A note on convergence factors

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1. In this note $\sum_{\nu=1}^{\infty} u_\nu$ denotes a divergent series of positive, decreasing terms for which $\lim_{n\to\infty} u_n = 0$. $e_1, e_2, \ldots$ are real numbers (convergence factors) such that $\sum_{\nu=1}^{\infty} e_\nu u_\nu$ is convergent. We put

$$t_n = \sum_{\nu=1}^{n} e_\nu, \qquad \sigma_n = \frac{1}{n} t_n.$$

H. Rademacher[1] has shown that

$$\underline{\lim}\, \sigma_n \leqq 0 \leqq \overline{\lim}\, \sigma_n.$$

He also proved

Theorem A. If $\underline{\lim}\, n u_n > 0$, *then for all sequences* e_ν *for which* $\Sigma\, e_\nu u_\nu$ *is convergent we must have* $\lim \sigma_n = 0$.

We shall now add

Theorem 1. *If* $\underline{\lim}\, n u_n = 0$, *we can find a sequence* e_ν *for which* $\Sigma\, e_\nu u_\nu$ *is convergent and* $\overline{\lim}\, \sigma_n > 0$. *This is possible, even if the* e_ν *may take only the values plus one or minus one.*

2. Proof of Theorem 1. We write

$$u_n = n^{-1}\, a(n).$$

Since $\underline{\lim}\, n u_n = 0$, we have $\underline{\lim}\, a(n) = 0$. It is therefore possible to select a subsequence $a(n_1), a(n_2), \ldots$ which tends to zero rapidly enough to ensure the convergence of the series

$$\sum_{k=1}^{\infty} a(n_k). \tag{1}$$

We may also assume that the conditions

$$k = o(n_k), \qquad n_{k+1} \geqq 2n_k$$

are satisfied. If this is not the case to start with, we need only omit a sufficient number of terms from (1) and renumber the remaining terms.

[1] Math. Zeitschrift, 11 (1921), 276.

We now choose $e_\nu = +1$, if

$$n_k \leqq \nu < 2n_k, \qquad (k = 1, 2 \ldots), \tag{2}$$

$e_\nu = (-1)^\nu$, if ν is not in one of the intervals (2).

A section $\sum_{\nu=m}^{N} e_\nu u_\nu$ of the series will in general consist of sums of alternating terms separated by sums of positive terms arising from values of ν given by (2). Since the u_ν are decreasing, the sum of a stretch of consecutive alternating terms will be less in absolute magnitude than its first term, so that the contribution of such sums to the value of $\sum_{\nu=m}^{N} e_\nu u_\nu$ is less than

$$u_m + \sum_{n_k \geq m} u_{2n_k} < u_m + \sum_{n_k \geq m} u_{n_k} < u_m + \sum_{n_k \geq m} a(n_k).$$

The contribution of one of the intervals (2) to the sum is

$$\sum_{n_k \leqq \nu < 2n_k} u_\nu < n_k u_{n_k} = a(n_k),$$

so that

$$\sum_{\nu=m}^{N} e_\nu u_\nu = O\left(u_m + \sum_{n_k \geq m} a(n_k)\right) = o(1),$$

as m tends to infinity. Therefore $\sum_{\nu=1}^{\infty} e_\nu u_\nu$ is convergent. But

$$\sigma_{2n_k} \geqq \frac{n_k - k - 1}{2n_k},$$

and therefore

$$\overline{\lim}\, \sigma_n \geqq \tfrac{1}{2}.$$

3. It is, of course, possible to ensure the existence of $\lim\limits_{n \to \infty} \sigma_n$ by imposing conditions on $\sum_{\nu=1}^{\infty} u_\nu$ and $\sum_{\nu=1}^{\infty} e_\nu u_\nu$. We prove in this direction

Theorem 2. If (i) $\left|\sum_{\nu=n}^{\infty} e_\nu u_\nu\right| < K u_n$ *for some* $K > 0$ $(n = 1, 2 \ldots)$

and if (ii) $\lim\limits_{n \to \infty} (u_{n+1}/u_n) = 1$

then $\lim\limits_{n \to \infty} \sigma_n = 0.$

Proof of Theorem 2. By condition (i)

$$|e_n u_n| = \left|\sum_{\nu=n}^{\infty} e_\nu u_\nu - \sum_{\nu=n+1}^{\infty} e_\nu u_\nu\right| < K(u_n + u_{n+1}) < 2K u_n,$$

and therefore $|e_n| < 2K$.

Suppose now that $\overline{\lim}\ \sigma_n = 3lK > 0$. Then we can find an infinite sequence of integers $N_1, N_2, \ldots$ such that

$$\sigma_{N_j} > 2lK, \qquad (j = 1, 2 \ldots).$$

Let m_j be the largest value of $n < N_j$ for which $\sigma_n \leqq lK$. Since $\underline{\lim}\ \sigma_n \leqq 0$, by Rademacher's result quoted above, m_j must tend to infinity with N_j. Writing m for m_j, N for N_j we have

$$lK < \sigma_N - \sigma_m = \frac{(t_N - t_m)\, m - (N - m)\, t_m}{mN}$$

$$\leqq \frac{1}{mN}\left\{m \sum_{\nu=m+1}^{N} |e_\nu| + (N - m) \sum_{\nu=1}^{m} |e_\nu|\right\}$$

$$\leqq \frac{1}{mN}\{2K\,(N - m)\, m + 2Km\,(N - m)\}$$

$$= \frac{4K\,(N - m)}{N}.$$

Hence $N - m > \frac{1}{4} l\, N \to \infty$ as $N \to \infty$.

Now

$$\left|\sum_{m+1}^{N-1} e_\nu\, u_\nu\right| \leqq K\,(u_{m+1} + u_N) < 2Ku_{m+1}. \tag{3}$$

But

$$\sum_{m+1}^{N-1} e_\nu\, u_\nu = \sum_{m+1}^{N-1} t_\nu\,(u_\nu - u_{\nu+1}) + t_N\, u_N - t_m\, u_{m+1}$$

$$= \sum_{m+1}^{N-1} \sigma_\nu\, \nu\,(u_\nu - u_{\nu+1}) + N\sigma_N\, u_N - m\sigma_m\, u_{m+1}$$

$$> lK\,\{(m + 1)\, u_{m+1} + u_{m+2} + \ldots + u_N\} + 2lKNu_N - lKNu_N - mlKu_{m+1}$$

$$> lK\,\{u_{m+1} + u_{m+2} \ldots + u_N\}.$$

It follows from condition (ii) of the Theorem that, given ϵ $(0 < \epsilon < \frac{1}{4}l)$, we have, for $\nu > n(\epsilon)$,

$$u_{\nu+1}/u_\nu > 1 - \epsilon.$$

Hence

$$\left|\sum_{m+1}^{N-1} e_\nu\, u_\nu\right| > lK\,(u_{m+1} + u_{m+2} + \ldots + u_N)$$

$$> lKu_{m+1}\,\{1 + (1 - \epsilon) + (1 - \epsilon)^2 + \ldots + (1 - \epsilon)^{N-m-1}\} \qquad (m \geqq n\,(\epsilon))$$

$$= lKu_{m+1}\,\frac{1 - (1 - \epsilon)^{N-m}}{\epsilon}$$

$$> lK\,\frac{u_{m+1}}{2\epsilon} > 2Ku_{m+1}$$

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for sufficiently large $N - m$. This is a contradiction of (3). Therefore $\overline{\lim}\, \sigma_n = 0$. Similarly we find that $\underline{\lim}\, \sigma_n$ cannot be negative; that is $\lim \sigma_n = 0$.

4. The restriction $\lim \frac{u_{m+1}}{u_m} = 1$ is necessary and the right hand side of the inequality (i) cannot be replaced by $K u_{N+1}^{a}$ with $a < 1$, as can be shown by gegenbeispiels constructed in the following way. Let $\sum_{\nu=1}^{\infty} v_\nu$ be a convergent series of decreasing, positive terms. We choose a sequence of integers $n_1, n_2, \ldots$ tending to infinity and insert between v_{n_k} and v_{n_k+1} new terms $v'_{n_k}, v''_{n_k}, \ldots$ satisfying $v_{n_k} > v'_{n_k} > v''_{n_k} > \ldots > v_{n_k+1}$. The number of these terms we take sufficiently large to ensure that $v'_{n_k} + v''_{n_k} + \ldots + v_{n_k+1} > 1$. Renumbering the terms we obtain a divergent series $\sum_{\nu=1}^{\infty} u_\nu$. The e_ν we choose all equal to $+1$ with the exception of those e_ν multiplying the newly inserted terms. To these terms we give the factor $e_\nu = (-1)^\nu$. It is plain that $\sum_{\nu=1}^{\infty} e_\nu u_\nu$ will be convergent and that we shall have $\overline{\lim}\, \sigma_n > 0$, provided only that the sequence $n_1, n_2, \ldots$ increases rapidly enough. If we take $v_n = q^n$ $(0 < q < 1)$, we obtain a series $\sum_{\nu=1}^{\infty} u_\nu$ satisfying condition (i), but not condition (ii) of Theorem 2. If $v_n = n^{-1/(1-a)}$ condition (ii) is satisfied and $\left| \sum_{\nu=n+1}^{\infty} e_\nu u_\nu \right| < K u_{n+1}^{a}$, but the conclusion of the theorem holds in neither case.

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