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ON A COMBINATORIAL PROOF FOR AN IDENTITY INVOLVING THE TRIANGULAR NUMBERS

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Abstract

In this paper, we present a combinatorial proof for an identity involving the triangular numbers. The proof resembles Franklin's proof of Euler's pentagonal number theorem.

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1. Introduction

In its analytical version Euler's pentagonal number theorem can be stated as follows.

THEOREM 1.1 (Euler's pentagonal number theorem).

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty,$$

where $(q;q)_\infty = (1-q)(1-q^2)(1-q^3)\cdots$.

This theorem has a beautiful combinatorial interpretation due to Legendre given below.

THEOREM 1.2 (Combinatorial version of Euler's pentagonal number theorem). *Let $D_e(n)$ denote the number of partitions of n into an even number of distinct parts, and let $D_o(n)$ denote the number of partitions of n into an odd number of distinct parts. Then*

$$D_e(n) - D_o(n) = \begin{cases} (-1)^j & \text{if } n = j(3j \pm 1)/2, \\ 0 & \text{otherwise.} \end{cases}$$

For this version of Euler's theorem there is a well-known combinatorial proof given by Franklin in 1881 (see [2]). In this paper we present a proof of the following identity that resembles Franklin's proof.

THEOREM 1.3. *For any complex number* $|q| < 1$,

$$1 + q + \sum_{n=1}^{\infty} \frac{(1-q^2)(1-q^4)\cdots(1-q^{2n})}{(1-q^3)(1-q^5)\cdots(1-q^{2n+1})} q^{2n+1} = \sum_{n=0}^{\infty} q^{n(n+1)/2}. \tag{1.1}$$

2. The bijective proof

The right-hand side of (1.1) is Ramanujan's partial theta function $\psi(q)$. This identity is already known (see [1, 2]). What we present here is a combinatorial proof for it. A different proof is presented in [3].

We know that the sum on the left-hand side of (1.1) generates partitions into parts greater than 1 where the even parts are distinct, the largest part is odd, and having weight $(-1)^m$, where m is the number of even parts. We also know that the exponents of q on the right-hand side of (1.1) are the triangular numbers.

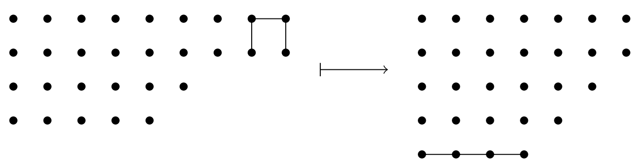
The identity (1.1) can be rewritten as

$$\sum_{n=1}^{\infty} \frac{(1-q^2)(1-q^4)\cdots(1-q^{2n})}{(1-q^3)(1-q^5)\cdots(1-q^{2n+1})} q^{2n+1} = \sum_{n=2}^{\infty} q^{n(n+1)/2}. \tag{2.1}$$

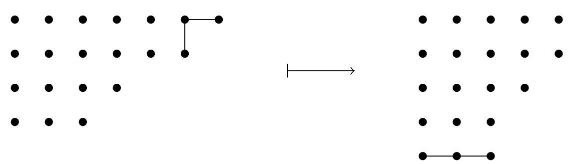
In order to prove the identity (2.1) we have to show that the coefficient of q^n on the left-hand side is either 1 if n is a triangular number or 0 otherwise. In other words, if $p_e(n)$ ($p_o(n)$) denotes the number of partitions of n generated by the left-hand side of (2.1) having an even (odd) number of even parts, then we have to show that

$$p_e(n) - p_o(n) = \begin{cases} 1 & \text{if } n \text{ is a triangular number,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

PROOF OF THEOREM 1.3. On the set $\mathcal{P}(n)$ of partitions of n into parts greater than 1 where the even parts are distinct and the largest part is odd we define the following map. Given a partition $\lambda = \lambda_1 + \cdots + \lambda_s \in \mathcal{P}(n)$ we look at the Ferrers graph of λ. From this diagram we remove the last two columns to form a new part λ_{s+1}, being left with a partition $\lambda' = \lambda'_1 + \cdots + \lambda'_s$ of $n - \lambda_{s+1}$. We insert this new part below the smallest part of λ' when either $\lambda_{s+1} \le \lambda'_s$ and λ'_s is odd or $\lambda_{s+1} < \lambda'_s$ and λ'_s is even, obtaining the partition $\lambda'' = \lambda'_1 + \cdots + \lambda'_s + \lambda_{s+1} \in \mathcal{P}(n)$. For example:



and



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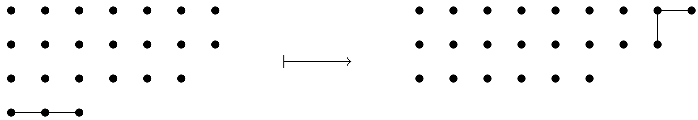
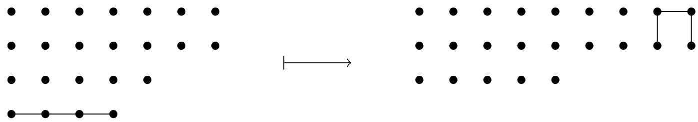
But if $\lambda_{s+1} > \lambda'_s$ or $\lambda_{s+1} = \lambda'_s$ and λ'_s is even, we cannot do this operation in order to obtain a partition in $\mathcal{P}(n)$. In this case, we remove the smallest part λ_s to form two new columns that are equal (when λ_s is even) or differ by 1 (when λ_s is odd). We add these columns to the Ferrers graph of the partition $\lambda_1 + \cdots + \lambda_{s-1}$. It is easy to verify that the partition obtained in this way is in $\mathcal{P}(n)$. For example:



and



We observe now that when n is a triangular number, $n = k(k+1)/2$, this procedure does not work for that partition in $\mathcal{P}(n)$ having $k/2$ parts $k+1$, when k is even, or $(k+1)/2$ parts k, if k is odd. In fact:

- if we remove the last two columns from the partition having $k/2$ parts $k+1$, when k is even, the new part will have length $2(k/2) = k$ which cannot be put below the smallest part $k+1-2 = k-1$. Also, by removing the smallest part of length $k+1$, we cannot put the new two columns (one having length $(k/2)+1$ and the other having length $k/2$) in front of the last column (that has length $(k/2)-1$) of the partition with the smallest part removed.
- if we remove the last two columns from the partition having $(k+1)/2$ parts k, when k is odd, the new part will have length $2(k+1)/2 = k+1$ which cannot be put below the new smallest part $k-2$. Also, by removing the smallest part of length k, we cannot put the new two columns (the largest one having length $(k+1)/2$) in front of the last column (that has length $(k-1)/2$) of the partition with the smallest part removed.

For example, it is not possible to apply any of the operations described above in the partitions of 15 $(k = 5)$ and 21 $(k = 6)$ below.

To finish the proof, we have to show that the operations on the Ferrers graphs of the partitions in $\mathcal{P}(n)$ described above change the parity of the number of even parts. By doing this we will have shown that (2.2) holds.

For those partitions in $\mathcal{P}(n)$ for which we can remove the two last columns of the Ferrers graphs and form new smallest parts, there are two possibilities: if the columns have the same length, we do not modify the number of odd parts and the new part has length even, which modifies the parity of the number of even parts; if the columns differ by one we lose one even part and obtain an odd part as the new part, modifying the parity of the number of even parts. These arguments can be easily reversed in order to show that the inverse operation also modifies the parity of the number of even parts. □

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