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FIVE MUTUALLY TANGENT SPHERES AND VARIOUS ASSOCIATED CONFIGURATIONS

BY

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ABSTRACT. If five spheres $\sigma_0, \sigma_1, \ldots, \sigma_4$ touch each other externally and have radii in geometrical progression, there is a dilative rotation mapping $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ to $\sigma_1, \sigma_2, \sigma_3, \sigma_4$; the dilatation factor is shown to be negative. The ten points of contact of the spheres lie by fours on 15 circles, forming a $(15_4 10_6)$ configuration in inversive space. In the corresponding configuration in the inversive plane, the 15 circles meet again in 60 points, which lie by fours on 45 circles touching by threes at each of the 60 points, and forming a configuration isomorphic to that of 60 Pascal lines (associated with six points on a conic) meeting by fours at 45 points. The 45 circles arise from ten Money–Coutts configurations of nine anti-tangent cycles. Conjectures are made about other circles through the 60 points.

1. **Introduction.** If five spheres $\sigma_0, \sigma_1, \ldots, \sigma_4$ all touch each other externally and have radii $1, r, r^2, r^3, r^4$ in geometric progression, then [2, p. 119] r satisfies the equation

$$r^2-(1+\sqrt{2})r+1=0, \tag{1}$$

and there is a similarity mapping $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ to $\sigma_1, \sigma_2, \sigma_3, \sigma_4$. This similarity, since it is not an isometry, is a dilative rotation (i.e. a rotation about a line l followed by a dilatation with centre O on l) [3, p. 102]. Coxeter tacitly assumes in [2] that the dilatation factor is positive, so that "the points of contact of consecutive spheres lie on a concho-spiral" [2, p. 119], but one of his research students, Mrs Asia Weiss, has recently shown that it is negative [6]. We give here a simple alternative proof of this fact.

Associated with five tangent spheres are a (10_6) configuration of spheres and points and also a $(15_6 10_9)$ configuration. We show that the two configurations exist also in a more general form.

Both these configurations give rise to a $(15_4 10_6)$ configuration of circles and points in three dimensions. Such a configuration exists also in two dimensions and it contains ten (9_4) configurations. The author has shown [5] that a (9_4) configuration gives rise to a Money–Coutts configuration of nine anti-tangent cycles, and we deduce from this that the $(15_4 10_6)$ configuration gives rise to a $(45_4 60_3)$ configuration of 45 circles touching by threes at 60 points.

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This last configuration is isomorphic to the configuration of 45 points of intersection of the 60 Pascal lines of six points on a conic. Other geometrical properties of the Pascal configuration suggest further fruitful investigations into the geometry of the circles and points. I am grateful to Professor Coxeter for reminding me about the Pascal configuration.

2. **The dilative rotation.** In the notation of section 1, take O as origin, l as the z-axis. The dilative rotation obtained by rotation about l through an angle $\theta+\pi$ followed by a negative dilatation with factor $-\lambda$ is equivalent to a rotation through an angle θ followed by a positive dilatation with factor λ followed by a reflection in the xy-plane. Suppose if possible that this dilative rotation maps $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ to $\sigma_1, \sigma_2, \sigma_3, \sigma_4$; then $\lambda = r$. Let us express the point (x, y, z) by $(x+iy, z)$ and write $e^{i\theta}=\kappa$. Denote the centre of σ_0 by (a, b), where we may suppose that a is real; then the centre of σ_i is $(ar^i\kappa^i, b(-r)^i)$ $(i=1, \ldots, 4)$ and its radius is r^i. The conditions for σ_0 to touch $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ externally are

$$a^2(r\kappa-1)(r\bar{\kappa}-1)+b^2(r+1)^2=(r+1)^2,$$
$$a^2(r^2\kappa^2-1)(r^2\bar{\kappa}^2-1)+b^2(r^2-1)^2=(r^2+1)^2,$$
$$a^2(r^3\kappa^3-1)(r^3\bar{\kappa}^3-1)+b^2(r^3+1)^2=(r^3+1)^2,$$
$$a^2(r^4\kappa^4-1)(r^4\bar{\kappa}^4-1)+b^2(r^4-1)^2=(r^4+1)^2.$$

If these conditions are satisfied, we see by applying the dilative rotation that all the spheres are mutually tangent. Let us write $b/a=c$, $1/a=d$; then the above conditions become

$$(r^2+1)-r(\kappa+\bar{\kappa})+c^2(r+1)^2=d^2(r+1)^2, \tag{2}$$
$$(r^4+1)-r^2(\kappa^2+\bar{\kappa}^2)+c^2(r^2-1)^2=d^2(r^2+1)^2, \tag{3}$$
$$(r^6+1)-r^3(\kappa^3+\bar{\kappa}^3)+c^2(r^3+1)^2=d^2(r^3+1)^2, \tag{4}$$
$$(r^8+1)-r^4(\kappa^4+\bar{\kappa}^4)+c^2(r^4-1)^2=d^2(r^4+1)^2. \tag{5}$$

If we eliminate c^2 and d^2 from equations (2) and (4), by taking (4)$-$ $(r^2-r+1)^2\times(2)$, we obtain

$$(r^6+1)-(r^2-r+1)^2(r^2+1)-r^3(\kappa^3+\bar{\kappa}^3)+(r^2-r+1)^2r(\kappa+\bar{\kappa})=0.$$

Dividing by r, and writing $\kappa+\bar{\kappa}=k$, we have

$$2(r^4-2r^3+2r^2-2r+1)-r^2(k^3-3k)+(r^2-r+1)^2k=0. \tag{6}$$

From (1), $r^2-r+1=\sqrt{2}r$, and $r^4-2r^3+2r^2-2r+1=(r^2-r+1)^2-r^2=2r^2-r^2=r^2$, so (6) becomes

$$k^3-5k-2\equiv(k+2)(k^2-2k-1)=0$$

on dividing through by $-r^2$.

Now $k=2\cos\theta$; hence the solution $k=1+\sqrt{2}$ is impossible. Hence $k=-2$ or $k=1-\sqrt{2}$.

If $k=-2$ then $\kappa=-1$, and (2) and (3) give

$$(1+\sqrt{2})+2+(3+\sqrt{2})c^2=(3+\sqrt{2})d^2,$$
$$(1+2\sqrt{2})-2+(-1+2\sqrt{2})c^2=(3+2\sqrt{2})d^2,$$

whence $c^2=-1$ and $d^2=0$, which is impossible.

Hence $k=1-\sqrt{2}=\kappa+\bar{\kappa}$, and we easily check that the unique solution of equations (2), ..., (5) is given by $c^2=3(3\sqrt{2}-2)/14$, $d^2=(3\sqrt{2}-2)/2$. Hence $a^2=(3\sqrt{2}+2)/7$, $b^2=3/7$. Also $2\cos\theta=\kappa+\bar{\kappa}=1-\sqrt{2}$, so $\theta=101°57'$. (If we return to the original point of view and take the dilatation factor to be $-r$, the angle of rotation will be $78°03'$ in the opposite direction.)

Is it possible to have a different arrangement of spheres with radii $1, r, \ldots, r^4$ giving a *positive* dilatation factor? No: since all five radii are known, once we have placed $\sigma_0, \ldots, \sigma_3$ all touching each other externally, there is only one possible position for σ_4 (since the second sphere touching $\sigma_0, \ldots, \sigma_3$ has radius r^{-1}).

3. **The points of contact of the spheres.** If we denote the dilative rotation by α, and denote the sphere $\sigma_0\alpha^r$ by σ_r ($r=\ldots,-2,-1,0,1,2,\ldots$) we obtain an infinite sequence of spheres, as described in [2], such that any five consecutive spheres are mutually externally tangent. Denote the centre of σ_i by C_i, and the z-coordinate of C_i by $z_i=b(-r)^i$. When $j=1$ or 3, we see that

$$\sigma_i \text{ and } \sigma_{i+j} \text{ touch externally,}$$
$$\sigma_i \text{ and } \sigma_{i+j} \text{ have radii } r^i \text{ and } r^{i+j},$$
$$z_{i+j}=-r^j z_i.$$

Hence the point of contact $P_{i,i+j}$ of σ_i and σ_{i+j} has z-coordinate 0. Thus all the points of contact $\ldots, P_{01}, P_{12}, P_{23}, P_{34}, \ldots$ and $\ldots, P_{03}, P_{14}, P_{25}, \ldots$ are coplanar. Also it is easy to see that $\ldots, P_{01}, P_{12}, P_{23}, \ldots$ lie on an equiangular spiral, and $\ldots, P_{03}, P_{14}, \ldots$ on a congruent spiral.

Since any infinite sequence of spheres (τ_r) ($r=\ldots,-2,-1,0,1,2,\ldots$), such that any five consecutive spheres are mutually tangent, is inversively equivalent to the sequence (σ_r), it follows that the points of contact $\ldots, Q_{01}, Q_{12}, Q_{23}, Q_{34}, \ldots$ and $\ldots, Q_{03}, Q_{14}, Q_{25}, \ldots$ all lie on a sphere (or plane), where Q_{ij} denotes the point of contact of τ_i and τ_j. Coxeter has given two simpler proofs of this result [4], but it is a nice consequence of the fact that the dilatation factor in the above discussion is negative.

4. **A (10_6) configuration.** Let us now consider again just five mutually tangent spheres $\tau_0, \tau_1, \ldots, \tau_4$. We have seen that $Q_{01}, Q_{12}, Q_{23}, Q_{34}, Q_{03}, Q_{14}$ lie on a sphere; these six points are the points of contact of τ_1 and τ_3 with the

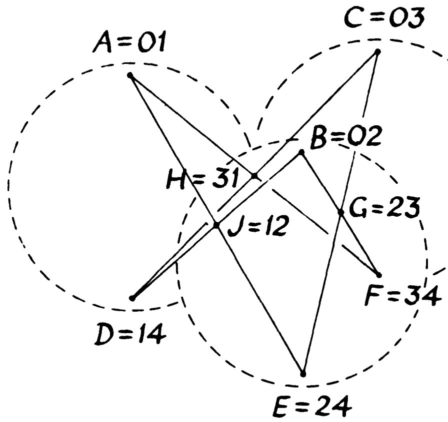

Figure 1

remaining spheres, so we can name the sphere on which they lie τ_{13} (or τ_{31}). The five mutually tangent spheres can be arranged in a sequence in 5! ways, but these sequences give us a total of only ten sets of six points of contact lying on ten spheres $\tau_{01}, \tau_{02}, \ldots, \tau_{34}$.

If we invert Q_{04} to infinity, then τ_0 and τ_4 become parallel planes, and τ_1, τ_2, τ_3 become tangent spheres of equal radius sandwiched between the planes as in Fig. 1. The existence of the ten spheres, each containing six points of contact, is now obvious. A list of the spheres is given below. In Fig. 1 we have written $Q_{01}=A$, $Q_{02}=B$, etc., and $Q_{04}=K$ is at infinity.

$$
\begin{array}{ll}
BCDJHK=\tau_{01} & ACDFJG=\tau_{13} \\
CAEGJK=\tau_{02} & BCEFJH=\tau_{23} \\
ABFHGK=\tau_{03} & AEFJHK=\tau_{14} \\
ABCDEF=\tau_{04} & BFDGJK=\tau_{24} \\
ABDEHG=\tau_{12} & CDEHGK=\tau_{34}
\end{array} \tag{7}
$$

(Since K is at infinity in Fig. 1, six of the spheres appear as planes.) Each of the ten points lies on six spheres; thus we have a (10_6) configuration of spheres and points. By permuting the suffixes $0, 1, \ldots, 4$ we obtain 5! symmetries of the configuration, and it is easily seen that there are no others: the symmetry group is S_5. Since we can permute the spheres $\tau_0, \tau_1, \ldots, \tau_4$ in any manner using products of inversions, we can realize all the symmetries of the configuration within the inversive group.

5. **A $(15_6 10_9)$ configuration.** We see from Fig. 1 that there are five more spheres each containing six points of contact. One of them is $ABCGHJ$, containing the points of contact of $\tau_0, \tau_1, \tau_2, \tau_3$; we shall denote this sphere by

τ_{45}. The complete list of five spheres is

$$DEFGHJ = \tau_{05} \qquad ABDEJK = \tau_{35}$$
$$BCEFGK = \tau_{15} \qquad ABCGHJ = \tau_{45} \tag{8}$$
$$CAFDHK = \tau_{25}$$

If $ijklm$ is any permutation of 01234, we shall give Q_{ij} the alternative name $Q_{ij5,klm}$, where the suffix is an unordered pair of unordered triplets. Then we find that τ_{01} contains the points $Q_{023,145}$, $Q_{024,135}$, $Q_{025,134}$, $Q_{034,125}$, $Q_{035,124}$, $Q_{045,123}$, with similar results for the remaining fourteen spheres. Also $Q_{012,345}$ lies on the nine spheres τ_{03}, τ_{04}, τ_{05}, τ_{13}, τ_{14}, τ_{15}, τ_{23}, τ_{24}, τ_{25}; etc.. Thus we have a $(15_6 10_9)$ configuration of spheres and points. By permuting the suffixes $0, 1, \ldots, 5$ we obtain 6! symmetries of the configuration, and it is easily seen that there are no others: the symmetry group is S_6.

Each of these two configurations contains fifteen circles with four points on each, each circle belonging to two spheres of the (10_6) configuration and to three spheres of the $(15_6 10_9)$ configuration. Using Fig. 1 we calculate that the three circles lying in one of the ten original spheres intersect at an angle of $2 \sin^{-1}(1/2\sqrt{2})$, whilst the three circles on the five other spheres intersect at right angles. Since inversions preserve angles, we can therefore realize only 5! of the symmetries of $(15_6 10_9)$ within the inversive group.

6. **The general $(15_6 10_9)$ configuration.** We have proved the existence of a special form of the $(15_6 10_9)$ configuration, derived from five mutually tangent spheres. We shall now investigate the most general manner of constructing the configuration. The general configuration is shown in Fig. 2, with K still at infinity. Let us analyse the figure to see how it can be constructed. The planes $BCEF$, $CAFD$, $ABDE$ meet at O, say; hence the lines AD, BE, CF meet at O. Denote the sphere $ABCDEF$ by σ. The points G, H, J lie in the polar plane π of O with respect to σ.

We begin therefore with a sphere σ, a point O not on σ, and two lines through O meeting σ at A, D and B, E; then J is determined. Now we require $BCJH$ to be concyclic; hence $DH.DC = DJ.DB = k_1$ say. Hence C lies on the inverse of π with respect to the sphere (real or imaginary) with centre D and radius $\sqrt{k_1}$. Similarly $EG.EC = EJ.EA = k_2$ say. Hence C lies on the inverse of π with respect to the sphere with centre E and radius $\sqrt{k_2}$. Thus C lies on two spheres as well as on σ. With suitable initial choices for σ, OAD and OBE, these three spheres will meet in two points, giving two possible positions for C; we choose either one of them. Now define $G = \pi \cap EC$, $H = \pi \cap DC$, $F = \sigma \cap OC$. To show that B, G, F are collinear, we observe that $BF \cap EC$ lies in π; but G is the point of EC in π, so $BF \cap EC = G$. Similarly H lies on AF.

Now $BCJH$ and $ACJG$ are concyclic; these circles have two points in

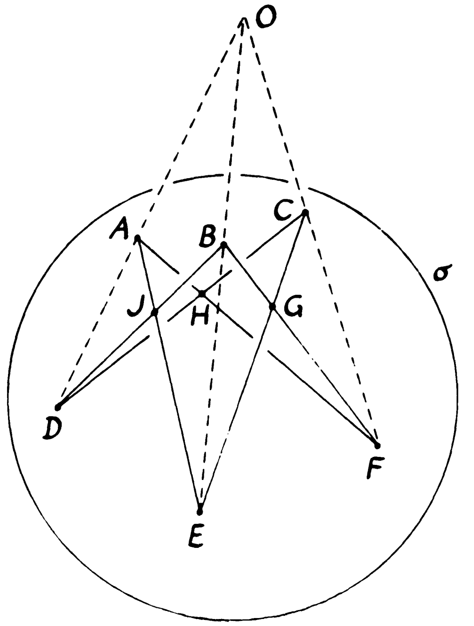

Figure 2

common, so *ABCGHJ* lie on a sphere. Hence *ABHG*, being coplanar also, are concyclic.

Now *BCEF* are concyclic (being coplanar and lying on σ) and *BCJH* are concyclic. Hence *BCEFJH* lie on a sphere. Similarly *CAFDGJ* and *ABDEHG* lie on spheres. From these spheres, cut by planes, we see that *FDGJ* and *DEHG* are concyclic. Hence *DEFGHJ* lie on a sphere.

We now have all the fifteen spheres, including the nine planes in the figure.

Since any configuration of five mutually tangent spheres can be mapped to any other by a product of inversions, we can say that such a configuration has no degrees of freedom *with respect to the inversive group.* It is easily seen that the general $(15_6 10_9)$ configuration has four degrees of freedom.

7. **The $(15_4 10_6)$ configuration in the plane.** If we consider only the circles and points in the $(15_6 10_9)$ configuration of spheres and points, we obtain a $(15_4 10_6)$ configuration of circles and points with symmetry group S_6. The three-dimensional nature of the configuration was an essential factor in its construction in section 6; we shall now consider whether this $(15_4 10_6)$ configuration exists in a plane. The circles are

	ABDE	*BCHJ*	*AFHK*
	ACDF	*DEGH*	*BDJK*
(9)	*BCEF*	*DFGJ*	*BFGK*
	ABGH	*EFHJ*	*CDHK*
	ACGJ	*AEJK*	*CEGK*.

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If we omit the point K and the six circles through K, we obtain a (9_4) configuration that can be expressed symbolically by the square array

$$\begin{matrix} A & B & C \\ D & E & F \\ G & H & J \end{matrix} \tag{10}$$

Any four points forming the vertices of a horizontal-vertical rectangle in the array lie on a circle of the (9_4) configuration.

THEOREM 1 [5, Theorems 3, 4]. *Given such a (9_4) configuration, the circles ABC, DEF, GHJ are coaxal, and the circles ADG, BEH, CFJ belong to the orthogonal coaxal system; conversely, given three circles of a coaxal system and three circles of the orthogonal system, if we choose nine of their eighteen points of intersection in a suitable manner (e.g. as in Fig. 3 where a limiting point of one system has been taken to be at infinity) they will lie by fours on nine circles to form a (9_4) configuration.*

THEOREM 2 [5, §4]. *If nine points lie by fours on eight circles in the manner of a (9_4) configuration, then the ninth circle of the configuration exists also.*

The author has shown [5, Theorem 5] that, in the (9_4) configuration given by (10), the six circles *AEJ, AFH, BDJ, BFG, CDH, CEG* (each circle containing one point from each row and column of the array) have a common radical centre, P say. This means that P has the same power with respect to the six circles; if this power is zero, then P lies on all the circles, so we can take $P=K$ to obtain the $(15_4 10_6)$ configuration. This shows that if a (9_4) configuration satisfies a single extra condition we can obtain a $(15_4 10_6)$ configuration, but it does not tell us how to construct such a (9_4) configuration, so we use a different approach.

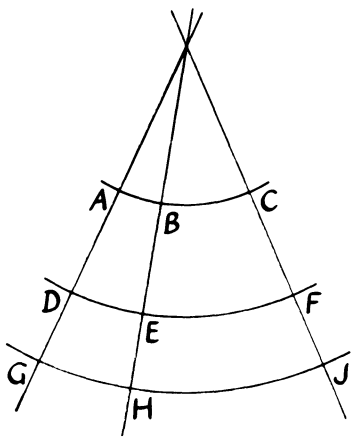

Figure 3

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Supposing that the $(15_4 10_6)$ configuration exists, let us omit A and the six circles through A; we then obtain a (9_4) configuration

$$(11) \qquad \begin{array}{ccc} B & K & C \\ F & G & E \\ J & D & H \end{array}$$

whence FGE is orthogonal to FBJ by Theorem 1. We shall show that (9_4) configurations (10) exist with this extra orthogonality property, and that this property is sufficient to ensure the existence of K.

In Fig. 4 the lines DG, EH, FJ meet at O, and DEF, GHJ are circles with centre O. The unique circle through F and J orthogonal to FGE meets HE twice, at B and B' say. (If F and J lie on the same side of HE, this orthogonal circle may not meet HE.) The circle through B with centre O meets DG, FJ at A, C, say. We now have three circles in each of two orthogonal coaxal systems (the limiting points of one system being O and the point at infinity), so we have a (9_4) configuration of type (10) with FGE orthogonal to FBJ.

Now $OB.OB' = OF.OJ = OE.OH = OD.OG = k$ say. Hence inversion in the circle with centre O and radius $\sqrt{k}$ (an imaginary circle in Fig. 4) maps the circle $FBJB'$ to itself and FGE to JDH. But inversion preserves orthogonality, so JDH is orthogonal to FBJ.

The reflection in the internal bisector of angle EOF maps FGE and JDH to themselves and FBJ to ECH. Hence FBJ, ECH are both orthogonal to FGE, JDH, so we have two circles from each of two orthogonal coaxal systems.

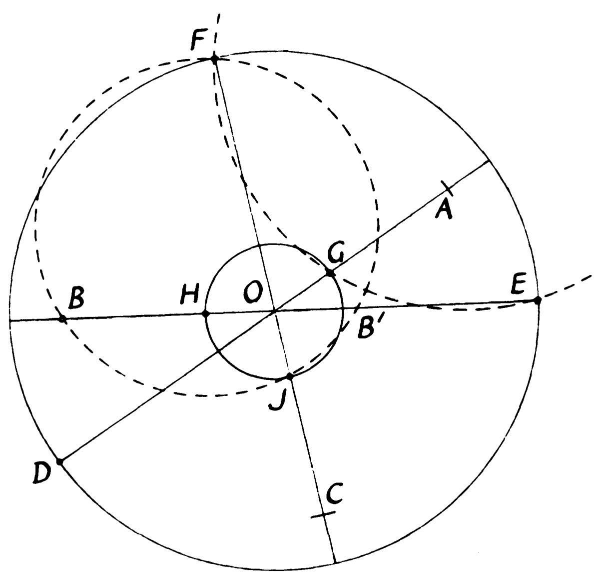

Figure 4

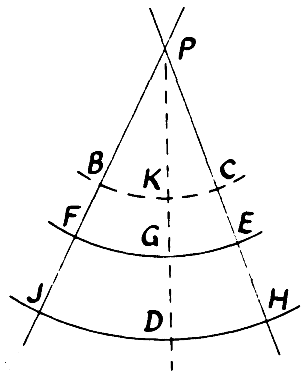

Figure 5

Assume without loss of generality that *FBJ*, *ECH* meet twice, at *P* and *Q*, and invert *Q* to infinity; we then have Fig. 5, in which *FGE*, *JDH* are circles with centre *P*. Since *BCEF* are concyclic, *B* and *C* lie on a circle with centre *P*; since *FGJD* are concyclic, the line *GD* passes through *P*. Let this circle and line meet at *K* as shown; then we have a (9_4) configuration of type (11), giving four circles through *K*, namely *BFGK*, *CEGK*, *BDJK*, *CDHK*.

To show that the (9_4) configuration

$$\begin{matrix} A & H & C \\ F & K & D \\ G & B & J \end{matrix}$$

exists, we observe that eight circles of this configuration already are known to exist, so by Theorem 2 the ninth circle exists; i.e. *AFHK* are concyclic. Similarly by considering

$$\begin{matrix} A & E & C \\ J & K & G \\ D & B & F \end{matrix}$$

we see that *AEJK* are concyclic. We now have all fifteen circles of a $(15_4 10_6)$ configuration.

The (9_4) configuration has four degrees of freedom with respect to the inversive group; the $(15_4 10_6)$ configuration has three, and its symmetry group is S_6.

8. **Ten Money–Coutts configurations.** The author's original interest in (9_4) configurations arose from the following result.

Theorem 3 [5, Theorem 4]. *The nine circles of a* (9_4) *configuration meet in pairs in eighteen points other than the points of the configuration. These eighteen*

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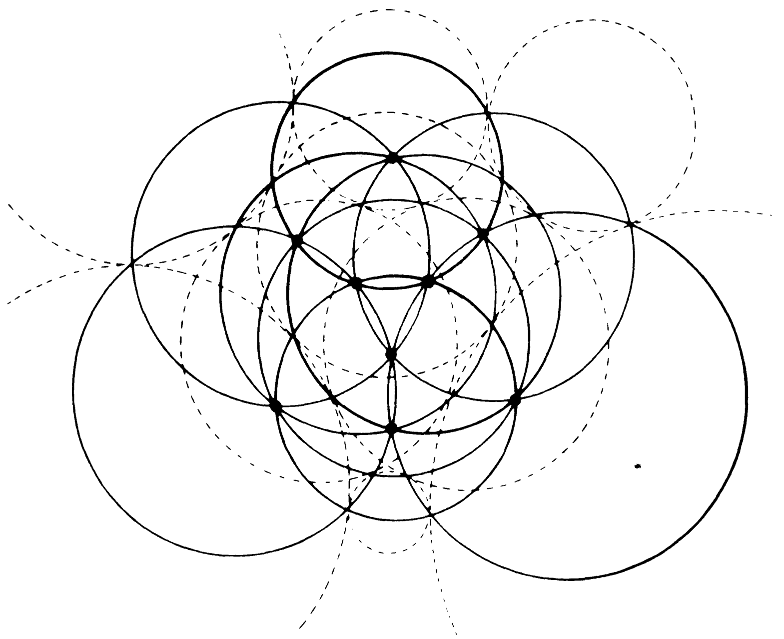

Figure 6

points lie by fours on nine circles, two of which touch at each of the eighteen points. These circles can be oriented to form a configuration of nine anti-tangent cycles, each anti-touching four others (a Money–Coutts configuration).

The situation is illustrated in Fig. 6. Now a $(15_4 10_6)$ configuration contains ten (9_4) configurations, obtained by omitting any one of the ten points and the six circles through it. Each (9_4) configuration gives a Money–Coutts configuration.

THEOREM 4. *The* 90 *circles in the ten Money–Coutts configurations derived from a* $(15_4 10_6)$ *configuration coincide in pairs to give a configuration of* 45 *circles touching by threes at* 60 *points.*

Proof. Some of the statements in this proof can be visualized by labelling the points in Fig. 6; they are proved in [5]. Consider the (9_4) configuration (10). We shall use the notation $BCEF \cap DEGH$ to denote the point of intersection, *other than E*, of the circles *BCEF* and *DEGH*, etc.. The four points $BCEF \cap DEGH$, $DEGH \cap BCHJ$, $BCHJ \cap DFGJ$, $DFGJ \cap BCEF$ lie on a circle, which we shall denote by *a*. We define similarly a total of nine circles $a, b, c, \ldots, j$ forming a Money–Coutts configuration, with one circle associated with each of the nine points. Two circles touch if the corresponding points lie in distinct rows and distinct columns of the array (10); for instance, *a* and *e* touch at $BCHJ \cap DFGJ$.

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Since this Money–Coutts configuration is obtained by omitting K from the $(15_4 10_6)$ configuration, we shall denote it by K and shall rename its nine circles $ka, kb, \ldots, kj$. The (9_4) configuration given by the array (11) gives the nine circles $ab, ac, \ldots, ak$ of the Money–Coutts configuration A; etc.. We easily check that $ka = ak$, etc., so there are 45 circles rather than 90.

We have seen that the circles ka and ke touch at $BCHJ \cap DFGJ$. The circles ek and ea touch at this same point, and so do ae and ak. Hence ka, ek, ae all touch at this point. We thus have a configuration of 45 circles touching by threes at 60 points, four of the points lying on each circle.

The converse of Theorem 3 is true: any Money–Coutts configuration has an associated (9_4) configuration from which it is derived. It is therefore easily seen that any configuration of 45 circles as described in Theorem 4 has an associated $(15_4 10_6)$ configuration from which it is derived. Such configurations therefore have three degrees of freedom and symmetry group S_6. We shall call them *T-configurations*. The particular T-configuration shown in Fig. 8 has six extra accidental points of triple contact, shown by black dots.

The cycles of a Money–Coutts configuration are anti-tangent; we cannot re-orient some of them to make them all tangent. However, if three circles all touch at the same point as in Theorem 4 we cannot orient them so that each anti-touches the other two. I conjecture that we can orient the cycles in each of the ten Money–Coutts configurations in such a way that the common circle of two Money–Coutts configurations is oriented in opposite directions in the two

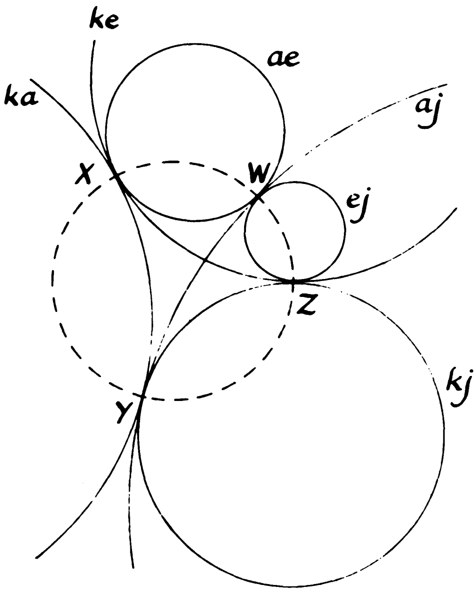

Figure 7

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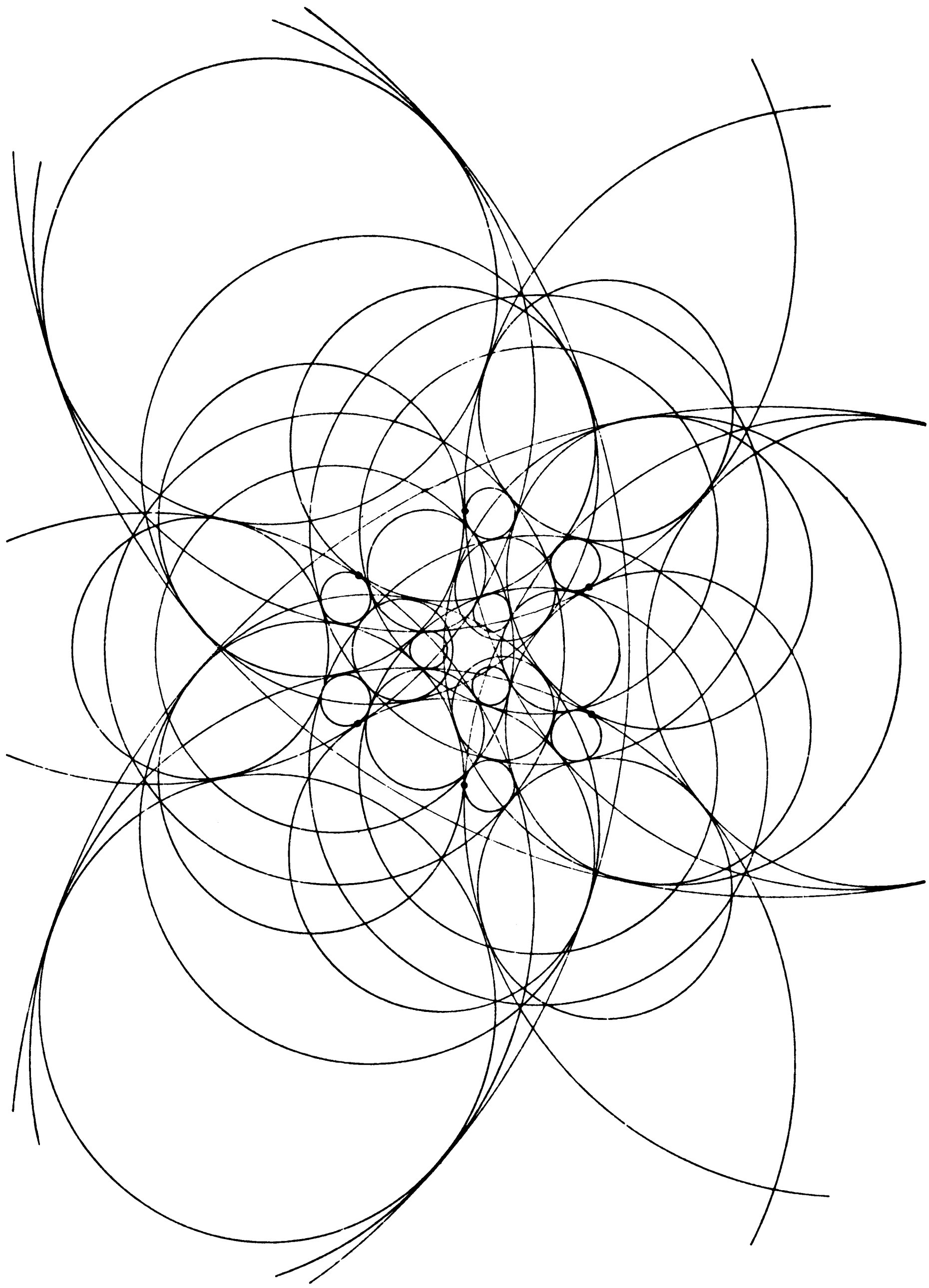

Figure 8

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configurations (e.g. *ka* and *ak* are opposite cycles). We should then have a configuration of 45 bicycles.

Write $BCHJ \cap DFGJ = X$, $DEGH \cap BCEF = Y$, $BFGK \cap CDHK = W$, $ACDF \cap ABGH = Z$. The cycles *ka* and *ke* anti-touch at X, *ka* and *kj* at Y, *ke* and *kj* at Z (Fig. 7). Hence the circle XYZ is orthogonal to *ka*, *ke*, *kj*. Similarly WXY is orthogonal to *ae*, *aj*, *ak* and WXZ is orthogonal to *ea*, *ej*, *ek*. Hence $WXYZ$ are concyclic and this circle is orthogonal to *ae*, *aj*, *ak*, *ej*, *ek*, *jk* (Fig. 7). The 60 points lie by fours on fifteen such *orthocircles*.

The eighteen points of any one of the Money–Coutts configurations lie on six of these orthocircles, and these six have a common orthogonal circle [5, Theorem 1], the *base circle* of the configuration, which may be imaginary. A T-configuration contains fifteen orthocircles and ten base circles, each orthocircle orthogonal to four base circles and each base circle orthogonal to six orthocircles: a type of $(15_4 10_6)$ configuration in which the incidence relation is orthogonality.

9. **The connection with the Pascal configurations.** The circle $ABDE$ in the three-dimensional $(15_6 10_9)$ configuration is the intersection of the three spheres $\tau_{04}, \tau_{12}, \tau_{35}$, so we can denote $ABDE$ by the *syntheme* $(04, 12, 35)$ using Sylvester's terminology [1, p. 220], a syntheme being an unordered triplet of unordered pairs or *duads*. A point, such as $A = Q_{015,234}$, lies on a circle if and only if the numbers in each triplet of the suffix belong one to each duad in the corresponding syntheme.

A typical Money–Coutts circle in section 8 is *ka* or *ak*. Now K and A are $Q_{045,123}$ and $Q_{015,234}$, so we may denote *ka* by $(05, 23)$. Similarly each of the 45 circles is denoted by an unordered pair of duads. The three circles *ka*, *ke*, *ae* touch at a common point. These circles are $(05, 23)$, $(13, 45)$, $(01, 24)$; we shall denote their common point by 013245, where only the *cyclic* order of the six numbers is important, and the opposite cyclic order 542310 represents the same point. The notation is explained by the fact that 05 and 23 are "opposite adjacent pairs" in the cyclic sequence 013245, and so are the pairs 13, 45 and 01, 24. This point is also the point of intersection (other than $Q_{521,430}$) of the circles $(54, 23, 10)$ and $(42, 31, 05)$. The 60 possible cyclic orders give the 60 points of tangency of the 45 circles.

Now, if three circles touch at a common point, their centres lie on a line. Hence the centres of the 45 Money–Coutts circles lie by threes on 60 lines, forming a $(60_3 45_4)$ configuration of lines and points. From the notation just developed we immediately obtain an isomorphism between this configuration and the configuration of 60 Pascal lines associated with six points on a conic, meeting by fours at 45 points: if we denote six points of a conic by $0, 1, 2, 3, 4, 5$, then for instance the pairs of opposite sides of the hexagon 013245 meet at the three points $(01, 24)$, $(13, 45)$, $(05, 23)$, which are collinear on the Pascal line of the hexagon.

The configuration obtained from six points on a conic is a special case of a more general configuration which we shall call a Cremona configuration [1, p. 220 et seq.]. This is derived from fifteen lines (obtained by projecting a certain configuration of fifteen lines and planes from 3-space onto a plane) denoted by duads formed from the numbers $0, 1, \ldots, 5$. Lines whose duads have no number in common meet at 45 points denoted by (01, 23) etc., and these points lie by threes on 60 Pascal lines as described above, giving a $(60_3 45_4)$ configuration.

In a Cremona configuration the six points (01, 23), (01, 24), (01, 25), (01, 34), (01, 35), (01, 45) are collinear, on the line (01), but in a T-configuration the centres of the six circles (01, 23), (01, 24) etc. are not collinear. (This last statement is not proved; it has merely been observed from a figure.) Hence not all $(60_3 45_4)$ configurations of points and lines are derived from Cremona configurations.

Also in a Cremona configuration the 60 Pascal lines are concurrent by threes in 20 Steiner points; for instance, 012345, 032541, 052143 are concurrent. I conjecture from observation that in a T-configuration the *Steiner circle* through the points 012345, 032541, 052143 is orthogonal to the nine Money–Coutts circles that pass, three by three, through these points. A tedious proof of this would be possible using the Argand diagram as in [5, §3], but I have not found a synthetic proof.

The 60 Pascal lines are also concurrent by threes in 60 Kirkman points; for instance, 012345, 120534, 201453 are concurrent. Let us call the circle through the three corresponding points of a T-configuration a *Kirkman circle*. We shall say that the point *acebfd* is *conjugate* to *abcdef*. By writing the symbol for a given point in all possible ways (either retaining or reversing the cyclic order) we find that there are just three points conjugate to it; also, if B is conjugate to A then A is conjugate to B. For instance, the three points conjugate to 031524 are 012345, 120534, 201453. These points lie on a Kirkman circle, and my final conjecture is that this circle passes through 031524 also.

If this conjecture is correct, there is a natural one-one correspondence between the 60 points of a T-configuration and the 60 Kirkman circles: to each point corresponds the Kirkman circle through the point and its three conjugate points. Since certain of the points lie on orthocircles, Steiner circles etc., we might expect to discover properties of the corresponding Kirkman circles. I have as yet obtained nothing significant from this train of thought, but an interesting property of the conjectured Kirkman circles is given in the next section.

10. **A property of the conjectured Kirkman circles.** We shall assume in this final section that each Kirkman circle does pass through a fourth point, as described above; the 60 Kirkman circles and the 60 points of the T-configuration then form a (60_4) configuration.

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Now the Pascal lines and Kirkman points in a Cremona configuration form a (60_3) configuration, but this splits up into six (10_3) configurations [1, p. 232], as is easily verified. Similarly our (60_4) configuration splits up into six (10_4) configurations. One of these consists of the ten points 012345, 013254, 014532, 015423, 021435, 024153, 025314, 031524, 034215, 043215, and the corresponding Kirkman circles. Let us denote this (10_4) configuration by K_1, and the other five by $K_2, \ldots, K_6$.

The symmetry groups of all the configurations in this paper are transitive on points and circles, or points and spheres, and the (10_4) configurations are self-dual, as are (10_6) and (9_4). However, those symmetries of K_1 that fix one circle do not permute the points of that circle transitively: they all fix the corresponding point also. In this respect the (10_4) differs from the previous configurations; it is less symmetrical.

The $(15_4\, 10_6)$ configuration from which a T-configuration is derived can be called the *auxiliary* configuration, consisting of the *auxiliary circles* and *auxiliary points* of the T-configuration. (This is preferable to the term "base points" used in [5].) Each of the 60 points of the T-configuration is the meet of two auxiliary circles; alternatively, two auxiliary circles pass through each of the 60 points.

If we consider those auxiliary circles that pass through the ten points of K_1, we find that there are just five of them, namely (02, 14, 35), (03, 15, 24), (04, 13, 25), (05, 12, 34), (01, 23, 45), and if we now adjoin these circles to K_1 we obtain a $(15_4\, 10_6)$ configuration, A_1 say, easily seen to be isomorphic to the auxiliary configuration. Similarly from $K_2, \ldots, K_6$ we obtain $(15_4\, 10_6)$ configurations $A_2, \ldots, A_6$. Thus our original auxiliary configuration, A say, gives rise to six configurations $A_1, \ldots, A_6$. These in turn can be regarded as auxiliary configurations, each giving rise to six $(15_4\, 10_6)$ configurations, one of which is found to be A in each case.

The final question is this: does this process "close up" in some manner, or does it lead to an infinite number of $(15_4\, 10_6)$ configurations?

Note: Since the acceptance of this paper, the author has verified the conjectures made in the paper. The proofs and further results about T-configurations will be presented in a subsequent paper.

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