $(-\infty, j_m]$, and hence strictly decreasing by Corollary 2.

Now we want to prove that if $x(t)$ is a solution of (1) with $x(0)$ in V, then there exist $K > 0$ and $\alpha > 0$ independent of $x(0)$ and s such that

$$(7) \qquad |x(t)| \le Ke^{-\alpha(t-s)}|x(s)| \quad \text{for } s \le t .$$

All we need show is that if $x(t)$ is a solution with $x(0)$ in V and $|x(s)| = 1$, then there exists $T > 0$ (independent of $x(0)$ and s) such that $|x(s+T)| < \frac{1}{2}$, for then we may take $\alpha = T^{-1} \log 2$ and $K = 2$. If this is not true there exist a sequence t_m and a sequence of solutions $x(t, m)$ of (1) with $x(0, m)$ in V and $|x(t_m, m)| = 1$, but $1 \ge |x(t_m+m, m)| \ge \frac{1}{2}$. Since $|x(t, m)|$ is strictly decreasing, $[t_m, t_m+m]$ contains a subinterval $[s_m, s_m+1]$ such that

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$$0 \le |x(t_1, m)| - |x(t_2, m)| < m^{-1} \quad \text{if} \quad s_m \le t_1 \le t_2 \le s_m+1 .$$

Put $\phi(t, m) = x(s_m+t, m)$. Then for $0 \le t \le 1$,

$$\frac{1}{2} \le |\phi(t, m)| \le 1 , \tag{8}$$

$$\phi'(t, m) = A(s_m+t)\phi(t, m) ,$$

and if $0 \le t_1 \le t_2 \le 1$,

$$0 \le |\phi(t_1, m)| - |\phi(t_2, m)| < m^{-1} . \tag{9}$$

Hence the set of functions $\phi(t, m)$ is uniformly bounded and equicontinuous on $[0, 1]$ and so we can find a subsequence (for which we use the same notation) $\phi(t, m) \to y(t)$ uniformly on $[0, 1]$. Because of (8) and (9), $|y(t)|$ is a constant β with $\frac{1}{2} \le \beta \le 1$.

We prove that $|y_i(t)| = \beta$ on $[0, 1]$ for $i = 1, 2, \ldots, n$. Firstly, suppose there exists t_0 in $[0, 1]$ such that $|y_i(t_0)| = \beta$ but $|y_j(t_0)| < \beta$ for $j \ne i$. For definiteness, we assume that $i = 1$. By continuity, there is a $\delta > 0$ and an interval J containing t_0 such that for t in J , $|y_1(t)| = \beta$ and $|y_i(t)| \le \beta - 2\delta$ for $i \ne 1$. Then if t is in J and m is sufficiently large, $|\phi_1(t, m)| \ge \beta/2$ and $|\phi_i(t, m)| \le |\phi_1(t, m)| - \delta$ for $i \ne 1$. So, putting $f_1 = 1$ if $y_1(t_0) = \beta$ and equal to -1 if $y_1(t_0) = -\beta$, and using Corollary 1,

$$\begin{aligned}
e_1 f_1 \phi_1'(t, m) \\
&= e_1 f_1 a_{11}(s_m+t)\phi_1(t, m) + \sum_{j=2}^{n} e_1 f_1 a_{1j}(s_m+t)\phi_j(t, m) \\
&\ge e_1 a_{11}(s_m+t)|\phi_1(t, m)| - \sum_{j=2}^{n} |a_{1j}(s_m+t)||\phi_j(t, m)| \\
&\ge \left(e_1 a_{11}(s_m+t) - \sum_{j=2}^{n} |a_{1j}(s_m+t)|\right)|\phi_1(t, m)| + \left(\sum_{j=2}^{n} |a_{1j}(s_m+t)|\right)\delta \\
&\qquad \text{if } t \text{ is in } J \text{ and } m \text{ is sufficiently large} \\
&\ge e_1 a_{11}(s_m+t)\varepsilon \quad \text{where } \varepsilon = \min\{\beta/2, \delta\} > 0 \\
&\ge \Delta\varepsilon .
\end{aligned}$$

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Thus if $t_2 > t_1$ are in J and m is sufficiently large,

$$e_1f_1(\phi_1(t_1, m)-\phi_1(t_2, m)) \geq \Delta\varepsilon(t_2-t_1) .$$

Letting $m \to \infty$,

$$0 = e_1f_1(y_1(t_1)-y_1(t_2)) \geq \Delta\varepsilon(t_2-t_1) > 0 .$$

This is a contradiction, and so for all t in $[0, 1]$ there are at least two i's for which $|y_i(t)| = \beta$.

Suppose now there exists t_0 in $[0, 1]$ such that $|y_i(t_0)| = |y_j(t_0)| = \beta$, where $i \neq j$, but $|y_k(t_0)| < \beta$ if k is different from i and j . We suppose for definiteness that $i = 1$ and $j = 2$. By continuity, there is a $\delta > 0$ and an interval J containing t_0 such that for t in J , $\sup\{y_1(t), y_2(t)\} = \beta$ and $|y_i(t)| \leq \beta - 2\delta$ if $i \neq 1, 2$. However, because of what we have just proved, $|y_1(t)| = |y_2(t)| = \beta$ for all t in J . Then if t is in J and m is sufficiently large, $|\phi_1(t, m)| \geq \beta/2$ and $|\phi_i(t, m)| \leq |\phi_1(t, m)| - \delta$ for $i \neq 1, 2$. So, putting $f_2 = 1$ if $y_2(t_0) = \beta$ and equals -1 if $y_2(t_0) = -\beta$,

$$\begin{aligned}
&e_1f_1\phi_1'(t, m) + e_2f_2\phi_2'(t, m) \\
&= e_1f_1 \sum_{j=1}^{n} a_{1j}(s_m+t)\phi_j(t, m) + e_2f_2 \sum_{j=1}^{n} a_{2j}(s_m+t)\phi_j(t, m) \\
&\geq \Big\{e_1a_{11}(s_m+t)|\phi_1(t, m)|+e_1f_1f_2a_{12}(s_m+t)|\phi_2(t, m)| \\
&\qquad - \sum_{j=3}^{n} |a_{1j}(s_m+t)||\phi_j(t, m)|\Big\} + \Big\{e_2f_1f_2a_{21}(s_m+t)|\phi_1(t, m)| \\
&\qquad + e_2a_{22}(s_m+t)|\phi_2(t, m)| - \sum_{j=3}^{n} |a_{2j}(s_m+t)||\phi_j(t, m)|\Big\} \\
&\geq \Big\{e_1(a_{11}(s_m+t)+f_1f_2a_{12}(s_m+t))+e_2(f_1f_2a_{21}(s_m+t)+a_{22}(s_m+t)) \\
&\quad - \sum_{j=3}^{n} (|a_{1j}(s_m+t)|+|a_{2j}(s_m+t)|)\Big\}|\phi_1(t, m)| + \sum_{j=3}^{n} (|a_{1j}(s_m+t)|+|a_{2j}(s_m+t)|)\delta \\
&\qquad + (e_1f_1f_2a_{12}(s_m+t)+e_2a_{22}(s_m+t))(|\phi_2(t, m)|-|\phi_1(t, m)|)
\end{aligned}$$

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if t is in J and m is sufficiently large.

Applying Corollary 1 to the matrix function obtained from $A(t)$ by multiplying the first row and column by f_1 and the second row and column by f_2, we see that the sum of the first and second terms in the last expression is greater than or equal to $\Delta\varepsilon$, where $\varepsilon = \min\{\beta/2, \delta\}$. The modulus of the third term is less than or equal to $\Delta\varepsilon/2$ if m is large enough and so we deduce that

$$e_1 f_1 \phi_1'(t, m) + e_2 f_2 \phi_2'(t, m) \geq \Delta\varepsilon/2$$

if t is in J and m is large enough. Then, proceeding as before, we get a contradiction.

So for all t in $[0, 1]$ there are at least three i's for which $|y_i(t)| = \beta$. After $(n-3)$ similar arguments we finally reach the conclusion that $|y_i(t)| = \beta$ for all t in $[0, 1]$ and $i = 1, 2, \ldots, n$.

Now, putting $f_i = 1$ if $y_i(t) = \beta$ and equal to -1 if $y_i(t) = -\beta$,

$$\begin{aligned}\sum_{i=1}^{n} e_i f_i \phi_i'(t, m) &= \sum_{i=1}^{n} e_i f_i \left(\sum_{j=1}^{n} a_{ij}(s_m+t)\phi_j(t, m)\right) \\ &= \left(\sum_{i=1}^{n} e_i \left(\sum_{j=1}^{n} f_i f_j a_{ij}(s_m+t)\right)\right)|\phi_1(t, m)| \\ &\quad + \sum_{i=1}^{n} e_i f_i \left(\sum_{j=1}^{n} f_j a_{ij}(s_m+t)\left(|\phi_j(t, m)| - |\phi_1(t, m)|\right)\right).\end{aligned}$$

Applying Corollary 1 to the matrix function $[f_i f_j a_{ij}(t)]$, we see that the first term is greater than or equal to $\Delta\beta/2$, if m is so large that $|\phi_1(t, m)| \geq \beta/2$. The second term converges uniformly to 0 on $[0, 1]$ and so its modulus is less than or equal to $\Delta\beta/4$ if m is large. Thus, for all t in $[0, 1]$,

$$\sum_{i=1}^{n} e_i f_i \phi_i'(t, m) \geq \Delta\beta/4$$

if m is large enough. Then, proceeding as before, we get a contradiction. So (7) must hold.

Similarly, there exists a subspace W of dimension $(n-q)$ such that if $x(t)$ is a nontrivial solution of (1) with $x(0)$ in W then $|x(t)|$ is strictly increasing on $(-\infty, \infty)$ and there exist $K > 0$ and $\alpha > 0$ such that

$$|x(t)| \le Ke^{-\alpha(s-t)}|x(s)| \quad \text{for} \quad s \ge t \; . \tag{10}$$

Clearly $W \cap V = \{0\}$. Let P be the projection with kernel W and range V . Then, as in Berkey [1], we can show as a consequence of (7), (10), and the boundedness of $A(t)$ that (1) has an exponential dichotomy with projection P .

REMARK. It may be thought that if $A(t)$ is complex and satisfies

(i) $A(t)$ is bounded,

(ii) $\inf\{|\det[A(t)-i\beta I]| : -\infty < t < \infty, -\infty < \beta < \infty\} > 0$, and

(iii) $|\text{re } a_{ii}(t)| \ge \sum_{j=1}^{n} {}_{(j\ne i)} |a_{ji}(t)|$ for all t and i , or

(iii)' $|\text{re } a_{ii}(t)| \ge \sum_{j=1}^{n} {}_{(j\ne i)} |a_{ij}(t)|$ for all t and i ,

then (1) has an exponential dichotomy. This is certainly true when $A(t)$ is constant but does not hold in general. This we see from the equation

$$x_1' = (-1-i)x_1 + e^{-it}x_2 \; ,$$

$$x_2' = e^{it}x_1 - x_2 \; ,$$

which satisfies the above conditions but has the nontrivial bounded solution $x_1(t) = e^{-it}$, $x_2(t) = 1$.

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