Canad. Math. Bull. Vol. **57** (2), 2014 pp. 326–334
http://dx.doi.org/10.4153/CMB-2012-036-6

On Zero-divisors in Group Rings of Groups with Torsion

S. V. Ivanov and Roman Mikhailov

Abstract. Nontrivial pairs of zero-divisors in group rings are introduced and discussed. A problem on the existence of nontrivial pairs of zero-divisors in group rings of free Burnside groups of odd exponent $n \gg 1$ is solved in the affirmative. Nontrivial pairs of zero-divisors are also found in group rings of free products of groups with torsion.

1 Introduction

Let G be a group and $\mathbb{Z}[G]$ denote the group ring of G over the integers. If $h \in G$ is an element of finite order $q > 1$ and $X, Y \in \mathbb{Z}[G]$, then we have the following equalities in $\mathbb{Z}[G]$:

$$X(1-h)\cdot(1+h+\cdots+h^{q-1})Y = 0,$$
$$X(1+h+\cdots+h^{q-1})\cdot(1-h)Y = 0.$$

Hence, $X(1-h)$ and $(1+h+\cdots+h^{q-1})Y$, $X(1+h+\cdots+h^{q-1})$ and $(1-h)Y$ are left and right zero-divisors of $\mathbb{Z}[G]$ (unless one of them is 0 itself), which we call trivial pairs of zero-divisors associated with an element $h \in G$ of finite order $q > 1$. Equivalently, $A, B \in \mathbb{Z}[G]$, with $AB = 0$, $A, B \neq 0$, is a *trivial* pair of zero-divisors in $\mathbb{Z}[G]$ if there are $X, Y \in \mathbb{Z}[G]$ and $h \in G$ of finite order $q > 1$ such that either $A = X(1-h)$ and $B = (1+h+\cdots+h^{q-1})Y$ or $A = X(1+h+\cdots+h^{q-1})$ and $B = (1-h)Y$.

An element $A \in \mathbb{Z}[G]$ is called a nontrivial left (right) zero-divisor if A is a left (right, resp.) zero-divisor and for every $B \in \mathbb{Z}[G]$ such that $B \neq 0$, $AB = 0$, the pair A, B is not a trivial pair of zero-divisors.

The notorious Kaplansky conjecture on zero-divisors claims that, for any torsion-free group G, its integral group ring $\mathbb{Z}[G]$ (or, more generally, its group algebra $\mathbb{F}[G]$ over a field $\mathbb{F}$) contains no zero-divisors. In this note, we are concerned with a more modest problem on the existence of zero-divisors in group rings of infinite groups with torsion that would be structured essentially differently from the above examples of trivial pairs of zero-divisors. We remark in passing that every pair of zero-divisors in $\mathbb{Z}[G]$ is trivial whenever G is cyclic (or locally cyclic).

Received by the editors July 3, 2012.
Published electronically December 29, 2012.
The first author is supported in part by NSF grant DMS 09-01782. This research of the second author is supported by the Chebyshev Laboratory (Department of Mathematics and Mechanics, St. Petersburg State University) under RF Government grant 11.G34.31.0026
AMS subject classification: **20C07**, 20E06, 20F05, 20F50.
Keywords: Burnside groups, free products of groups, group rings, zero-divisors.

Note that if G is a finite group, then every nonzero element X in the augmentation ideal of $\mathbb{Z}[G]$ is a left (right) zero-divisor, because the linear operator $L_X\colon \mathbb{Q}[G] \to \mathbb{Q}[G]$, given by multiplication $Y \to XY$ ($Y \to YX$, resp.), has a nontrivial kernel as follows from $\dim L_X(\mathbb{Q}[G]) < \dim \mathbb{Q}[G]$. Hence, $2 - g_1 - g_2$, where $g_1, g_2 \in G$, is a left (right) zero-divisor of $\mathbb{Z}[G]$ unless $g_1 = g_2 = 1$. On the other hand, the element $2 - g_1 - g_2 \in \mathbb{Z}[G]$ is not a trivial left (right) zero-divisor unless g_1, g_2 generate a cyclic subgroup of G. Hence, for a finite group G, the group ring $\mathbb{Z}[G]$ of G contains no nontrivial zero-divisors if and only if G is cyclic. More generally, if G is a group with a noncyclic finite subgroup H, then the element $2-h_1-h_2 \in \mathbb{Z}[G]$, where $h_1, h_2 \in H$, is a nontrivial zero-divisor of $\mathbb{Z}[G]$ unless h_1, h_2 generate a cyclic subgroup of H (for details, see the proof of Theorem 1.2).

However, if G is an infinite torsion (or periodic) group all of whose finite subgroups are cyclic, then the existence of nontrivial pairs of zero-divisors in $\mathbb{Z}[G]$ is not clear. For instance, let $B(m, n)$ be the free Burnside group of rank m and exponent n; that is, $B(m, n)$ is the quotient F_m/F_m^n of a free group F_m of rank m. It is known [8,13] that if $m \geq 2$ and $n \gg 1$ is odd, then every noncyclic subgroup of $B(m, n)$ contains a subgroup isomorphic to the free Burnside group $B(\infty, n)$ of countably infinite rank; in particular, every finite subgroup of $B(m, n)$ is cyclic. Note that this situation is dramatically different for even $n \gg 1$; see [6].

In this regard and because of other properties of $B(m, n)$, analogous to properties of absolutely free groups (see [13]), the first author asked the following question [10, Problem 11.36d]: Suppose $m \geq 2$ and odd $n \gg 1$. Is it true that every pair of zero-divisors in $\mathbb{Z}[B(m, n)]$ is trivial, *i.e.,* if $AB = 0$ in $\mathbb{Z}[B(m, n)]$, then $A = XC$, $B = DY$, where $X, Y, C, D \in \mathbb{Z}[B(m, n)]$ such that $CD = 0$ and the set $\operatorname{supp}(C) \cup \operatorname{supp}(D)$ is contained in a cyclic subgroup of $B(m, n)$?

In this paper we will give a negative answer to this question by constructing a nontrivial pair of zero-divisors in $\mathbb{Z}[B(m, n)]$ as follows.

Theorem 1.1 *Let $B(m, n)$ be the free Burnside group of rank $m \geq 2$ and odd exponent $n \gg 1$, and let a_1, a_2 be free generators of $B(m, n)$. Denote $c := a_1 a_2 a_1^{-1} a_2^{-1}$ and let*

$$A := (1 + c + \cdots + c^{n-1})(1 - a_1 a_2 a_1^{-1}),$$
$$B := (1 - a_1)(1 + a_2 + \cdots + a_2^{n-1}).$$

Then $AB = 0$ in $\mathbb{Z}[B(m, n)]$, and A, B is a nontrivial pair of zero-divisors in $\mathbb{Z}[B(m, n)]$.

It seems of interest to look at other classes of groups with torsion all of whose finite subgroups are cyclic and ask a similar question on the existence of nontrivial pairs of zero-divisors in their group rings. From this viewpoint, we consider free products of cyclic groups, all of whose finite subgroups are cyclic by the Kurosh subgroup theorem [11], and show the existence of nontrivial pairs of zero-divisors in their group rings. More generally, we will prove the following theorem.

Theorem 1.2 *Let a group G contain a subgroup isomorphic either to a finite noncyclic group or to the free product $C_q * C_r$, where C_n denotes a cyclic group of order n (perhaps, $n = \infty$), and $1 < \min(q, r) < \infty$. Then the integer group ring $\mathbb{Z}[G]$ of G has a nontrivial pair of zero-divisors.*

On the one hand, in view of Theorems 1.1 and 1.2, one might wonder if there exists a nonlocally cyclic group G with torsion without nontrivial pairs of zero-divisors in $\mathbb{Z}[G]$; in particular, whether there is a free Burnside group $B(m,n)$, where $m,n > 1$, with this property. Note that, for every even $n \geq 2$ and $m \geq 2$, the free Burnside group $B(m,n)$ contains a dihedral subgroup, hence, by Theorem 1.2, $\mathbb{Z}[B(m,n)]$ does have a nontrivial pair of zero-divisors.

On the other hand, our construction of nontrivial pairs of zero-divisors in $\mathbb{Z}[C_q * C_r]$, where $1 < q < \infty$, $r \in \{2,\infty\}$, and $C_q = \langle a\rangle_q$ is generated by a, produces nontrivial pairs of zero-divisors of the form $AB = 0$, where $A = (1-a)U$, $B = U^{-1}(\sum_{i=1}^{q} a^i)$, and U is a unit of $\mathbb{Z}[C_q * C_r]$. Thus, our nontrivial pairs of zero-divisors in $\mathbb{Z}[C_q * C_r]$ are still rather restrictive and could be named *primitive*.

Generalizing the definition of a trivial pair of zero-divisors, we say that $A, B \in \mathbb{Z}[G]$, where $A, B \neq 0$, $AB = 0$, is a *primitive* pair of zero-divisors in $\mathbb{Z}[G]$ if there exists a unit U of $\mathbb{Z}[G]$ such that $A = XU$, $B = U^{-1}Y$, and X, Y is a trivial pair of zero-divisors in $\mathbb{Z}[G]$. One might conjecture that all pairs of zero-divisors in $\mathbb{Z}[G]$ are primitive whenever G is a free product of cyclic groups. Results and techniques of Cohn [1,2] (see also [3,4]) on units and zero-divisors in free products of rings could be helpful in the investigation of this conjecture.

2 Three Lemmas

Lemma 2.1 *Suppose that G is a group, $h \in G$, $H = \langle h\rangle$, $X \in \mathbb{Z}[G]$, and $C \in \mathbb{Z}[H]$ is not invertible in $\mathbb{Z}[G]$. Then, for every $g \in G$, the left coset gH of G by H is either disjoint from* $\operatorname{supp}(XC)$ *or* $|gH \cap \operatorname{supp}(XC)| \geq 2$.

Proof Let $X = \sum_{i=1}^{k} x_i C_i$, where $C_i \in \mathbb{Z}[H]$ and $x_i \in G$, so that $x_i H \neq x_j H$ for $i \neq j$. Then $XC = \sum_{i=1}^{k} x_i C_i C$ and $\operatorname{supp}(XC) = \bigcup_{i=1}^{k} x_i \operatorname{supp}(C_i C)$ is a disjoint union. Since C is not invertible in $\mathbb{Z}[H]$, $|\operatorname{supp}(C_i C)| > 1$, and the result follows. ■

Recall that a subgroup K of a group G is called *antinormal* if, for every $g \in G$, the inequality $gKg^{-1} \cap K \neq \{1\}$ implies that $g \in K$.

Lemma 2.2 *Suppose that G is a group, $a, b \in G$, the elements $b, c := aba^{-1}b^{-1}$ have order $n > 1$, $d := aba^{-1}$, the cyclic subgroups $\langle c\rangle$, $\langle ab^i\rangle$, $i = 0, 1 \dots, n-1$, are nontrivial, antinormal, and $d \notin \langle c\rangle$, $c^j d \notin \langle ab^i\rangle$ for all $i, j \in \{0, 1 \dots, n-1\}$. Then equalities*

$$(1 + c + \cdots + c^{n-1})(1 - d) = XC, \tag{2.1}$$

$$(1 - a)(1 + b + \cdots + b^{n-1}) = DY, \tag{2.2}$$

where $X, Y \in \mathbb{Z}[G]$, $C, D \in \mathbb{Z}[H]$, H is a cyclic subgroup of G, and $CD = 0$, are impossible.

Proof Arguing on the contrary, assume that equalities (2.1) and (2.2) hold true. Denote $H = \langle h\rangle$. Note that neither C nor D is invertible in $\mathbb{Z}[H]$, because, otherwise, $CD = 0$ would imply that one of C, D is 0, which contradicts one of (2.1) and (2.2) and the assumptions $a \notin \langle c\rangle$, $c \neq 1$.

https://doi.org/10.4153/CMB-2012-036-6 Published online by Cambridge University Press

Hence, Lemma 2.1 applies to equality (2.1) and yields that the set

$$\operatorname{supp}(XC) = \{1, c, \dots, c^{n-1}, d, cd, \dots, c^{n-1}d\}$$

can be partitioned into subsets of cardinality > 1 that are contained in distinct left cosets gH, $g \in G$.

Assume that $c^{i_1}, c^{i_2} \in gH$, where $0 \le i_1 < i_2 \le n-1$. Then $c^{i_1-i_2} = h^k \ne 1$ and, by antinormality of $\langle c\rangle$, we have $h = c^i$ for some i. Since $d \in \operatorname{supp}(XC)$, it follows from Lemma 2.1 that $dh^j = dc^{ij} \in \operatorname{supp}(XC)$ with $h^j \ne 1$. Hence, either $dc^{ij} = c^{i'}$ or $dc^{ij} = c^{i'}a$ with $c^{ij} \ne 1$. In either case, we have a contradiction to $d \notin \langle c\rangle$ and antinormality of $\langle c\rangle$.

Now assume that $c^{i_1}d, c^{i_2}d \in gH$, where $0 \le i_1 < i_2 \le n-1$. Then $d^{-1}c^{i_1-i_2}d = h^k \ne 1$ and, by antinormality of $\langle c\rangle$, we have $h = d^{-1}c^i d$ for some i. Since $1 \in \operatorname{supp}(XC)$, it follows from Lemma 2.1 that $h^j = d^{-1}c^{ij}d \in \operatorname{supp}(XC)$ with $h^j \ne 1$. Hence, either $d^{-1}c^{ij}d = c^{i'} \ne 1$ or $d^{-1}c^{ij}d = c^{i'}d$. In either case, we have a contradiction to antinormality of $\langle c\rangle$ and $d \notin \langle c\rangle$.

The contradictions obtained above prove that the foregoing partition of the set $\operatorname{supp}(XC)$ consists of two element subsets so that one element belongs to $\{1, c, \dots, c^{n-1}\}$ and the other belongs to $\{d, cd, \dots, c^{n-1}d\}$. In particular, it follows from $1 \in \{1, c, \dots, c^{n-1}\}$ that

$$h^{k_1} = c^{i_1}d \ne 1 \tag{2.3}$$

for some k_1, i_1.

Applying a "right-hand" version of Lemma 2.1 to the equality (2.2), we analogously obtain that the set

$$\operatorname{supp}(DY) = \{1, b, \dots, b^{n-1}, a, ab, \dots, ab^{n-1}\}$$

can be partitioned into subsets of cardinality > 1 which are contained in distinct right cosets Hg, $g \in G$.

Assume that $b^{i_1}, b^{i_2} \in Hg$, where $0 \le i_1 < i_2 \le n-1$. Then $b^{i_1-i_2} = h^k \ne 1$, and, by antinormality of $\langle b\rangle$, we have $h = b^i$ for some i. Since $a \in \operatorname{supp}(DY)$, it follows from the analog of Lemma 2.1 (in the "right-hand" version) that $h^j a = b^{ij}a \in \operatorname{supp}(DY)$ with $h^j \ne 1$. Hence, either $b^{ij}a = b^{i'}$ or $b^{ij}a = ab^{i'}$. In either case, we have a contradiction to $c \ne 1$ and antinormality of $\langle b\rangle$.

Now assume that $ab^{i_1}, ab^{i_2} \in Hg$, where $0 \le i_1 < i_2 \le n-1$. Then $ab^{i_1-i_2}a^{-1} = h^k \ne 1$, and, by antinormality of $\langle b\rangle$, we have $h = ab^i a^{-1}$ for some i. Since $1 \in \operatorname{supp}(DY)$, it follows from the analog of Lemma 2.1 that $h^j = ab^{ij}a^{-1} \in \operatorname{supp}(DY)$ with $h^j \ne 1$. Hence, either $ab^{ij}a^{-1} = b^{i'} \ne 1$ or $ab^{ij}a^{-1} = ab^{i'}$. In either case, we have a contradiction to antinormality of $\langle b\rangle$ and $c \ne 1$.

The contradictions obtained above prove that the foregoing partition of the set $\operatorname{supp}(DY)$ consists of two element subsets so that one element belongs to $\{1, b, \dots, b^{n-1}\}$ and the other belongs to $\{a, ab, \dots, ab^{n-1}\}$. In particular, it follows from $1 \in \{1, b, \dots, b^{n-1}\}$ that

$$h^{k_2} = ab^{i_2} \ne 1 \tag{2.4}$$

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for some k_2, i_2.

In view of equalities (2.3) and (2.4), we obtain $c^{i_1}dab^{i_2} = ab^{i_2}c^{i_1}d$. Since the subgroup $\langle ab^{i_2}\rangle$ is antinormal, we conclude that $c^{i_1}d \in \langle ab^{i_2}\rangle$. This, however, is impossible by assumption, and Lemma 2.2 is proved. ∎

Lemma 2.3 *Suppose $a, b \in G$ are elements of a group G such that the subgroup $\langle a, b\rangle$, generated by a, b, is isomorphic to the free product $\langle a\rangle_q * \langle b\rangle_r$, where $\langle c\rangle_s$ denotes a cyclic group of order s generated by c (perhaps, $s = \infty$), $1 < q < \infty$, $r \in \{2, \infty\}$, and $(q, r) = (2, 2)$ if $r = 2$. Then the elements*

$$A := (1-a)\Big(1 + (1-a)b\Big(\sum_{i=1}^{q} a^i\Big)\Big), \tag{2.5}$$

$$B := \Big(1 - (1-a)b\Big(\sum_{i=1}^{q} a^i\Big)\Big)\Big(\sum_{i=1}^{q} a^i\Big) \tag{2.6}$$

satisfy $AB = 0$ and form a nontrivial pair of zero-divisors in $\mathbb{Z}[G]$.

Proof Since

$$\Big(1 + (1-a)b\Big(\sum_{i=1}^{q} a^i\Big)\Big) \cdot \Big(1 - (1-a)b\Big(\sum_{i=1}^{q} a^i\Big)\Big) = 1,$$

it follows that $AB = (1-a)(\sum_{i=1}^{q} a^i) = 0$ and $A, B \neq 0$, hence A, B is a pair of zero-divisors in $\mathbb{Z}[G]$. We need to show that A, B is a nontrivial pair of zero-divisors. Arguing on the contrary, assume that A, B is a trivial pair of zero-divisors in $\mathbb{Z}[G]$. Then there is an element $h \in G$ of finite order $s > 1$ and $X, Y \in \mathbb{Z}[G]$ such that either

$$A = X(1-h) \quad \text{and} \quad B = \Big(\sum_{i=1}^{s} h^i\Big)Y \tag{2.7}$$

or

$$A = X\Big(\sum_{i=1}^{s} h^i\Big) \quad \text{and} \quad B = (1-h)Y. \tag{2.8}$$

Let $\sigma\colon \mathbb{Z}[G] \to \mathbb{Z}$ denote the augmentation homomorphism and $H = \langle h\rangle_s$. It follows from definitions (2.5) and (2.6) that $\sigma(A) = 0$ and $\sigma(B) = q$. On the other hand, it follows that if (2.7) are true, then $\sigma(A) = 0$ and if (2.8) hold, then $\sigma(B) = 0$. Hence, equalities (2.7) are true. Looking again at (2.5) and (2.6), we see that

$$\operatorname{supp} A = \big\{\, 1, a, a^i b a^j \mid i \in \{0,1\},\ j \in \{0, 1, \dots, q-1\} \big\}. \tag{2.9}$$

By Lemma 2.1, $\operatorname{supp} A$ can be partitioned into subsets of cardinality > 1 that are contained in distinct left cosets gH, $g \in G$. Since $1 \in \operatorname{supp} A$, there is also an element $h^\ell \neq 1$ in $\operatorname{supp} A$. Now we consider two cases: $r = \infty$ and $(q, r) = (2, 2)$.

Suppose $r = \infty$. Since for all i, j elements $a^i b a^j \in \operatorname{supp} A$ have infinite orders, it follows that $a = h^\ell$ for some ℓ.

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Assume that $(q, r) = (2, 2)$. Then (2.9) turns into

$$\operatorname{supp} A = \{1, a, b, aba, ba, ab\}.$$

Recall that $\operatorname{supp} A$ can be partitioned into some k subsets $S_1, \dots, S_k$ of cardinality greater than 1 that are contained in distinct left cosets gH, $g \in G$. Hence, $k \le 3$. Note that if $g_1, g_2 \in \{1, ba, ab\}$ are distinct, then $g_1^{-1} g_2$ has infinite order in the free product $\langle a \rangle_2 * \langle b \rangle_2$, whence $g_1^{-1} g_2 \notin H$ and g_1, g_2 belong to different sets $S_1, \dots, S_k$. Therefore, $k = 3$.

Now we can verify that there is only one partition $\operatorname{supp} A = S_1 \cup S_2 \cup S_3$ such that $S_1 = \{1, g_2\}$, $S_2 = \{g_3, g_4\}$, $S_3 = \{g_5, g_6\}$, $1 \in S_1$, $ba \in S_2$, $ab \in S_3$, and elements g_2, $g_3^{-1} g_4$, $g_5^{-1} g_6$ commute pairwise. This unique partition is the following: $S_1 = \{1, a\}$, $S_2 = \{b, ba\}$, $S_3 = \{ab, aba\}$. Hence, $a = h^\ell$.

Thus in either case we have proved that $a = h^\ell$ for some ℓ. Then $\ell q = s$ and

$$\sum_{i=1}^{s} h^i = \Big(\sum_{j=0}^{\ell-1} h^j\Big)\Big(\sum_{k=0}^{q-1} h^{\ell k}\Big) = \Big(\sum_{j=0}^{\ell-1} h^j\Big)\Big(\sum_{k=1}^{q} a^k\Big). \tag{2.10}$$

Hence,

$$A\Big(\sum_{i=1}^{s} h^i\Big) = X(1-h)\Big(\sum_{i=1}^{s} h^i\Big) = 0.$$

On the other hand, it follows from (2.10) that

$$\begin{aligned} A\Big(\sum_{i=1}^{s} h^i\Big) &= A\Big(\sum_{k=1}^{q} a^k\Big)\Big(\sum_{j=0}^{\ell-1} h^j\Big) = (1-a)b\Big(\sum_{k=1}^{q} a^k\Big)^2\Big(\sum_{j=0}^{\ell-1} h^j\Big) \\ &= q(1-a)b\Big(\sum_{k=1}^{q} a^k\Big)\Big(\sum_{j=0}^{\ell-1} h^j\Big) = q(1-a)b\Big(\sum_{i=1}^{s} h^i\Big). \end{aligned}$$

Hence, $(1-a)b(\sum_{i=1}^{s} h^i) = 0$ in $\mathbb{Z}[G]$, and, for every product bh^i, $i = 1, \dots, s$, there is j such that $bh^i = abh^j$. This equality implies that $b^{-1}ab = h^{i-j}$, hence $a = h^\ell$ commutes with $b^{-1}ab$ in the free product $\langle a \rangle_q * \langle b \rangle_r$. This is a contradiction, which completes the proof. ■

3 Proofs of Theorems

Proof of Theorem 1.1 Let $F_m = \langle b_1, b_2, \dots, b_m \rangle$ be a free group of rank m with free generators $b_1, b_2, \dots, b_m$ and let $B(m, n) = F_m / F_m^n$ be a free m-generator Burnside group $B(m, n)$ of exponent n, where F_m^n is the (normal) subgroup generated by all n-th powers of elements of F_m. Let $a_1, a_2, \dots, a_m$ be free generators of $B(m, n)$, where a_i is the image of b_i, $i = 1, \dots, m$, under the natural homomorphism $F_m \to B(m, n) = F_m / F_m^n$.

Note that if $G = \langle g_1, g_2 \rangle$ is generated by elements g_1, g_2, and G has exponent n, *i.e.*, $G^n = \{1\}$, then G is a homomorphic image of $B(m, n)$ if $m \ge 2$. Also, there is a nilpotent group $G_{2,n} = \langle g_1, g_2 \rangle$ of exponent n and class 2 in which elements

$[g_1, g_2] := g_1 g_2 g_1^{-1} g_2^{-1}$, g_2, $g_1 g_2^i$, $i = 0, \dots, n-1$, have order n. Therefore, elements $[a_1, a_2] := a_1 a_2 a_1^{-1} a_2^{-1}$, a_2, $a_1 a_2^i$, $i = 0, \dots, n-1$, have order n in $B(m, n)$ if $m \geq 2$. In addition, since $g_1 g_2 g_1^{-1} \notin \langle [g_1, g_2] \rangle$ and $[g_1, g_2]^j g_1 g_2 g_1^{-1} \notin \langle g_1 g_2^i \rangle$ in $G_{2,n}$ for all $i, j \in \{0, \dots, n-1\}$, it follows that $a_1 a_2 a_1^{-1} \notin \langle [a_1, a_2] \rangle$ and $[a_1, a_2]^j a_1 a_2 a_1^{-1} \notin \langle a_1 a_2^i \rangle$ in $B(m, n)$ for all $i, j \in \{0, \dots, n-1\}$.

Recall that if $n \gg 1$ is odd (*e.g.*, $n > 10^{10}$ as in [12]), then every maximal cyclic subgroup of $B(m, n)$ is antinormal in $B(m, n)$ (this is actually shown in the proof of [13, Theorem 19.4]; similar arguments can be found in [5,9]). Since cyclic subgroups $\langle [a_1, a_2] \rangle$, $\langle a_1 a_2^i \rangle$, $i = 0, \dots, n-1$, are of order n and $B(m, n)$ has exponent n, it follows that these subgroups $\langle [a_1, a_2] \rangle$, $\langle a_1 a_2^i \rangle$, $i = 0, \dots, n-1$, are maximal cyclic and hence are antinormal. Now we can see that all the conditions of Lemma 2.2 are satisfied for elements $a = a_1$, $b = a_2$, $c = [a_1, a_2]$, $d = a_1 a_2 a_1^{-1}$ of $B(m, n)$. Hence, Lemma 2.2 applies and yields that equalities (2.1) and (2.2) are impossible. Furthermore, it is easy to see that $(1 + c + \cdots + c^{n-1})(1 - d) \neq 0$, because $c^i d \neq 1$, $i = 0, \dots, n-1$, and $(1-a)(1+b+\cdots+b^{n-1}) \neq 0$, because $ab^j \neq 1$, $j = 0, \dots, n-1$.

Finally, we need to show that

$$(1 + c + \cdots + c^{n-1})(1 - d)(1 - a)(1 + b + \cdots + b^{n-1}) = 0.$$

Note that $d = cb$ and $da = ab$, hence, assuming that $i_1, j_1, \dots, i_4, j_4$ are arbitrary integers that satisfy $0 \leq i_1, j_1, \dots, i_4, j_4 \leq n-1$, we have

$$\begin{aligned}
&(1 + c + \cdots + c^{n-1})(1 - d)(1 - a)(1 + b + \cdots + b^{n-1}) \\
&\quad = \Big(\sum_{i_1} c^{i_1} - \sum_{i_2} c^{i_2} d - \sum_{i_3} c^{i_3} a + \sum_{i_4} c^{i_4} da \Big) \Big(\sum_{j_1} b^{j_1} \Big) \\
&\quad = \Big(\sum_{i_1} c^{i_1} - \sum_{i_2} c^{i_2} cb - \sum_{i_3} c^{i_3} a + \sum_{i_4} c^{i_4} ab \Big) \Big(\sum_{j_1} b^{j_1} \Big) \\
&\quad = \sum_{i_1, j_1} c^{i_1} b^{j_1} - \sum_{i_2, j_2} c^{i_2+1} b^{j_2+1} - \sum_{i_3, j_3} c^{i_3} ab^{i_3} + \sum_{i_4, j_4} c^{i_4} ab^{j_4+1} = 0.
\end{aligned}$$

Thus $(1+c+\cdots+c^{n-1})(1-d)$ and $(1-a)(1+b+\cdots+b^{n-1})$ is a pair of zero-divisors in $\mathbb{Z}[B(m, n)]$, which is not trivial by Lemma 2.2, and Theorem 1.1 is proved. ■

The idea of the above construction of a nontrivial pair of zero-divisors in $\mathbb{Z}[B(m, n)]$ could be associated with Fox derivatives (which is somewhat analogous to [7], however, no mention of Fox derivatives is made in [7]) and may be described as follows. As above, let $F_2 = F(b_1, b_2)$ be a free group with free generators b_1, b_2. For $w \in F_2$, consider Fox derivatives $\frac{\partial w}{\partial b_i} \in \mathbb{Z}[F_2]$, $i = 1, 2$. Then

$$w - 1 = \frac{\partial w}{\partial b_1}(b_1 - 1) + \frac{\partial w}{\partial b_2}(b_2 - 1) \tag{3.1}$$

in $\mathbb{Z}[F_2]$. Letting $w := [b_1, b_2]^n$, we observe that

$$\frac{\partial [b_1, b_2]^n}{\partial b_i} = \Big(\sum_{j=0}^{n-1} [b_1, b_2]^j \Big) \frac{\partial [b_1, b_2]}{\partial b_i}, \quad i = 1, 2.$$

Hence,

$$\frac{\partial [b_1,b_2]^n}{\partial b_1} = \Big(\sum_{j=0}^{n-1}[b_1,b_2]^j\Big)(1-b_1b_2b_1^{-1}),$$

$$\frac{\partial [b_1,b_2]^n}{\partial b_2} = \Big(\sum_{j=0}^{n-1}[b_1,b_2]^j\Big)(b_1-b_1b_2b_1^{-1}b_2^{-1}).$$

Therefore, taking the image of the equality (3.1) in $\mathbb{Z}[B(2,n)]$, we obtain

$$0=[a_1,a_2]^n-1=\Big(\sum_{j=0}^{n-1}[a_1,a_2]^j\Big)(1-a_1a_2a_1^{-1})(a_1-1) + \Big(\sum_{j=0}^{n-1}[a_1,a_2]^j\Big)(a_1-a_1a_2a_1^{-1}a_2^{-1})(a_2-1).$$

Now multiplication on the right by $\sum_{i=0}^{n-1} a_2^i$ yields

$$\Big(\sum_{j=0}^{n-1}[a_1,a_2]^j\Big)(1-a_1a_2a_1^{-1})(a_1-1)\Big(\sum_{i=0}^{n-1}a_2^i\Big)=0,$$

and this is what we have in Theorem 1.1.

Analogously, let a group $G=\langle a_1,a_2\rangle$ be generated by a_1,a_2, let $a_2^n=1$ in G, let $w(b_1,b_2)\in F(b_1,b_2)$ be a word with the property that $w(a_1,a_2)=1$ in G, and let $\theta\colon \mathbb{Z}[F(b_1,b_2)]\to\mathbb{Z}[G]$, where $\theta(b_i)=a_i$, $i=1,2$, denote the natural epimorphism. As above, we can obtain

$$\theta\Big(\frac{\partial w(b_1,b_2)}{\partial b_1}(b_1-1)\Big(\sum_{j=0}^{n-1}b_2^j\Big)\Big)=\theta\Big(\frac{\partial w(b_1,b_2)}{\partial b_1}\Big)(a_1-1)\Big(\sum_{j=0}^{n-1}a_2^j\Big)=0.$$

This equation can be used for constructing other potentially nontrivial pairs of zero-divisors in $\mathbb{Z}[G]$ (which, however, does not work in case when G is a free product of the form $\langle a_1\rangle * \langle a_2\rangle$).

Proof of Theorem 1.2 Suppose G is a group and G contains a subgroup H isomorphic either to a finite noncyclic group or to the free product $C_q * C_r$ of cyclic groups C_q, C_r, where $1<\min(q,r)<\infty$.

First assume that H is a finite noncyclic group. Then there are $h_1,h_2\in H$ such that the subgroup $\langle h_1,h_2\rangle$, generated by h_1,h_2, is not cyclic. Since $2-h_1-h_2$ is a left (right) zero-divisor in $\mathbb{Z}[H]$, $2-h_1-h_2$ is also a left (right, resp.) zero-divisor in $\mathbb{Z}[G]$. If $2-h_1-h_2$ is a trivial left (right, resp.) zero-divisor in $\mathbb{Z}[G]$, then it follows from Lemma 2.1 that elements $1,h_1,h_2$ belong to the same coset gH_0 (H_0g, resp.), where $H_0=\langle h_0\rangle$ is cyclic. But then $g\in H_0$ and $h_1,h_2\in H_0$, whence the subgroup $\langle h_1,h_2\rangle$ is cyclic. This contradiction completes the proof in the case where H is finite noncyclic.

Suppose $C_q * C_r$ is a subgroup of G, $1<\min(q,r)<\infty$. We may assume that q is finite. Denote $C_q=\langle a\rangle_q$ and $C_r=\langle b\rangle_r$. Note that the subgroup $\langle a, babab\rangle$ of C_q*C_r is isomorphic to the free product C_q*C_∞ unless $q=r=2$. Therefore, we may assume that G contains a subgroup isomorphic to $C_{q'}*C_{r'}$, where $q'=q>1$ is finite and either $r'=\infty$ or $q'=r'=2$. Now Theorem 1.2 follows from Lemma 2.3. ■

https://doi.org/10.4153/CMB-2012-036-6 Published online by Cambridge University Press

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Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801, USA
e-mail: ivanov@math.uiuc.edu

Steklov Mathematical Institute, Gubkina 8, Moscow, 119991, Russia

and

Chebyshev Laboratory, St. Petersburg State University, 14th Line, 29b, Saint Petersburg, 199178 Russia
e-mail: romanvm@mi.ras.ru